Approximate Dynamic Programming via a Smoothed Linear Program (Electronic Companion)

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A. Proofs for Sections 4.2–4.4

Lemma 1. For any $r \in \mathbb{R}^K$ and $\theta \geq 0$:

(i) $\ell(r, \theta)$ is a finite-valued, decreasing, piecewise linear, convex function of $\theta$.

(ii) 
$$
\ell(r, \theta) \leq \frac{1 + \alpha}{1 - \alpha} \| J^* - \Phi r \|_\infty.
$$

(iii) The right partial derivative of $\ell(r, \theta)$ with respect to $\theta$ satisfies
$$
\frac{\partial^+}{\partial \theta^+} \ell(r, 0) = - \left( (1 - \alpha) \sum_{x \in \Omega(r)} \pi_{\mu^*, \nu}(x) \right)^{-1},
$$

where
$$
\Omega(r) \triangleq \argmax_{\{x \in \mathcal{X} : \pi_{\mu^*, \nu}(x) > 0\}} \Phi r(x) - T \Phi r(x).
$$

Proof. (i) Given any $r$, clearly $\gamma \triangleq \| \Phi r - T \Phi r \|_\infty$, $s \triangleq 0$ is a feasible point for (9), so $\ell(r, \theta)$ is feasible. To see that the LP is bounded, suppose $(s, \gamma)$ is feasible. Then, for any $x \in \mathcal{X}$ with $\pi_{\mu^*, \nu}(x) > 0$,
$$
\gamma \geq \Phi r(x) - T \Phi r(x) - s(x) \geq \Phi r(x) - T \Phi r(x) - \theta / \pi_{\mu^*, \nu}(x) > -\infty.
$$
Letting \((\gamma_1, s_1)\) and \((\gamma_2, s_2)\) represent optimal solutions for the LP \((9)\) with parameters \((r, \theta_1)\) and \((r, \theta_2)\) respectively, it is easy to see that \(((\gamma_1 + \gamma_2)/2, (s_1 + s_2)/2)\) is feasible for the LP with parameters \((r, (\theta_1 + \theta_2)/2)\). It follows that \(\ell(r, (\theta_1 + \theta_2)/2) \leq (\ell(r, \theta_1) + \ell(r, \theta_2))/2\). The remaining properties are simple to check.

(ii) Let \(\epsilon \triangleq \|J^* - \Phi r\|_{\infty}\). Then, since \(T\) is an \(\alpha\)-contraction under the \(\|\cdot\|_{\infty}\) norm,

\[
\|T \Phi r - \Phi r\|_{\infty} \leq \|J^* - T \Phi r\|_{\infty} + \|J^* - \Phi r\|_{\infty} \leq \alpha \|J^* - \Phi r\|_{\infty} + \epsilon = (1 + \alpha)\epsilon.
\]

Since \(\gamma \triangleq \|T \Phi r - \Phi r\|_{\infty}\), \(s \triangleq 0\) is feasible for \((9)\), the result follows.

(iii) Fix \(r \in \mathbb{R}^K\), and define

\[
\Delta \triangleq \max_{\{x \in X : \pi_{\mu^*, \nu}(x) > 0\}} (\Phi r(x) - T \Phi r(x)) - \max_{\{x \in X \setminus \Omega(r) : \pi_{\mu^*, \nu}(x) > 0\}} (\Phi r(x) - T \Phi r(x)) > 0.
\]

Consider the program for \(\ell(r, \delta)\). It is easy to verify that for \(\delta \geq 0\) and sufficiently small, viz. \(\delta \leq \Delta \sum_{x \in \Omega(r)} \pi_{\mu^*, \nu}(x)\), \((\bar{s}_\delta, \bar{\gamma}_\delta)\) is an optimal solution to the program, where

\[
\bar{s}_\delta(x) \triangleq \begin{cases} 
\frac{\delta}{\sum_{x \in \Omega(r)} \pi_{\mu^*, \nu}(x)} & \text{if } x \in \Omega(r), \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
\bar{\gamma}_\delta \triangleq \gamma_0 - \frac{\delta}{\sum_{x \in \Omega(r)} \pi_{\mu^*, \nu}(x)},
\]

so that

\[
\ell(r, \delta) = \ell(r, 0) - \frac{\delta}{(1 - \alpha) \sum_{x \in \Omega(r)} \pi_{\mu^*, \nu}(x)}.
\]

