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Online Appendix For 'Dynamic Pricing with a Prior on Market Response'

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Results in this appendix are numbered consistently with those in the main paper. Results that do not appear in the paper (auxiliary Lemmas or additional theorems omitted from the exposition in the main paper) are numbered using the convention 'SectionLetter.Number' (eg. Theorem E.1).

We recall the following assumptions in several proofs that follow and so find it convenient to repeat them here.

Assumption 1.

1. *$F(\cdot)$ has a differentiable density $f(\cdot)$ with support $\mathbb{R}^+$.*
2. *F has a non-decreasing hazard rate. That is, $\rho(p) = \frac{f(p)}{\bar{F}(p)}$ is non-decreasing in p.*

Assumption 2. *$J^*_\lambda(x)$ is a differentiable function of λ on $\mathbb{R}^+$ for all $x \in \mathbb{N}$.*

A Proofs for Section 3

Lemma 1. *$\pi^*_\lambda(x)$ is decreasing in x (on $\mathbb{N}$) and non-decreasing in λ (on $\mathbb{R}_+$).*

Proof: We find it convenient to prove the following sub-homogeneity property for $J^*_\lambda(x)$ viewed as function of λ: For $\lambda_2 \geq \lambda_1 > 0$, $J^*_{\lambda_2}(x) \leq \frac{\lambda_2}{\lambda_1} J^*_{\lambda_1}(x)$. To see this, consider a system beginning with x units of inventory facing arrivals at rate λ_2. Every arrival to the system is marked as either 'real' or 'fictitious' with probability $\frac{\lambda_1}{\lambda_2}$ and $1 - \frac{\lambda_1}{\lambda_2}$ respectively, independent of all other arrivals. Consider using the pricing policy $\pi^*_{\lambda_2}(\cdot)$, and denote by $J^{*,f}_{\lambda_2}(x)$ and $J^{*,r}_{\lambda_2}(x)$ the expected revenues earned under this policy from sales to arrivals marked as fictitious and real respectively. By construction, we have $J^*_{\lambda_2}(x) = J^{*,f}_{\lambda_2}(x) + J^{*,r}_{\lambda_2}(x)$ and further, $J^{*,r}_{\lambda_2}(x) = \frac{\lambda_1}{\lambda_2} J^*_{\lambda_2}(x)$. But $J^{*,r}_{\lambda_2}(x)$ is the expected revenue earned under a randomized non-anticipatory policy for a system beginning with x units of inventory and arrival rate λ_1, so that $J^{*,r}_{\lambda_2}(x) \leq J^*_{\lambda_1}(x)$. Thus $\frac{\lambda_1}{\lambda_2} J^*_{\lambda_2}(x) \leq J^*_{\lambda_1}(x)$ which is the inequality we require.

We now turn to the proof of the Lemma. We have from the HJB equation for the case of a known arrival rate and $x > 0$:

$$\frac{\alpha J^*_\lambda(x)}{\lambda} = \sup_p \bar{F}(p)(p + J^*_\lambda(x-1) - J^*_\lambda(x))$$

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Now $\frac{\alpha J^*_\lambda(x)}{\lambda}$ is trivially increasing in x. $\frac{\alpha J^*_\lambda(x)}{\lambda}$ is non-increasing in λ by the inequality we have just shown (i.e. since $J^*_\lambda(x)$ is a sub-homogenous function of λ). Further, observe that $\sup_p \bar{F}(p)(p-c)$ is decreasing in c. It follows that $J^*_\lambda(x) - J^*_\lambda(x-1)$ is decreasing in x and non-decreasing in λ. But, $\pi^*_\lambda(x) - \frac{1}{\rho(\pi^*_\lambda(x))} = J^*_\lambda(x) - J^*_\lambda(x-1)$ and $p - 1/\rho(p)$ is an increasing function of p by Assumption 1. The claim follows. □

Lemma 2. *For all $x \in \mathbb{N}$, $J^*_\lambda(x)$ is an increasing, concave function of λ on $\mathbb{R}_+$.*

Proof: Consider two systems with $\lambda_1 < \lambda_2$. We will show that $\frac{d}{d\lambda} J^*_\lambda(x)|_{\lambda=\lambda_1} \geq \frac{d}{d\lambda} J^*_\lambda(x)|_{\lambda=\lambda_2}$. Delaying a proof until later in our argument, we have:

$$\frac{d}{d\lambda} J^*_\lambda(x)\bigg|_{\lambda=\bar{\lambda}} = E\left[T_\alpha \pi^*_{\bar{\lambda}}(x_{T_\alpha}) \bar{F}\left(\pi^*_{\bar{\lambda}}(x_{T_\alpha})\right)\right] \tag{1}$$

where T_α is exponentially distributed with mean $1/\alpha$. Now, the instantaneous rate at which a sale occurs in a system with arrival rate λ and x units of inventory on hand is given by $\lambda \bar{F}(\pi^*_\lambda(x)) = \lambda \frac{\alpha J^*_\lambda(x) \rho(\pi^*_\lambda(x))}{\lambda} = \alpha J^*_\lambda(x) \rho(\pi^*_\lambda(x))$, which is an increasing function of λ, since $\pi^*_\lambda(x)$ and $J^*_\lambda(x)$ are increasing functions of λ (see Lemma 1) and $\rho(\cdot)$ is a non-decreasing function by Assumption 1. Thus, letting $x^{\lambda_i}_{T_\alpha}$ be the inventory on hand at time T_α in the ith system (for $i = 1, 2$), we must have that $x^{\lambda_1}_{T_\alpha}$ stochastically dominates $x^{\lambda_2}_{T_\alpha}$. We consequently have:

$$\begin{aligned}
\frac{d}{d\lambda} J^*_\lambda(x)\bigg|_{\lambda=\lambda_2} &= E\left[T_\alpha \pi^*_{\lambda_2}(x^{\lambda_2}_{T_\alpha}) \bar{F}\left(\pi^*_{\lambda_2}(x^{\lambda_2}_{T_\alpha})\right)\right] \\
&\leq E\left[T_\alpha \pi^*_{\lambda_2}(x^{\lambda_1}_{T_\alpha}) \bar{F}\left(\pi^*_{\lambda_2}(x^{\lambda_1}_{T_\alpha})\right)\right] \\
&\leq E\left[T_\alpha \pi^*_{\lambda_1}(x^{\lambda_1}_{T_\alpha}) \bar{F}\left(\pi^*_{\lambda_1}(x^{\lambda_1}_{T_\alpha})\right)\right] \\
&= \frac{d}{d\lambda} J^*_\lambda(x)\bigg|_{\lambda=\lambda_1}
\end{aligned}$$

The first inequality follows from the fact that $\pi^*_\lambda(x)$ is decreasing in x by Lemma 1 and since $p\bar{F}(p)$ is decreasing in p for $p \geq p^*$ (the static revenue maximizing price). The second inequality follows from the fact that $\pi^*_\lambda(x)$ is increasing in λ by Lemma 1 and since $p\bar{F}(p)$ is decreasing in p for $p \geq p^*$. That $p\bar{F}(p)$ is decreasing in p for $p \geq p^*$ follows from the fact that $\frac{d}{dp} p\bar{F}(p) = f(p)(1/\rho(p) - p)$ which by Assumption 1 is negative for $p > p^*$ and 0 at $p = p^*$.

That $J^*_\lambda(x)$ is increasing in λ follows from the positivity of the right hand side in (1).