Thus,

\[
\frac{\ell(r, \delta) - \ell(r, 0)}{\delta} = - \left(1 - \alpha \sum_{x \in \Omega(r)} \pi_{\mu^*, \nu}(x)\right)^{-1}.
\]

Taking a limit as \(\delta \searrow 0\) yields the result. \(\blacksquare\)

**Lemma 2.** Suppose that the vectors \(J \in \mathbb{R}^X\) and \(s \in \mathbb{R}^X\) satisfy

\[
J \leq T_{\mu^*} J + s.
\]

Then,

\[
J \leq J^* + \Delta^* s,
\]

where

\[
\Delta^* \triangleq \sum_{k=0}^{\infty} (\alpha P_{\mu^*})^k = (I - \alpha P_{\mu^*})^{-1},
\]
and $P_{\mu^*}$ is the transition probability matrix corresponding to an optimal policy.

**Proof.** Note that the $T_{\mu^*}$, the Bellman operator corresponding to the optimal policy $\mu^*$, is monotonic and is a contraction. Then, repeatedly applying $T_{\mu^*}$ to the inequality $J \leq T_{\mu^*}J + s$ and using the fact that $T_{\mu^*}^k J \to J^*$, we obtain

$$J \leq J^* + \sum_{k=0}^{\infty} (\alpha P_{\mu^*})^k s = J^* + \Delta^* s.$$ 

\[\blacksquare\]

**Lemma 3.** For the autonomous queue with basis functions $\phi_1(x) \triangleq 1$ and $\phi_2(x) \triangleq x$, if $N$ is sufficiently large, then

$$\inf_{r, \psi \in \tilde{\Psi}} \frac{2\nu^T \psi}{1 - \alpha\beta(\psi)} \|J^* - \Phi r\|_{\infty, 1/\psi} \geq \frac{3\rho_2 q}{32(1 - q)} (N - 1).$$

**Proof.** We have:

$$\inf_{r, \psi \in \tilde{\Psi}} \frac{2\nu^T \psi}{1 - \alpha\beta(\psi)} \|J^* - \Phi r\|_{\infty, 1/\psi} \geq \inf_{\psi \in \Phi} \inf_r \|\psi\|_\infty \|J^* - \Phi r\|_\infty.$$

We will produce lower bounds on the two infima on the right-hand side above. Observe that

$$\inf_r \|J^* - \Phi r\|_\infty = \inf_r \max_x |\rho_2 x^2 + \rho_1 x + \rho_0 - r_1 x - r_0|$$

$$\geq \inf_r \max \left( \max_x |\rho_2 x^2 + (\rho_1 - r_1) x| - |\rho_0 - r_0|, |\rho_0 - r_0| \right)$$

$$= \inf_{r_0} \max \left( \inf_{r_1} \max_x |\rho_2 x^2 + (\rho_1 - r_1) x| - |\rho_0 - r_0|, |\rho_0 - r_0| \right),$$

which follows from the triangle inequality and the fact that

$$\max_x |\rho_2 x^2 + \rho_1 x + \rho_0 - r_1 x - r_0| \geq |\rho_0 - r_0|.$$ 

Routine algebra verifies that

$$\inf_{r_1} \max_x |\rho_2 x^2 + (\rho_1 - r_1) x| \geq \frac{3}{16} \rho_2 (N - 1)^2.$$ 