We now establish the equality (1). Consider a system with arrival rate λ. The expected revenue from this system is equal to the expected revenue from an un-discounted system, where after a random time $T_\alpha \sim \exp(1/\alpha)$, no revenues are recorded. This can be seen by simply noting that the HJB equations for the respective problems are identical and given by

$$\alpha J^*_\lambda(x) = \begin{cases} \sup_{p \geq 0} \lambda \overline{F}(p)(p + J^*_\lambda(x-1) - J^*_\lambda(x)) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In particular,

$$J^*_\lambda(x) = E\left[\int_0^{T_\alpha} \pi^*_\lambda(x_t)\bar{F}(\pi^*_\lambda(x_t))\lambda dt \Big| x_0 = x\right] \tag{3}$$

Next, we observe that increasing λ to $\lambda+\delta$ is equivalent to decreasing α to $\alpha(\frac{\lambda}{\lambda+\delta})$. That is,

$$J^{*,\alpha}_{\lambda+\delta}(x) = J^{*,\alpha(\frac{\lambda}{\lambda+\delta})}_\lambda(x)$$

which is immediate from the HJB equation for a known arrival rate. This in turn is equivalent to increasing T_α on each sample path to $T_\alpha(1+\delta/\lambda)$. In particular, we have:

$$\begin{aligned} J^{*,\alpha}_{\lambda+\delta}(x) &= E\left[\int_0^\infty \pi^*_{\lambda+\delta}(x_t)\bar{F}(\pi^*_{\lambda+\delta}(x_t))(\lambda+\delta)\exp(-\alpha t)dt \Big| x_0 = x\right] \\ &= E\left[\int_0^\infty \pi^*_{\lambda+\delta}(x_t)\bar{F}(\pi^*_{\lambda+\delta}(x_t))\lambda \exp\left(-\alpha\frac{\lambda}{\lambda+\delta}t\right) dt \Big| x_0 = x\right] \\ &= E\left[\int_0^{T_\alpha(1+\delta/\lambda)} \pi^*_{\lambda+\delta}(x_t)\bar{F}(\pi^*_{\lambda+\delta}(x_t))\lambda dt \Big| x_0 = x\right] \end{aligned} \tag{4}$$

where the second equality follows by noting that the optimal policy for the system with arrival rate $\lambda+\delta$ and discount factor α is identical to that for the system with arrival rate λ and discount factor $\alpha\frac{\lambda}{\lambda+\delta}$ which in turn follows from the fact that the HJB equations for the two systems are identical. The third equality follows as in (3).

Now $\pi^*_\lambda(x)$ is a differentiable function of λ for all $x \in \mathbb{N}$. To see this we note that $\pi^*_\lambda(x)$ is given implicitly by

$$\pi^*_\lambda(x) = 1/\rho(\pi^*_\lambda(x)) + J^*_\lambda(x) - 1_{x>0}J^*_\lambda(x-1).$$

Since $p - 1/\rho(p)$ is increasing on $p \geq 0$ with $\mathbb{R}^+$ in its range (and therefore invertible on $\mathbb{R}^+$) and differentiable in p (all of which follows from Assumptions 1) and since $J^*_\lambda(x)$ was assumed differentiable in λ (Assumption 2) we may invoke the Inverse Function Theorem to conclude that $\pi^*_\lambda(x)$ is a differentiable function of λ on $\mathbb{R}^+$.

Let x'_t denote the inventory on hand at time t in an optimally controlled system with arrival rate $\lambda+\delta$. Let us couple the sales processes in the systems with arrival rate λ and $\lambda+\delta$ as follows: assume the prevailing prices in the two systems are p and p' respectively. If $\lambda\bar{F}(p) \leq (\lambda+\delta)\bar{F}(p')$ then the system with arrival rate λ will witness its next sale no sooner than the system with arrival rate $\lambda+\delta$; the next sale to the system with arrival rate $\lambda+\delta$ will arrive at rate $(\lambda+\delta)\bar{F}(p')$ and will constitute a sale in the system with arrival rate λ with probability $\lambda\bar{F}(p)/(\lambda+\delta)\bar{F}(p')$. The situation is reversed if $(\lambda+\delta)\bar{F}(p') < \lambda\bar{F}(p)$. By the continuity of π^*_λ in λ, we have $x'_{T_\alpha} \to x_{T_\alpha}$ in probability under this coupling. Then, by the Cauchy-Schwarz inequality,

$$|E[T_\alpha\pi^*_{\lambda+\delta}(x_{T_\alpha})\bar{F}(\pi^*_{\lambda+\delta}(x_{T_\alpha}))] - E[T_\alpha\pi^*_{\lambda+\delta}(x'_{T_\alpha})\bar{F}(\pi^*_{\lambda+\delta}(x'_{T_\alpha}))]| \leq 2\sup_p p\bar{F}(p)\sqrt{\Pr(x'_{T_\alpha} \neq x_{T_\alpha})E[T^2_\alpha]},$$

where $\Pr(\cdot)$ is the joint measure induced by our coupling. Since $\sup_p p\bar{F}(p) < \infty$ by Assumption 1, we thus have:

$$E[T_\alpha \pi^*_{\lambda+\delta}(x_{T_\alpha})\bar{F}(\pi^*_{\lambda+\delta}(x_{T_\alpha}))] - E[T_\alpha \pi^*_{\lambda+\delta}(x'_{T_\alpha})\bar{F}(\pi^*_{\lambda+\delta}(x'_{T_\alpha}))] \to 0$$

Again, via the continuity of π^*_λ in λ, the dominated convergence theorem yields

$$E[T_\alpha \pi^*_{\lambda}(x_{T_\alpha})\bar{F}(\pi^*_{\lambda}(x_{T_\alpha}))] - E[T_\alpha \pi^*_{\lambda+\delta}(x_{T_\alpha})\bar{F}(\pi^*_{\lambda+\delta}(x_{T_\alpha}))] \to 0$$

by considering the dominating random variable $2T_\alpha \sup_p p\bar{F}(p)$. Together, the preceding two limits let us conclude that

$$E[T_\alpha \pi^*_{\lambda}(x_{T_\alpha})\bar{F}(\pi^*_{\lambda}(x_{T_\alpha}))] - E[T_\alpha \pi^*_{\lambda+\delta}(x'_{T_\alpha})\bar{F}(\pi^*_{\lambda+\delta}(x'_{T_\alpha}))] \to 0$$

Together with (3) and (4) this yields:

(5)

$$\begin{aligned}
&\frac{d}{d\lambda} J^*_\lambda(x)\Big|_{\lambda=\bar{\lambda}} \\
&= \lim_{\delta\to 0}\left(E\left[\int_0^{T_\alpha(1+\delta/\bar{\lambda})} \pi^*_{\bar{\lambda}+\delta}(x_t)\bar{F}(\pi^*_{\bar{\lambda}+\delta}(x_t))\bar{\lambda}dt \Big| x_0 = x\right] - E\left[\int_0^{T_\alpha} \pi^*_{\bar{\lambda}}(x_t)\bar{F}(\pi^*_{\bar{\lambda}}(x_t))\bar{\lambda}dt \Big| x_0 = x\right]\right)/\delta \\
&= \frac{d}{d\lambda} E\left[\int_0^{T_\alpha} \pi^*_\lambda(x_t)\bar{F}(\pi^*_\lambda(x_t))\bar{\lambda}dt \Big| x_0 = x\right]\Big|_{\lambda=\bar{\lambda}} + \lim_{\delta\to 0}\left(E\left[T_\alpha \pi^*_{\bar{\lambda}+\delta}(x'_{T_\alpha})\bar{F}\left(\pi^*_{\bar{\lambda}+\delta}(x'_{T_\alpha})\right)\right] + O(\delta)\right) \\
&= \frac{d}{d\lambda} E\left[\int_0^{T_\alpha} \pi^*_\lambda(x_t)\bar{F}(\pi^*_\lambda(x_t))\bar{\lambda}dt \Big| x_0 = x\right]\Big|_{\lambda=\bar{\lambda}} + E\left[T_\alpha \pi^*_{\bar{\lambda}}(x_{T_\alpha})\bar{F}\left(\pi^*_{\bar{\lambda}}(x_{T_\alpha})\right)\right]
\end{aligned}$$

We note that

$$E\left[\int_0^{T_\alpha} \pi_\lambda(x_t)\bar{F}(\pi_\lambda(x_t))\bar{\lambda}dt \Big| x_0 = x\right]$$

is differentiable with respect to $\pi_\lambda(\cdot)$. This follows from the differentiability of $E[\exp(-\alpha\tau)]$ with respect to η when τ is distributed as an exponential random variable with parameter η, and since $\bar{F}$ is differentiable by Assumption 1.

Now,

$$\begin{aligned}
&\frac{d}{d\lambda} E\left[\int_0^{T_\alpha} \pi^*_\lambda(x_t)\bar{F}(\pi^*_\lambda(x_t))\bar{\lambda}dt \Big| x_0 = x\right]\Big|_{\lambda=\bar{\lambda}} \\
&= \sum_{X=0}^{x}\left(\frac{d}{d\lambda}\pi^*_\lambda(X)\right)\left(\frac{d}{d\pi_\lambda(X)} E\left[\int_0^{T_\alpha} \pi_\lambda(x_t)\bar{F}(\pi_\lambda(x_t))\bar{\lambda}dt \Big| x_0 = x\right]\Big|_{\pi_\lambda(X)=\pi^*_\lambda(X)}\right)\Big|_{\lambda=\bar{\lambda}} \\
&= 0
\end{aligned}$$

where we use fact that since $\pi^*_{\bar{\lambda}}$ attains maximum revenue for an arrival rate $\lambda = \bar{\lambda}$,

$$\frac{d}{d\pi_\lambda(X)} E\left[\int_0^{T_\alpha} \pi_\lambda(x_t)\bar{F}(\pi_\lambda(x_t))\bar{\lambda}dt \Big| x_0 = x\right]\Big|_{\pi_\lambda(X)=\pi^*_{\bar{\lambda}}(X)} = 0$$

With (5), this yields equality (1) and the proof. $\square$

B Proofs for Section 4

Lemma 3. *For all $z \in \mathcal{S}, \alpha > 0$*

$$J^*(z) \leq \tilde{J}(z) \leq J^*_{\mu(z)}(x) \leq \frac{\overline{F}(p^*)p^*\mu(z)}{\alpha}.$$

where p^* is the static revenue maximizing price.

Proof: Since $J^*_\lambda(x)$ is concave in λ by Lemma 2, Jensen's inequality gives us that $J^*_{a/b}(x) = J^*_{E[\lambda]}(x) \geq E[J^*_\lambda(x)] = \tilde{J}(z)$. Note that $J^*_\lambda(x)$ is bounded above by the value of a system with customer arrival rate λ but without a finite capacity constraint. The optimal policy in such a system is simply to charge the static revenue maximizing price, p^*, garnering a value of $\frac{\overline{F}(p^*)p^*\lambda}{\alpha}$ yielding $J^*_\lambda(x) \leq \frac{\overline{F}(p^*)p^*\lambda}{\alpha}$. $\square$

Lemma 4. *For all $z \in \mathcal{S}$, there is a unique $p \geq 0$ such that $\frac{\overline{F}(p)}{\rho(p)}\mu(z) = \alpha\tilde{J}(z)$.*

Proof: Note that $\frac{\overline{F}(p)p\mu(z)}{\alpha}$ is a continuous, monotone decreasing function of p for $p \geq p^*$ under Assumption 1. But since $\frac{\overline{F}(\pi^*(z))\pi^*(z)\mu(z)}{\alpha} = J^*(z)$, the result is immediate from Lemma 3; in fact the unique solution to $\frac{\overline{F}(p)}{\rho(p)}\mu(z) = \alpha\tilde{J}(z)$ must be in $[p^*, \pi^*(z)]$. $\square$

C Proofs for Section 6

Lemma 5. *Let $\pi : \mathcal{S} \to \mathbb{R}_+$ be an arbitrary policy and let $\pi' : \mathcal{S} \to \mathbb{R}_+$ be defined according to $\pi'(x, a, b) = \pi(x, a, b/\alpha)$. Then, for all $z \in \mathcal{S}, \alpha > 0$, $J^{\pi,\alpha}(z) = J^{\pi',1}(x, a, \alpha b)$, and, in particular, $J^{*,\alpha}(z) = J^{*,1}(x, a, \alpha b)$.*

Proof: Let $\hat{z} \equiv (\hat{x}, \hat{a}, \hat{b}) \in \mathcal{S}$ be arbitrary. Restricting attention to the pricing policy π, we have that $J^{\pi,\alpha}$ is given by the unique solution to the HJB equation $H^\pi J = 0$. That is, $J^{\pi,\alpha}$ uniquely satisfies

$$\overline{F}(\pi(x,a,b))\left(\frac{a}{b}\left(\pi(x,a,b) + J(x-a,a+1,b) - J(x,a,b)\right) + \frac{d}{db}J(x,a,b)\right) - \alpha J(x,a,b) = 0, \tag{6}$$

for all $z \in \mathcal{S}_{\hat{x},\hat{a},\hat{b}}$ and similarly for $J^{\pi',1}$. In particular,

$$\overline{F}(\pi(x,a,b))\left(\frac{a}{b}\left(\pi(x,a,b) + J^{\pi,\alpha}(x-a,a+1,b) - J^{\pi,\alpha}(x,a,b)\right) + \frac{d}{db}J^{\pi,\alpha}(x,a,b)\right) - \alpha J^{\pi,\alpha}(x,a,b) = 0,$$

for all $z \in \mathcal{S}_{\hat{x},\hat{a},\hat{b}}$ and

$$\overline{F}(\pi'(x,a,b))\left(\frac{a}{b}\left(\pi'(x,a,b) + J^{\pi',1}(x-a,a+1,b) - J^{\pi',1}(x,a,b)\right) + \frac{d}{db}J^{\pi',1}(x,a,b)\right) - J^{\pi',1}(x,a,b) = 0$$

for all $z \in \mathcal{S}_{\hat{x},\hat{a},\alpha\hat{b}}$.

Now, in order to prove our claim it will suffice to show that $\bar{J}(z)$ defined according to $\bar{J}(x,a,b) = J^{\pi',1}(x,a,\alpha b)$ satisfies (6). But, identifying the change of variables $b' = \alpha b$, we have:

$$
\begin{aligned}
&\overline{F}(\pi(x,a,b))\left(\frac{a}{b}\left(\pi(x,a,b) + J^{\pi',1}(x-a,a+1,\alpha b) - J^{\pi',1}(x,a,\alpha b)\right) + \frac{d}{db}J^{\pi',1}(x,a,\alpha b)\right) \\
&- \alpha J^{\pi',1}(x,a,\alpha b) \\
&= \overline{F}(\pi(x,a,b'/\alpha))\left(\frac{a\alpha}{b'}\left(\pi(x,a,b'/\alpha) + J^{\pi',1}(x-a,a+1,b') - J^{\pi',1}(x,a,b')\right) + \frac{d}{db}J^{\pi',1}(x,a,b')\right) \\
&\quad - \alpha J^{\pi',1}(x,a,b') \\
&= \alpha\left(\overline{F}(\pi'(x,a,b'))\left(\frac{a}{b'}\left(\pi'(x,a,b') + J^{\pi',1}(x-a,a+1,b') - J^{\pi',1}(x,a,\alpha b)\right) + \frac{d}{db'}J^{\pi',1}(x,a,b')\right)\right) \\
&\quad - \alpha J^{\pi',1}(x,a,b') \\
&= 0.
\end{aligned}
$$

This suffices for the proof. $\square$

Lemma 6. *Let $J \in \mathcal{J}$ satisfy $J(0,a,b) = 0$. Let $\tau = \inf\{t : J(z_t) = 0\}$. Let $z_0 \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$. Then,*

$$E\left[\int_0^\tau e^{-\alpha t} H^\pi J(z_t)dt\right] = J^\pi(z_0) - J(z_0)$$

Let $J : \mathbb{N} \to \mathbb{R}$ be bounded and satisfy $J(0) = 0$. Let $\tau = \inf\{t : J(x_t) = 0\}$. Let $x_0 \in \mathbb{N}$. Then,

$$E\left[\int_0^\tau e^{-\alpha t} H^\pi_\lambda J(x_t)dt\right] = J^\pi_\lambda(x_0) - J(x_0)$$