It thus follows that

$$\inf_r \|J^* - \Phi r\|_\infty \geq \inf_{r_0} \max \left( \frac{3}{16} \rho_2 (N - 1)^2 - |\rho_0 - r_0|, |\rho_0 - r_0| \right) \geq \frac{3}{32} \rho_2 (N - 1)^2.$$ 

We next note that any $\psi \in \tilde{\Psi}$ must satisfy $\psi \in \text{span}(\Phi)$ and $\psi \geq 1$. Thus, $\psi \in \tilde{\Psi}$ must take the form $\psi(x) = \alpha_1 x + \alpha_0$ with $\alpha_0 \geq 1$ and $\alpha_1 \geq (1 - \alpha_0)/(N - 1)$. Thus, $\|\psi\|_\infty = \max(\alpha_1 (N - 1) + \alpha_0)$. 

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Define $\kappa(N)$ to be the expected queue length under the distribution $\nu$, i.e.,
\[
\kappa(N) \triangleq \sum_{x=0}^{N-1} \nu(x)x = \frac{1 - q}{1 - q^N} \sum_{x=0}^{N-1} x q^x = \frac{q}{1 - q} \left[ 1 - Nq^{N-1}(1 - q) - q^N \right],
\]
so that $\nu^\top \psi = \alpha_1 \kappa(N) + \alpha_0$. Thus,
\[
\inf_{\psi \in \Psi} \frac{2\nu^\top \psi}{\|\psi\|_\infty} \inf_r \|J^* - \Phi r\|_\infty \geq \frac{2}{16} \rho_2 \left( \inf_{\alpha \geq 1} \frac{\alpha_1 \kappa(N) + \alpha_0}{\alpha_0} \max(\alpha_1(N - 1) + \alpha_0, \alpha_0) (N - 1)^2 \right)
\]
When $(1 - \alpha_0)/(N - 1) \leq \alpha_1 \leq 0$, we have
\[
\frac{\alpha_1 \kappa(N) + \alpha_0}{\max(\alpha_1(N - 1) + \alpha_0, \alpha_0)} (N - 1)^2 = \frac{\alpha_1 \kappa(N) + \alpha_0}{\alpha_0} (N - 1)^2 \geq \frac{(1 - \alpha_0)\kappa(N)/(N - 1) + \alpha_0(N - 1)^2}{\alpha_0} \geq \left( 1 - \frac{\kappa(N)}{N - 1} \right) (N - 1)^2.
\]
When $\alpha_1 > 0$, we have
\[
\frac{\alpha_1 \kappa(N) + \alpha_0}{\max(\alpha_1(N - 1) + \alpha_0, \alpha_0)} (N - 1)^2 = \frac{\alpha_1 \kappa(N) + \alpha_0}{\alpha_1(N - 1) + \alpha_0} (N - 1)^2 \geq (N - 1)\kappa(N),
\]
where the inequality follows from the fact that $\kappa(N) \leq N - 1$ and $\alpha_0 > 0$. It then follows that
\[
\inf_{\psi \in \Psi} \frac{2\nu^\top \psi}{\|\psi\|_\infty} \inf_r \|J^* - \Phi r\|_\infty \geq \frac{3}{16} \rho_2 \min \left( \kappa(N)(N - 1), \left( 1 - \frac{\kappa(N)}{N - 1} \right) (N - 1)^2 \right).
\]
Now, observe that $\kappa(N)$ is increasing in $N$. Also, by assumption, $p < 1/2$, so $q < 1$ and thus $\kappa(N) \to q/(1 - q)$ as $N \to \infty$. Then, for $N$ sufficiently large, $\frac{1}{2}q/(1 - q) \leq \kappa(N) \leq q/(1 - q)$. Therefore, for $N$ sufficiently large,
\[
\inf_{\psi \in \Psi} \frac{2\nu^\top \psi}{\|\psi\|_\infty} \inf_r \|J^* - \Phi r\|_\infty \geq \frac{3\rho_2 q}{32(1 - q)} (N - 1),
\]
as desired.