Proof: Define for $J \in \mathcal{J}$, and $\pi \in \Pi$,

$$\mathcal{A}_{\pi,z}J(z) = \lim_{t>0,t\to 0} \frac{e^{-\alpha t}E_{z,\pi}[J(z(t))] - J(z)}{t}.$$

Further, define

$$H^\pi J(z) = \overline{F}(\pi(z))\frac{a}{b}\pi(z) + \mathcal{A}_{\pi,z}J(z)$$

Lemma E.5 verifies that this definition is in agreement with our previous definition provided $J \in \mathcal{J}$. Let τ be a stopping time of the filtration $\sigma(z^t)$ (where $z^t = \{z_{t'} : 0 \le t' \le t\}$). We then have:

$$
\begin{aligned}
E\left[\int_0^\tau e^{-\alpha t} H^\pi J(z_t)dt\right] &= E\left[\int_0^\tau e^{-\alpha t}\left(\overline{F}(\pi(z_t))\frac{a_t}{b_t}\pi(z_t) + \mathcal{A}_{\pi,z}J(z_t)\right)dt\right] \\
&= J^\pi(z_0) + E_{z_0}\left[e^{-\alpha\tau}J(z_\tau)\right] - J(z_0) \\
&= J^\pi(z_0) - J(z_0)
\end{aligned}
$$

The second equality follows from the fact that

$$E\left[\int_0^\tau e^{-\alpha t}\mathcal{A}_{\pi,z}J(z_t)dt\right] = E_{z_0}\left[e^{-\alpha\tau}J(z_\tau)\right] - J(z_0)$$

which is Dynkin's formula for Markov processes (see III.10 in Rogers and Williams (2000)). The third equality follows by the definition of τ and the assumption that $J(0,a,b) = 0$. The proof of the second assertion is identical. $\square$

Lemma 7. *If $\lambda < \mu$, $J_\lambda^{\pi^{nl}}(x) \geq (\lambda/\mu) J_\mu^*(x)$ for all $x \in \mathbb{N}$.*

Proof: Letting $\tau = \inf\{t : n_t = x_0\}$ as usual, we have

$$\begin{aligned}
-E\left[\int_0^\tau e^{-\alpha t} H_\lambda^{\pi^{nl}} J_\mu^*(x_t) dt\right] &= E\left[\int_0^\tau e^{-\alpha t}(1-\lambda/\mu)\alpha J_\mu^*(x_t) dt\right] \\
&\leq E\left[\int_0^\tau e^{-\alpha t}(1-\lambda/\mu)\alpha J_\mu^*(x_0) dt\right] \\
&\leq (1-\lambda/\mu) J_\mu^*(x_0)
\end{aligned}$$

where the inequality follows from the fact that $J_\mu^*(x)$ is decreasing in x and since $\lambda < \mu$ here. So, from Lemma 6, we immediately have:

$$J_\mu^*(x_0) - J_\lambda^{\pi^{nl}}(x_0) \leq (1-\lambda/\mu) J_\mu^*(x_0)$$

which is the result. $\square$

Lemma 8. *If $\lambda \geq \mu$, $J_\lambda^{\pi^{nl}}(x) \geq J_\mu^*(x)$ for all $x \in \mathbb{N}$.*

Proof: Here,

$$-E\left[\int_0^\tau e^{-\alpha t} H_\lambda^{\pi^{nl}} J_\mu^*(x(t)) dt\right] \leq 0$$

so the result follows immediately from Lemma 6. $\square$

Corollary 1. *For all $z \in \mathcal{S}$, and exponential reservation price distributions with parameter r:*

$$\frac{1}{1+\log \kappa(a)} \leq \frac{\pi_{\text{db}}(z)}{\pi^*(z)} \leq 1$$

For all $z \in \mathcal{S}$, and logit reservation price distributions with parameter r:

$$\frac{1.27}{1.27+\log \kappa(a)} \leq \frac{\pi_{\text{db}}(z)}{\pi^*(z)} \leq 1$$

Proof: The decay balancing equation for exponential reservation prices yields:

$$\begin{aligned}
\frac{\pi_{\text{db}}(z)}{\pi^*(z)} &= \frac{r \log \frac{ra}{be^{-1}\tilde{J}(z)}}{r \log \frac{ra}{be^{-1}J^*(z)}} \\
&\geq \frac{\log \frac{ra}{be^{-1}\tilde{J}(z)}}{\log \frac{ra\kappa(a)}{be^{-1}\tilde{J}(z)}} \\
&= \frac{\log \frac{ra}{be^{-1}\tilde{J}(z)}}{\log \frac{ra}{be^{-1}\tilde{J}(z)} + \log \kappa(a)} \\
&\geq \frac{1}{1+\log \kappa(a)}
\end{aligned}$$

where the first inequality follows from Theorem 1 and the second inequality follows from the fact that by Lemma 3, $\tilde{J}(z) \leq \frac{a}{b} r$. That $\pi_{\text{db}}(z) \leq \pi^*(z)$ is immediate from the decay balance equation and the fact that $\tilde{J}(z) \geq J^*(z)$. The proof of the bound for logit reservation prices is identical; we employ the fact that for logit reservation prices, $\bar{F}(p^*)p^* = e^{-1.27} r$, so that $\tilde{J}(z) \leq \frac{a}{b} r e^{-0.27}$. $\square$

Lemma 9. *For all $z \in \mathcal{S}$, and reservation price distributions satisfying Assumptions 1 and 2,*

$$J^{\text{ub}}(z) \geq J^*(z)$$

Proof: Define the operator:

$$(H^{\text{ub}} J)(z) = \overline{F}(\pi_{\text{db}}(z)) \left(\frac{a}{b} \left(\pi^*(z) + J(z') - J(z) \right) + \frac{d}{db} J(z) \right) - e^{-1} J(z).$$

Analogous to the proof of Theorem E.1, one may verify that J^{ub} is the unique bounded solution to $(H^{\text{ub}} J)(z) = 0$ for all $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$ satisfying $J^{\text{ub}}(0, a, b) = 0$. Identically to the proof of Lemma 6, we can then show for $J \in \mathcal{J}$ satisfying $J(0, a, b) = 0$, and $z_0 \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$ that

$$E \left[\int_0^\tau e^{-\alpha t} H^{\text{ub}} J(z_t) dt \right] = J^{\text{ub}}(z_0) - J(z_0) \tag{7}$$

Now, observe that for $x > 0$,

$$\begin{aligned}
&(H^{\text{ub}} J^*)(z) \\
&= \overline{F}(\pi_{\text{db}}(z)) \left(\frac{a}{b} \left(\pi^*(z) + J^*(z') - J^*(z) \right) + \frac{d}{db} J^*(z) \right) - e^{-1} J^*(z) \\
&\geq \overline{F}(\pi^*(z)) \left(\frac{a}{b} \left(\pi^*(z) + J^*(z') - J^*(z) \right) + \frac{d}{db} J^*(z) \right) - e^{-1} J^*(z) \\
&= 0
\end{aligned}$$

where for the inequality, we use the fact that

$$\pi^*(z) + J^*(z') - J^*(z) + \frac{b}{a} \frac{d}{db} J^*(z) = 1/\rho(\pi^*(z)) \geq 0$$

and that $\pi_{\text{db}}(z) \leq \pi^*(z)$ from Corollary 1. The equality is simply the HJB equation. We consequently have

$$H^{\text{ub}} J^*(z) \geq 0 \quad \forall z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$$

so that (7) applied to J^* immediately gives:

$$J^{\text{ub}}(x, a, b) \geq J^*(x, a, b)$$

$\square$

Lemma 10. *Let $\pi : \mathcal{S} \to \mathbb{R}_+$ be an arbitrary policy and let $\pi' : \mathcal{S} \to \mathbb{R}_+$ be defined according to $\pi'(z) = (1/r)\pi(z)$. Then, for all $z \in \mathcal{S}, \alpha > 0, r > 0$, $J^{\pi,\alpha,r}(z) = r J^{\pi',\alpha,1}(z)$ and, in particular, $J^{*,\alpha,r}(z) = r J^{*,\alpha,1}(z)$.*

Proof: Consider the following coupling of the r system starting at state $z = (x, a, b)$, and of the 1 system starting at state z. The first system is controlled by the price function $\pi(\cdot)$ while the second is controlled by the price function $\pi'(\cdot) = (1/r)\pi(\cdot)$. Consider the evolution of both systems under a sample path with arrivals at $\{t_k\}$ and a corresponding binary valued sequence $\{\psi_k\}$ indicating whether or not the consumer chose to make a purchase. Let $E[\cdot]$ be a joint expectation over $\{t_k, \psi_k; k \leq x\}$ assuming $\{t_k\}$ are the points of a Poisson(λ) process where $\lambda \sim \Gamma(a, b)$, and ψ_k is a Bernoulli random variable with parameter $\exp(-\pi(t_k^-)/r) = \exp(-\pi'(t_k^-))$. We then have:

$$\begin{aligned} J^{\pi,\alpha,r}(z) &= E\left[\sum_{k=1}^{x} \psi_k \pi(t_k^-) \exp(-\alpha(t_k))\right] \\ &= rE\left[\sum_{k=1}^{x} \psi_k \pi'(t_k^-) \exp(-\alpha(t_k))\right] \\ &= rJ^{\pi',\alpha,1}(z) \end{aligned}$$

The result follows. □

Lemma 11. *For all $z \in \mathcal{S}$,*

$$J^*(z|\tau) \leq e^{-e^{-1}\tau}\left(e^{-(\pi^*-\pi_{\rm db})}\left[\pi^* + J^*(x-1, a+1, b_\tau^{\rm db})\right] + (1 - e^{-(\pi^*-\pi_{\rm db})})J^*(x, a+1, b_\tau^{\rm db})\right)$$

where $\pi^* = \pi^*(x, a, b_\tau^*)$ and $\pi_{\rm db} = \pi_{\rm db}(x, a, b_\tau^{\rm db})$.

Proof: Since $\pi^*(\cdot) \geq \pi_{\rm db}(\cdot)$, and further since $\pi_{\rm db}(\cdot)$ is decreasing in b [1], we must have that $\pi_t^* \geq \pi_{{\rm db}t}$ on $t < \tau$. Thus, by our coupling we must have that $n_t^* \leq n_t^{\rm db}$ on $t \leq \tau$; $n_\tau^* = 1$ with probability $e^{-(\pi^*-\pi_{\rm db})}$ and $n_\tau^* = 0$ with the remaining probability. Moreover, conditioned on τ and n_τ^*, λ is distributed as a Gamma random variable with shape parameter $a+1$ and scale parameter $b_\tau^{\rm db}$.

We thus have

$$\begin{aligned} &J^*(z|\tau) \\ &= E\left[\int_{t=0}^{\infty} e^{-e^{-1}t}\pi^*(z_t^*)\lambda\overline{F}(\pi^*(z_t^*))dt \middle| \tau, z_0^* = z\right] \\ &= e^{-e^{-1}\tau}e^{-(\pi^*-\pi_{\rm db})}\pi^* + e^{-(\pi^*-\pi_{\rm db})}E\left[\int_{t=\tau}^{\infty} e^{-e^{-1}t}\pi^*(z_t^*)\lambda\overline{F}(\pi^*(z_t^*))dt \middle| \tau, x_\tau^* = x-1, z_0^* = z\right] \\ &\quad + (1 - e^{-(\pi^*-\pi_{\rm db})})E\left[\int_{t=\tau}^{\infty} e^{-e^{-1}t}\pi^*(z_t^*)\lambda\overline{F}(\pi^*(z_t^*))dt \middle| \tau, x_\tau^* = x, z_0^* = z\right] \end{aligned}$$

[1] This follows easily from the fact that for any positive constant k, X/k is distributed as a Gamma random variable with parameters (a, bk) if X is distributed as a Gamma random variable with parameters (a, b).

But by our observation on the posterior statistics of λ given τ and n^*_τ,

$$\begin{aligned}
&E\left[\int_{t=\tau}^{\infty} e^{-e^{-1}t}\pi^*(z^*_t)\lambda\overline{F}(\pi^*(z^*_t))dt\Big|\tau, x^*_\tau = x-1, z^*_0 = z\right]\\
&\le \sup_{\pi_t: t\ge\tau} E\left[\int_{t=\tau}^{\infty} e^{-e^{-1}t}\pi_t\lambda\overline{F}(\pi_t)dt\Big|\tau, x^*_\tau = x-1, z^*_0 = z\right]\\
&= e^{-e^{-1}\tau}J^*(x-1, a+1, b^{\mathrm{db}}_\tau)
\end{aligned}$$

and similarly

$$\begin{aligned}
&E\left[\int_{t=\tau}^{\infty} e^{-e^{-1}t}\pi^*(z^*_t)\lambda\overline{F}(\pi^*(z^*_t))dt\Big|\tau, x^*_\tau = x, z^*_0 = z\right]\\
&\le \sup_{\pi_t: t\ge\tau} E\left[\int_{t=\tau}^{\infty} e^{-e^{-1}t}\pi_t\lambda\overline{F}(\pi_t)dt\Big|\tau, x^*_\tau = x, z^*_0 = z\right]\\
&= e^{-e^{-1}\tau}J^*(x, a+1, b^{\mathrm{db}}_\tau)
\end{aligned}$$

This yields the result. □

Lemma 12. *For $x > 1, a > 1, b > 0$, $J^*(x,a,b) \le 2.05J^*(x-1,a,b)$.*

Proof: We establish this result for the case where $\alpha = e^{-1}$. This is without loss since by Lemma 5 we know that for all $x > 1, a > 1, b > 0$, $J^{*,\alpha}(x,a,b) \le 2.05J^{*,\alpha}(x-1,a,b) \Leftrightarrow J^{*,e^{-1}}(x,a,\alpha b/e^{-1}) \le 2.05J^{*,e^{-1}}(x-1,a,\alpha b/e^{-1})$.

Let $\tau_1 = \inf\{t : n^*(t) = x-1\}$, and define

$$J^{*,\tau_1}(z) = E_{z,\pi^*}\left[\sum_{k=1}^{x-1} e^{-e^{-1}t_k}\pi_{t_k^-}\right].$$

Now,

$$J^*(z) = J^{*,\tau_1}(z) + E\left[e^{-e^{-1}\tau_1}J^*(1, a+x-1, b_{\tau_1})\right] \tag{8}$$

We will show that $E\left[e^{-e^{-1}\tau_1}J^*(1,a+x-1,b_{\tau_1})\right] \le 1.05J^*(x-1,a,b)$. Since we know by definition that $J^*(x-1,a,b) \ge J^{*,\tau_1}(z)$, the result will then follow immediately from (8).

To show $E\left[e^{-e^{-1}\tau_1}J^*(1,a+x-1,b_{\tau_1})\right] \le 1.05J^*(x-1,a,b)$, we will first establish a lower bound on

$$\pi^*(2, a+x-2, b_{\tau_1})/J^*(1, a+x-1, b_{\tau_1}).$$

Let $a+x-2 \equiv k, a+x-1 \equiv k'$. Certainly, $k' \le 2k$ since $a > 1$. Now,

$$\pi^*(2,k,b) = 1 + \log k/b - \log J^*(2,k,b) \ge 1 + \log k/b - \log J^*_{k/b}(2)$$

and $J^*(1,k',b) \le J^*(1,2k,b) \le J^*_{2k/b}(1)$ so that

$$\frac{\pi^*(2,k,b)}{J^*(1,k',b)} \ge \frac{1+\log k/b - \log J^*_{k/b}(2)}{J^*_{2k/b}(1)}$$

But,

$$\inf_{y\in(0,\infty)} \frac{1+\log y - \log J_y^*(2)}{J_{2y}^*(1)} = \inf_{y\in(0,\infty)} \frac{1+\log y - \log W(ye^{W(y)})}{W(2y)} \geq 0.96$$

recalling the expression for $J_y^*(x)$ from Section 3.1.

so that

$$\frac{\pi^*(2, a+x-2, b_{\tau_1})}{J^*(1, a+x-1, b_{\tau_1})} \geq 0.96$$

It follows that

$$\begin{aligned} J^*(x-1, a, b) &\geq J^{*,\tau_1}(z) \\ &\geq E[e^{-e^{-1}\tau_1}\pi^*(2, a+x-2, b_{\tau_1})] \\ &\geq 0.96\ E[e^{-e^{-1}\tau_1}J^*(1, a+x-1, b_{\tau_1})] \end{aligned}$$

Substituting in (8), we have the result. □

A Remark on the Proof of Lemma 12.