\textbf{Lemma 4.} For every $\lambda \geq 0$, there exists a $\hat{\theta} \geq 0$ such that an optimal solution $(r^*, s^*)$ to
\begin{equation}
\text{(A.1)} \quad \max_{r,s} \quad \nu^\top \Phi r - \lambda \pi^\top_{r,s} \nu s
\end{equation}
subject to $\Phi r \leq T \Phi r + s$, $s \geq 0$.

is also an optimal solution the SALP $[\S]$ with $\theta = \hat{\theta}$. 

\[4\]
Proof. Let \( \hat{\theta} \triangleq \pi_\star^\top \mu s \). It is then clear that \((r^\star, s^\star)\) is a feasible solution to (8) with \( \theta = \hat{\theta} \). We claim that it is also an optimal solution. To see this, assume to the contrary that it is not an optimal solution, and let \((\tilde{r}, \tilde{s})\) be an optimal solution to (8). It must then be that \( \pi_\star^\top \mu_s \tilde{s} \leq \pi_\star^\top \mu_s s^\star \) and moreover, \( \nu^\top \Phi \tilde{r} > \nu^\top \Phi r^\star \) so that
\[
\nu^\top \Phi r^\star - \lambda \pi_\star^\top \mu s^\star < \nu^\top \Phi \tilde{r} - \lambda \pi_\star^\top \mu \tilde{s}.
\]
This, in turn, contradicts the optimality of \((r^\star, s^\star)\) for (A.1) and yields the result. \[\blacksquare\]

B. Proof of Theorem 4

Our proof of Theorem 4 is based on uniformly bounding the rate of convergence of sample averages of a certain class of functions. We begin with some definitions: consider a family \( F \) of functions from a set \( S \) to \( \{0, 1\} \). Define the Vapnik-Chervonenkis (VC) dimension \( \dim_{VC}(F) \) to be the cardinality \( d \) of the largest set \( \{x_1, x_2, \ldots, x_d\} \subset S \) satisfying:
\[
\forall e \in \{0, 1\}^d, \exists f \in F \text{ such that } \forall i, f(x_i) = 1 \text{ iif } e_i = 1.
\]

Now, let \( F \) be some set of real-valued functions mapping \( S \) to \([0, B]\). The pseudo-dimension \( \dim_P(F) \) is the following generalization of VC dimension: for each function \( f \in F \) and scalar \( c \in \mathbb{R} \), define a function \( g: S \times \mathbb{R} \to \{0, 1\} \) according to:
\[
g(x, c) \triangleq \mathbb{I}(f(x) - c \geq 0).
\]
Let \( G \) denote the set of all such functions. Then, we define \( \dim_P(F) \triangleq \dim_{VC}(G) \).

In order to prove Theorem 4, define the \( F \) to be the set of functions \( f: \mathbb{R}^K \times \mathbb{R} \to [0, B] \), where, for all \( x \in \mathbb{R}^K \) and \( y \in \mathbb{R} \),
\[
f(y, z) \triangleq \zeta \left(r^\top y + z\right).
\]
Here, \( \zeta(t) \triangleq \max(\min(t, B), 0) \), and \( r \in \mathbb{R}^K \) is a vector that parameterizes \( f \). We will show that \( \dim_P(F) \leq K + 2 \). We will use the following standard result from convex geometry:

**Lemma 5 (Radon’s Lemma).** A set \( A \subset \mathbb{R}^m \) of \( m + 2 \) points can be partitioned into two disjoint sets \( A_1 \) and \( A_2 \), such that the convex hulls of \( A_1 \) and \( A_2 \) intersect.