The infimum in Lemma 12 is computed as follows. We first observe that

$$\frac{1+\log y - \log W(ye^{W(y)})}{W(2y)} \geq \frac{1+\log y - \log 2W(y)}{W(2y)}.$$

Some simple algebra establishes that

$$\frac{1+\log y - \log 2W(y)}{W(2y)} = \frac{1-\log 2 + W(y)}{W(2y)} \geq \frac{1-\log 2 + W(y)}{2W(y)} \geq 1$$

for $y < 0.1$ using the fact that $W(\cdot)$ is concave increasing and $W(0.1) < 0.092$. In addition, using the fact that $W(x)/W(2x)$ is increasing in x and by evaluating $W(2\times 10^8)/W(4\times 10^8) > 0.961$, we can conclude that

$$\frac{1-\log 2 + W(y)}{W(2y)} \geq \frac{1-\log 2 + W(y)}{1.041W(y)} \geq 0.961$$

for $y > 2\times 10^8$. It is then straightforward to numerically minimize $\frac{1+\log y - \log W(ye^{W(y)})}{W(2y)}$ over the compact interval $[0.1, 2\times 10^8]$ to any finite precision since it is Lipschitz over that interval.

D Auxiliary Results for Section 6

In what follows we derive an approximation bound for decay balancing prices when reservation prices satisfy the following assumption in addition to Assumption 1:

Assumption 3.

1. *$\frac{\rho(p)}{\overline{F}(p)}$ is a differentiable, convex function of p with support $\mathbb{R}_+$.*

2. *There exists a unique static revenue maximizing price $p^* > 0$ with $\frac{d}{dp}\frac{\rho(p)}{\overline{F}(p)}\Big|_{p=p^*} \geq 1/\overline{F}(p^*)p^{*2}$.*

Corollary D.1. *For all $z \in \mathcal{S}$, and reservation price distributions satisfying Assumptions 1 and 3*

$$\frac{1}{\kappa(a)} \leq \frac{\pi_{\text{db}}(z)}{\pi^*(z)} \leq 1$$

Proof: Recall that the decay balance equation implies that $\frac{\overline{F}(p^*)p^*\rho(\pi^*(z))}{\overline{F}(\pi^*(z))} = \frac{\overline{F}(p^*)p^*a}{J^*(z)b\alpha} \equiv r^*$. Let $\tilde{r} = \frac{\overline{F}(p^*)p^*a}{\tilde{J}(z)b\alpha}$. Lemma 3 implies that $r^* \geq \tilde{r} \geq 1$.

Define a function $g : [p^*, \pi^*(z)] \to [1, r^*]$ according to $g(p) = \frac{\overline{F}(p^*)p^*\rho(p)}{\overline{F}(p)}$. Observe that $g(p^*) = 1, g(\pi^*(z)) = r^*$ and further by Assumptions 1 and 3, $g(\cdot)$ is an increasing convex function of p on $[p^*, \pi^*(z)]$ with range $[1, r^*]$. It follows that the inverse function g^{-1} is a concave increasing function on $[1, r^*]$ with range $[p^*, \pi^*(z)]$.

Now we have that $\pi^{\text{db}}(z) = g^{-1}(\tilde{r}) = p^* + \frac{\pi^{\text{db}}(z)-p^*}{\tilde{r}-1}(\tilde{r}-1)$ and by the concavity of g^{-1}, we have $\pi^*(z) = g^{-1}(r^*) \leq g^{-1}(\tilde{r}) + \frac{g^{-1}(\tilde{r})-g^{-1}(1)}{\tilde{r}-1}(r^* - \tilde{r}) = p^* + \frac{\pi^{\text{db}}(z)-p^*}{\tilde{r}-1}(r^*-1)$.

Consequently,

$$\begin{aligned}
\frac{\pi_{\text{db}}(z)}{\pi^*(z)} &\geq \frac{p^* + \frac{\pi_{\text{db}}(z)-p^*}{\tilde{r}-1}(\tilde{r}-1)}{p^* + \frac{\pi_{\text{db}}(z)-p^*}{\tilde{r}-1}(r^*-1)} \\
&\geq \frac{p^* + \frac{\pi_{\text{db}}(z)-p^*}{\tilde{r}-1}(\tilde{r}-1)}{p^* + \frac{\pi_{\text{db}}(z)-p^*}{\tilde{r}-1}(\kappa(a)\tilde{r}-1)} \\
&\geq \frac{p^* + (\tilde{r}-1)/(\overline{F}(p^*)p^* \frac{d}{dp}\frac{\rho(p)}{\overline{F}(p)}\big|_{p=p^*})}{p^* + (\kappa(a)\tilde{r}-1)/(\overline{F}(p^*)p^* \frac{d}{dp}\frac{\rho(p)}{\overline{F}(p)}\big|_{p=p^*})} \\
&\geq \frac{1}{\kappa(a)}
\end{aligned}$$

where the second inequality follows from Theorem 1. The third inequality follows from the fact that by Assumption 3, $\frac{\tilde{r}-1}{\pi_{\text{db}}(z)-p^*} \geq g'(p)|_{p=p^*} = \overline{F}(p^*)p^* \frac{d}{dp}\frac{\rho(p)}{\overline{F}(p)}\big|_{p=p^*}$. The final inequality follows from part 2 of Assumption 3: $\overline{F}(p^*)p^* \frac{d}{dp}\frac{\rho(p)}{\overline{F}(p)}\big|_{p=p^*} \geq 1/p^*$. That $\frac{\pi_{\text{db}}(z)}{\pi^*(z)} \leq 1$ is immediate from the fact that $J^*(z) \leq \tilde{J}(z)$. □

Armed with this result, we can derive a performance bound analogous to Theorem 2, but for general reservation price distributions:

Theorem D.1. *For all $z \in \mathcal{S}$, and reservation price distributions satisfying Assumptions 1 and 3,*

$$\frac{1}{\kappa(a)} \leq \frac{J^{\pi_{\text{db}}}(z)}{J^*(z)} \leq 1.$$

E Existence and Uniqueness of solutions to the HJB equation

Our analysis thus far has been predicated on using the HJB equation to characterize the optimal value function J^*. This section makes this argument rigorous for the case of a Gamma prior (which is the focus of our analysis). In particular, we establish the following theorems for this special case:

Theorem E.1. *The value function J^* is the unique solution in $\mathcal{J}$ to $HJ = 0$.*

Theorem E.2. *A policy $\pi \in \Pi$ is optimal if and only if $H^\pi J^* = 0$.*

Our proofs to both Theorems E.1 and E.2 will rely on showing the existence of a bounded solution to the HJB Equation $(HJ)(z) = 0$ for $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$. We restrict attention to exponential reservation prices (which are the primary focus of our analysis). All of the arguments that follow are easily extended to the case of general reservation prices satisfying Assumption 1 , but doing so is notationally quite cumbersome.

E.1 Existence of Solutions to the HJB Equation

We will demonstrate the existence of a solution to the HJB Equation wherein price is restricted to some bounded interval. We will later show that the solution obtained is in fact a solution to the original HJB Equation. Throughout, this section, we will let r denote the mean of the reservation price.

Define $B = r + \frac{r}{\tilde{b}}\left(1 + \frac{e^{-1}(\tilde{a}+\tilde{x})}{\tilde{a}\alpha} + \frac{e^{-1}(\tilde{a}+\tilde{x})}{\alpha}\right)$. Let Π_B be the set of admissible price functions bounded by B, and define the Dynamic programming operator

$$(H^B J)(z) = \sup_{\pi \in \Pi_B} (H^\pi J)(z)$$

We will first illustrate the existence of a bounded solution to the HJB Equation:

$$(H^B J)(z) = 0 \tag{9}$$

for $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$.

For some arbitrary $N > \tilde{b}$ we first obtain a solution on the compact set $\mathcal{S}^N_{\tilde{x},\tilde{a},\tilde{b}} \equiv \{(x,a,b) \in \mathcal{S} : x + a = \tilde{x} + \tilde{a}; \tilde{b} \le b \le N\}$ with the boundary conditions $J(x,a,N) = 0$ and $J(0,a,b) = 0$:

Lemma E.1. (9) *has a unique bounded solution on $\mathcal{S}^N_{\tilde{x},\tilde{a},\tilde{b}}$ satisfying $J(x,a,N) = 0$ and $J(0,a,b) = 0$.*

The proof is analogous to that of Theorem VII.T3 in Bremaud (1981); upon setting $J(0,a,b) = 0$, (9) can be interpreted as an initial value problem of the form $\dot{J} = f(J,b)$ with $J(N) = 0$, in the space $\mathbb{R}^{\tilde{x}-1}$ equipped with the max-norm.

The following two Lemma's construct a solution to (9) on $\mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$ using solutions constructed on $\mathcal{S}^N_{\tilde{x},\tilde{a},\tilde{b}}$.

Lemma E.2. *Let J^N be the unique solution to* (9) *on $\mathcal{S}^N_{\tilde{x},\tilde{a},\tilde{b}}$ with $J(x,a,N) = 0$ and $J(0,a,b) = 0$. Moreover, let $J^{N'}$ be the unique solution to* (9) *on $\mathcal{S}^{N'}_{\tilde{x},\tilde{a},\tilde{b}}$ for some $N' > N$ with $J(x,a,N') = 0$ and $J(0,a,b) = 0$. Then, for $(x,a,b) \in \mathcal{S}^N_{\tilde{x},\tilde{a},\tilde{b}}$,*

$$|J^N(x,a,b) - J^{N'}(x,a,b)| \le r\frac{\tilde{a}+\tilde{x}}{\tilde{b}}\exp(-\alpha(N-b))$$

Moreover, $J^N(x,a,b) \le \frac{re^{-1}(\tilde{a}+\tilde{x})}{\alpha\tilde{b}}$

Proof: Define $\tau_N = \inf\{t : n_t = x\} \wedge \inf\{t : b_t = N\}$. Similarly, define $\tau_{N'}$. Let $\pi^{*,N}(\cdot)$, defined on $\mathcal{S}^N_{\tilde{x},\tilde{a},\tilde{b}}$, be the greedy price with respect to J^N. Finally, define the 'revenue' function $r^{*,N}_t = \frac{a_t e^{-\pi^{*,N}_t/r}\pi^{*,N}_t}{b_t}$. We then have, via an application of Lemma 6,

$$\begin{aligned}
J^N(x,a,b) &= E_{z,\pi^{*,N}}\left[\int_0^{\tau_N} e^{-\alpha t} r^{*,N}_t dt\right] + E_{z,\pi^{*,N}}\left[e^{-\alpha\tau_N} J^N(x_{\tau_N}, a_{\tau_N}, b_{\tau_N})\right] \\
&= E_{z,\pi^{*,N}}\left[\int_0^{\tau_N} e^{-\alpha t} r^{*,N}_t dt\right]
\end{aligned}$$

Note that this immediately yields:

$$J^N(x,a,b) \leq J^*(x,a,b) \leq J^*_{a/b}(x) \leq \frac{re^{-1}(\tilde{a}+\tilde{x})}{\alpha\tilde{b}}.$$

Now, for an arbitrary $\pi \in \Pi^B$, and the corresponding revenue function r, we have (again, via Lemma 6)

$$\begin{aligned}
J^{N'}(x,a,b) &\geq E_{z,\pi}\left[\int_0^{\tau_{N'}} e^{-\alpha t} r_t dt\right] + E_{z,\pi}\left[e^{-\alpha\tau_{N'}} J^{N'}(x_{\tau_{N'}}, a_{\tau_{N'}}, b_{\tau_{N'}})\right] \\
&= E_{z,\pi}\left[\int_0^{\tau_{N'}} e^{-\alpha t} r_t dt\right]
\end{aligned}$$

In particular, using the price function $\pi = \pi^{*,N}$ for $b \leq N$ and 0 otherwise, yields,

$$J^{N'}(x,a,b) \geq E_{z,\pi^{*,N}}\left[\int_0^{\tau_N} e^{-\alpha t} r^{*,N}_t dt\right] = J^N(x,a,b) \tag{10}$$

The same argument, applied to J^N, with the price function $\pi^{*,N'}$, yields

$$E_{z,\pi^{*,N'}}\left[\int_0^{\tau_N} e^{-\alpha t} r^{*,N'}_t dt\right] \leq J^N(x,a,b)$$

Finally, noting that on $\{\tau_{N'} > \tau_N\}$, $\tau_N \geq N - b$, we have

$$E_{z,\pi^{*,N'}}\left[\int_{\tau_N}^{\tau_{N'}} e^{-\alpha t} r^{*,N'}_t dt\right] \leq r\frac{\tilde{a}+\tilde{x}}{\tilde{b}}\exp(-\alpha(N-b))$$

Adding the two preceding inequalities, yields

$$J^{N'}(x,a,b) - r\frac{\tilde{a}+\tilde{x}}{\tilde{b}}\exp(-\alpha(N-b)) \leq J^N(x,a,b).$$

Since $J^{N'}(x,a,b) \geq J^N(x,a,b)$ by (10), the result follows.

□

This yields as a corollary the following result:

Lemma E.3. $\lim_{N\to\infty} J^N$ *exists on* $\mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$, *is bounded, and solves system* (9)

Proof: From Lemma E.2, we have $\lim_{N\to\infty} J^N(x,a,b)$ exists and is bounded for all $(x,a,b) \in \mathcal{S}$.

We posit that this limit is a solution to system (9). First note that by the continuity of

$$f(x,a,J,b) \equiv \inf_{p\in[0,B]} \left[e^{\gamma p}\alpha J(x,a) - \frac{a}{b}p + \frac{a}{b}\left(J(x-1,a+1) - J(x,a)\right)\right]$$

in J, we have:

$$\lim_{N\to\infty} f(x,a,J^N,b) = f(x,a,\lim_{N\to\infty} J^N,b)$$

for each x,a,b. It remains for us to show that

$$\lim_{\delta\to 0}\lim_{N\to\infty} \frac{J^N(x,a,b+\delta) - J^N(x,a,b)}{\delta}$$

exists and equals $\lim_{N\to\infty} dJ^N(x,a,b)/db$. Note however by the Mean Value Theorem that

$$J^N(x,a,b+\delta) - J^N(x,a,b)/\delta = dJ^N(x,a,b)/db + R_N$$

where

$$\begin{aligned}
|R_N| &\leq \sup_{b'\in[b,b+\delta]} dJ^N(x,a,y)/dy|_{y=b'} - \inf_{b'\in[b,b+\delta]} dJ^N(x,a,y)/dy|_{y=b'} \\
&= \sup_{b'\in[b,b+\delta]} f(x,a,J^N(x,a,b'),b') - \inf_{b'\in[b,b+\delta]} f(x,a,J^N(x,a,b'),b')
\end{aligned}$$

But $J^N(x,a,b)$ converges uniformly to its limit on $[b,b+\delta]$ by Lemma E.2, and f is uniformly continuous on $[b,b+\delta]$ being a continuous function restricted to a compact set, so that

$$\limsup_N |R_N| \leq \sup_{b'\in[b,b+\delta]} f(x,a,J^*(x,a,b'),b') - \inf_{b'\in[b,b+\delta]} f(x,a,J^*(x,a,b'),b')$$

Finally, by the continuity of J^* in b,

$$\lim_{\delta\to 0}\limsup_N |R_N| = 0$$

Similarly,

$$\lim_{\delta\to 0}\liminf_N |R_N| = 0$$

This completes the proof. □

The previous Lemma constructs a bounded solution to (9). We now show that this solution is in fact a solution to the original HJB Equation $(HJ)(z) = 0$ for $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$.