**Lemma 6.** \( \dim_P(F) \leq K + 2 \)

**Proof.** Assume, for the sake of contradiction, that \( \dim_P(F) > K + 2 \). It must be that there exists a ‘shattered’ set
\[
\left\{(y^{(1)}, z^{(1)}, c^{(1)}), (y^{(2)}, z^{(2)}, c^{(2)}), \ldots, (y^{(K+3)}, z^{(K+3)}, c^{(K+3)})\right\} \subset \mathbb{R}^K \times \mathbb{R} \times \mathbb{R},
\]

such that, for all \( e \in \{0, 1\}^{K+3} \), there exists a vector \( r_e \in \mathbb{R}^K \) with
\[
\zeta \left( r_e^\top y^{(i)} + z^{(i)} \right) \geq c^{(i)} \text{ iff } e_i = 1, \quad \forall \ 1 \leq i \leq K + 3.
\]

Observe that we must have \( c^{(i)} \in (0, B] \) for all \( i \), since if \( c^{(i)} \leq 0 \) or \( c^{(i)} > B \), then no such shattered set can be demonstrated. But if \( c^{(i)} \in (0, B] \), for all \( r \in \mathbb{R}^K \),
\[
\zeta \left( r^\top y^{(i)} + z^{(i)} \right) \geq c^{(i)} \implies r_e^\top y^{(i)} \geq c^{(i)} - z^{(i)},
\]
and
\[
\zeta \left( r^\top y^{(i)} + z^{(i)} \right) < c^{(i)} \implies r_e^\top y^{(i)} < c^{(i)} - z^{(i)}.
\]

For each \( 1 \leq i \leq K + 3 \), define \( x^{(i)} \in \mathbb{R}^{K+1} \) component-wise according to
\[
x^{(i)}_j \triangleq \begin{cases} 
y^{(i)}_j & \text{if } j < K + 1, \\
c^{(i)} - z^{(i)} & \text{if } j = K + 1.
\end{cases}
\]
Let \( A = \{x^{(1)}, x^{(2)}, \ldots, x^{(K+3)}\} \subset \mathbb{R}^{K+1} \), and let \( A_1 \) and \( A_2 \) be subsets of \( A \) satisfying the conditions of Radon’s lemma. Define a vector \( \bar{e} \in \{0, 1\}^{K+3} \) component-wise according to
\[
\bar{e}_i \triangleq \mathbb{I}_{\{x^{(i)} \in A_1\}}.
\]
Define the vector \( \tilde{r} \triangleq r_{\bar{e}} \). Then, we have
\[
\sum_{j=1}^{K} \tilde{r}_j x_j \geq x_{K+1}, \quad \forall \ x \in A_1,
\]
and
\[
\sum_{j=1}^{K} \tilde{r}_j x_j < x_{K+1}, \quad \forall \ x \in A_2.
\]

Now, let \( \bar{x} \in \mathbb{R}^{K+1} \) be a point contained in both the convex hull of \( A_1 \) and the convex hull of \( A_2 \). Such a point must exist by Radon’s lemma. By virtue of being contained in the convex hull of \( A_1 \), we must have
\[
\sum_{j=1}^{K} \bar{r}_j \bar{x}_j \geq \bar{x}_{K+1}.
\]
Yet, by virtue of being contained in the convex hull of \( A_2 \), we must have
\[
\sum_{j=1}^{K} \tilde{r}_j \bar{x}_j < \bar{x}_{K+1},
\]
which is impossible. ■
With the above pseudo-dimension estimate, we can establish the following lemma, which provides a Chernoff bound for the uniform convergence of a certain class of functions:

**Lemma 7.** Given a constant $B > 0$, define the function $\zeta: \mathbb{R} \to [0, B]$ by

$$\zeta(t) \triangleq \max(\min(t, B), 0).$$

Consider a pair of random variables $(Y, Z) \in \mathbb{R}^K \times \mathbb{R}$. For each $i = 1, \ldots, n$, let the pair $(Y^{(i)}, Z^{(i)})$ be an i.i.d. sample drawn according to the distribution of $(Y, Z)$. Then, for all $\epsilon \in (0, B]$,

$$P \left( \sup_{r \in \mathbb{R}^K} \left| \frac{1}{n} \sum_{i=1}^{n} \zeta \left( r^\top Y^{(i)} + Z^{(i)} \right) - \mathbb{E} \left[ \zeta \left( r^\top Y + Z \right) \right] \right| > \epsilon \right) \leq 8 \left( \frac{32eB}{\epsilon} \log \frac{32eB}{\epsilon} \right)^{K+2} \exp \left( - \frac{\epsilon^2 n}{64B^2} \right).$$