Lemma E.4. *Let $\tilde{J}$ be a bounded solution to (9). Then, $\tilde{J}$ is a solution to $(HJ)(z) = 0$ for $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$.*

Proof: We show the claim by demonstrating that the greedy price (in Π^B) with respect to $\tilde{J}$ is in fact attained in $[0,B)$. We begin by proving a bound on such a greedy price. Let $\pi^{\mathrm{b}} \in \Pi^B$ be the greedy price with respect to $\tilde{J}$, and $\tau = \inf\{t : N_t = x_0\}$. Letting $\tilde{r}_t = \frac{a_t e^{-\pi_t^{\mathrm{b}}/r}\pi_t^{\mathrm{b}}}{b_t}$, we have, via

Lemma 6,

$$\begin{aligned}\tilde{J}(z) &= E_{z,\pi^{\mathrm{b}}}\left[\int_0^{\tau} e^{-\alpha t}\tilde{r}_t dt\right] + E_{z,\pi^{\mathrm{b}}}\left[e^{-\alpha\tau}\tilde{J}(z_\tau)\right]\\ &= E_{z,\pi^{\mathrm{b}}}\left[\int_0^{\tau} e^{-\alpha t}\tilde{r}_t dt\right]\\ &\le J^*(z)\\ &\le \frac{re^{-1}(\tilde{a}+\tilde{x})}{\alpha\tilde{b}}.\end{aligned}$$

Now let $\tilde{J}^\delta$ be the solution to (9) when the discount factor is $\alpha(1+\delta/b)$. Let $\pi^{\mathrm{b},\delta}$ be the corresponding greedy price and $\tilde{r}_t^\delta = \frac{a_t e^{-\pi_t^{\mathrm{b},\delta}/r}\pi_t^{\mathrm{b},\delta}}{b_t}$. We then have from Lemma 6 and using the fact that $\tilde{J}(x,a,b+\delta) = \tilde{J}^\delta(x,a,b)$,

$$\begin{aligned}\tilde{J}(x,a,b+\delta) &= E_{z,\pi^{\mathrm{b},\delta}}\left[\int_0^{\tau^\delta} e^{-\alpha(1+\delta/b)t}\tilde{r}_t^\delta dt\right]\\ &\ge E_{z,\pi^{\mathrm{b}}}\left[\int_0^{\tau} e^{-\alpha(1+\delta/b)t}\tilde{r}_t dt\right]\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{J}(z) - \tilde{J}(x,a,b+\delta) &\le E_{z,\pi^{\mathrm{b}}}\left[\int_0^{\tau}(e^{-\alpha t} - e^{-\alpha(1+\delta/b)t})\tilde{r}_t dt\right]\\ &\le \int_0^\infty (e^{-\alpha t} - e^{-\alpha(1+\delta/b)t})\frac{re^{-1}(a+x)}{b}dt\end{aligned}$$

so that

$$\frac{d}{db}\tilde{J}(z) \ge -\frac{r\alpha}{b}\frac{e^{-1}(a+x)}{b\alpha^2}$$

Putting the two bounds together yields

$$\tilde{J}(x-1,a+1,b) - \tilde{J}(z) + \frac{b}{a}\frac{d}{db}\tilde{J}(z) \ge -\frac{re^{-1}(\tilde{a}+\tilde{x})}{\alpha\tilde{b}} - \frac{re^{-1}(\tilde{a}+\tilde{x})}{\tilde{a}\tilde{b}\alpha} \tag{11}$$

Now observe that the greedy price $\pi^{\mathrm{b}} \in \Pi$ with respect to $\tilde{J}$ is given by

$$p = \left(r - \tilde{J}(x-1,a+1,b) + \tilde{J}(z) - \frac{b}{a}\frac{d}{db}\tilde{J}(z)\right)^+$$

which by (11) is in $[0,B)$, so that we have that $\tilde{J}$ is, in fact, a solution to $(HJ)(z) = 0$ for $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$.
□

E.2 Proofs for Theorems E.1 and E.2

Lemma E.5. *For $J \in \mathcal{J}$, and $\pi \in \Pi$, let*

$$\mathcal{A}_{\pi,z}J(z) = \lim_{t>0,t\to 0}\frac{e^{-\alpha t}E_{z,\pi}[J(z(t))] - J(z)}{t}.$$

We have:

$$\mathcal{A}_{\pi,z}J(z) = e^{-\pi(z)/r}\frac{a}{b}\left(J(z') - J(z) + \frac{b}{a}\frac{d}{db}J(z)\right) - \alpha J(z)$$

Proof: As in Theorem T1 in Section VII.2 of Bremaud (1981), one may show for $J \in \mathcal{J}$, and an arbitrary $z_0 \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$,

$$\begin{aligned}J(z_t) =& J(z_0) + \int_0^t \left[\frac{b_s}{a_s}\frac{d}{db_s}J(z_s) + J(x_s - 1, a_s + 1, b_s) - J(z_s)\right]\frac{a_s}{b_s}e^{-p_s/r}ds \\ &+ \int_0^t \left[J(x_{s^-} - 1, a_{s^-} + 1, b_{s^-}) - J(z_{s^-})\right](dN_s - \frac{a_s}{b_s}e^{-p_s/r}ds)\end{aligned}$$

It is not hard to show that that $N_s - \frac{a_s}{b_s}e^{-p_s/r}$ is a zero-mean $\sigma(z^s, p^s)$ martingale, so that we may conclude

$$\begin{aligned}&e^{-\alpha t}E[J(z_t)] - J(z_0) = \\ &e^{-\alpha t}E\left[\int_0^t \left[\frac{b_s}{a_s}\frac{d}{db_s}J(z_s) + J(x_s - 1, a_s + 1, b_s) - J(z_s)\right]\frac{a_s}{b_s}e^{-p_s/r}ds\right] + (e^{-\alpha t} - 1)J(z_0)\end{aligned}$$

Dividing by t and taking a limit as $t \to 0$ yields, via bounded convergence, the result. □

Lemma E.6. *(Verification Lemma) If there exists a solution, $\tilde{J} \in \mathcal{J}$ to*

$$(HJ)(z) = 0$$

for all $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$, we have:

1. *$\tilde{J}(\cdot) = J^*(\cdot)$*
2. *Let $\pi^*(\cdot)$ be the greedy policy with respect to $\tilde{J}$. Then $\pi^*(\cdot)$ is an optimal policy.*

Proof:

Let $\pi \in \Pi$ be arbitrary. By Lemma 6,

$$\begin{aligned}J^\pi(z_0) - \tilde{J}(z_0) =& E\left[\int_0^\tau e^{-\alpha s}H^\pi \tilde{J}(z_s)ds\right] \\ \leq& 0\end{aligned} \tag{12}$$

with equality for $\pi^*(\cdot)$, since $H^{\pi^*}\tilde{J}(z) = (H\tilde{J})(z) = 0$ for all $z \in \mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$. □

Now we have shown the existence of a bounded solution, $\tilde{J}$ to $(HJ)(z) = 0$ on $\mathcal{S}_{\tilde{x},\tilde{a},\tilde{b}}$ in the previous section, so that the first conclusion of the Verification Lemma gives

Theorem D.1. *The value function J^* is the unique solution in $\mathcal{J}$ to $HJ = 0$.*

The second conclusion and (12) in the Verification Lemma give

Theorem D.2. *A policy $\pi \in \Pi$ is optimal if and only if $H^\pi J^* = 0$.*

References

Bremaud, P. 1981. *Point Processes and Queues: Martingale Dynamics*. 1st ed. Springer-Verlag.

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