Moreover, given $\delta \in (0, 1)$, if

$$n \geq \frac{64B^2}{\epsilon^2} \left( 2(K + 2) \log \frac{16eB}{\epsilon} + \log \frac{8}{\delta} \right),$$

then this probability is at most $\delta$.

**Proof.** Given Lemma 6, this follows immediately from Corollary 2 of of Haussler (1992, Section 4).

We are now ready to prove Theorem 4.

**Theorem 4.** Under the conditions of Theorem 2, let $r_{SALP}$ be an optimal solution to the SALP \((14)\), and let $\hat{r}_{SALP}$ be an optimal solution to the sampled SALP \((28)\). Assume that $r_{SALP} \in \mathcal{N}$. Further, given $\epsilon \in (0, B]$ and $\delta \in (0, 1/2]$, suppose that the number of sampled states $S$ satisfies

$$S \geq \frac{64B^2}{\epsilon^2} \left( 2(K + 2) \log \frac{16eB}{\epsilon} + \log \frac{8}{\delta} \right).$$

Then, with probability at least $1 - \delta - 2^{-383\delta^{128}}$,

$$\|J^* - \Phi_{\hat{r}_{SALP}}\|_{1, \nu} \leq \inf_{\substack{r \in \mathcal{N} \\psi \in \Psi}} \|J^* - \Phi_r\|_{\infty, 1/\psi} \left( \nu^\top \psi + \frac{2(\pi^\top r \nu \psi)(\alpha \beta(\psi) + 1)}{1 - \alpha} \right) + \frac{4\epsilon}{1 - \alpha}.$$

**Proof.** Define the vectors

$$\hat{s}_{\mu^*} \triangleq (\Phi_{\hat{r}_{SALP}} - T_{\mu^*} \Phi_{SALP})^+, \quad \text{and} \quad \hat{s} \triangleq (\Phi_{\hat{r}_{SALP}} - T\Phi_{\hat{r}_{SALP}})^+.$$

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Note that $\hat{s}_{\mu^*} \leq \hat{s}$. One has, via Lemma 2, that
\[ \Phi_{\hat{r}_{\text{SALP}}} - J^* \leq \Delta^* \hat{s}_{\mu^*}. \]

Thus, as in the last set of inequalities in the proof of Theorem 1, we have
\[
\|J^* - \Phi_{\hat{r}_{\text{SALP}}}\|_{1,\nu} \leq \nu^T (J^* - \Phi_{\hat{r}_{\text{SALP}}}) + \frac{2\pi_{\mu^*,\nu}^T \hat{s}_{\mu^*}}{1 - \alpha}.
\]

(B.1)

Now, let $\hat{\pi}_{\mu^*,\nu}$ be the empirical measure induced by the collection of sampled states $\hat{X}$. Given a state $x \in X$, define a vector $Y(x) \in \mathbb{R}^K$ and a scalar $Z(x) \in \mathbb{R}$ according to
\[ Y(x) \triangleq \Phi(x)^T - \alpha P_{\mu^*} \Phi(x)^T, \quad Z(x) \triangleq -g(x, \mu^*(x)), \]
so that, for any vector of weights $r \in \mathcal{N}$,
\[ (\Phi r(x) - T_{\mu^*} \Phi r(x))^+ = \zeta \left( r^T Y(x) + Z(x) \right). \]

Then,
\[
\left| \hat{\pi}_{\mu^*,\nu}^T \hat{s}_{\mu^*} - \pi_{\mu^*,\nu}^T \hat{s}_{\mu^*} \right| \leq \sup_{r \in \mathcal{N}} \frac{1}{S} \sum_{x \in \hat{X}} \zeta \left( r^T Y(x) + Z(x) \right) - \sum_{x \in X} \pi_{\mu^*,\nu}(x) \zeta \left( r^T Y(x) + Z(x) \right).
\]

Applying Lemma 7 we have that
\[
(B.2) \quad P \left( \left| \hat{\pi}_{\mu^*,\nu}^T \hat{s}_{\mu^*} - \pi_{\mu^*,\nu}^T \hat{s}_{\mu^*} \right| > \epsilon \right) \leq \delta.
\]

Next, suppose $(r_{\text{SALP}}, \hat{s})$ is an optimal solution to the SALP (14). Then, with probability at least $1 - \delta$,
\[
\nu^T (J^* - \Phi_{\hat{r}_{\text{SALP}}}) + \frac{2\pi_{\mu^*,\nu}^T \hat{s}_{\mu^*}}{1 - \alpha} \leq \nu^T (J^* - \Phi_{\hat{r}_{\text{SALP}}}) + \frac{2\pi_{\mu^*,\nu}^T \hat{s}_{\mu^*}}{1 - \alpha} + \frac{2\epsilon}{1 - \alpha}
\]

(B.3)
\[
\leq \nu^T (J^* - \Phi_{\hat{r}_{\text{SALP}}}) + \frac{2\hat{\pi}_{\mu^*,\nu}^T \hat{s}}{1 - \alpha} + \frac{2\epsilon}{1 - \alpha}
\]
\[
\leq \nu^T (J^* - \Phi_{r_{\text{SALP}}}) + \frac{2\hat{\pi}_{\mu^*,\nu}^T \hat{s}}{1 - \alpha} + \frac{2\epsilon}{1 - \alpha}.
\]

where the first inequality follows from (B.2), and the final inequality follows from the optimality of $(\hat{r}_{\text{SALP}}, \hat{s})$ for the sampled SALP (28).

Notice that, without loss of generality, we can assume that $\hat{s}(x) = (\Phi r_{\text{SALP}}(x) - T \Phi r_{\text{SALP}}(x))^+$,
for each $x \in \mathcal{X}$. Thus, $0 \leq \bar{s}(x) \leq B$. Further,

$$\hat{\pi}_{\mu^*,\nu}^\top \bar{s} - \pi_{\mu^*,\nu}^\top \bar{s} = \frac{1}{S} \sum_{x \in \mathcal{X}} \left( \bar{s}(x) - \pi_{\mu^*,\nu}^\top \bar{s} \right),$$

where the right-hand-side is of a sum of zero-mean bounded i.i.d. random variables. Applying Hoeffding’s inequality,

$$P \left( \left| \hat{\pi}_{\mu^*,\nu}^\top \bar{s} - \pi_{\mu^*,\nu}^\top \bar{s} \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{2\epsilon^2}{B^2} \right) < 2^{-383 \delta^{128}},$$

where final inequality follows from our choice of $S$. Combining this with (B.1) and (B.3), with probability at least $1 - \delta - 2^{-383 \delta^{128}}$, we have

$$\| J^* - \Phi \hat{r}_{\text{SALP}} \|_{1,\nu} \leq \nu^\top (J^* - \Phi r_{\text{SALP}}) + \frac{2\hat{\pi}_{\mu^*,\nu}^\top \bar{s}}{1 - \alpha} + \frac{2\epsilon}{1 - \alpha}$$

$$\leq \nu^\top (J^* - \Phi r_{\text{SALP}}) + \frac{2\pi_{\mu^*,\nu}^\top \bar{s}}{1 - \alpha} + \frac{4\epsilon}{1 - \alpha}.$$

The result then follows from (17)–(19) in the proof of Theorem 2.

References