Internet Appendix for "A Bayesian Approach to Real Options: The Case of Distinguishing Between Temporary and Permanent Shocks"

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1 Proofs and Derivations

Smooth-pasting conditions in Eqs. (37) and (38).

Let $\mathfrak{M}$ denote the set of $(\mathcal{G}_t)$-stopping times and $\tau \in \mathfrak{M}$ be an element of this set. By definition, the value of the option $G(X(t), p(t))$ satisfies

$$\begin{aligned} G(X(t), p(t)) = \sup\nolimits_{\tau\in\mathfrak{M}} \mathbb{E}\Big[\textstyle\int_t^\tau e^{-r(s-t)}\lambda_3 p(s)\, e^{-\int_t^s \lambda_3 p(u)du} H\left(\frac{X(s)}{1+\varphi}\right) ds \\ + e^{-\int_t^\tau \lambda_3 p(s)ds} e^{-r(\tau-t)} \left(S(X(\tau), p(\tau)) - I\right) |\mathcal{G}_t\Big]. \end{aligned} \tag{A1}$$

The right-hand side of (A1) consists of two terms. The first term corresponds to the payoff if the shock reverts before the firm invests. The second term corresponds to the payoff if the firm invests before the shock reverses. Let

$$D(X(t), p(t)) \equiv \mathbb{E}\left[\int_t^{+\infty} e^{-r(s-t)}\lambda_3 p(s)\, e^{-\int_t^s \lambda_3 p(u)du} H\left(\frac{X(s)}{1+\varphi}\right) ds|\mathcal{G}_t\right], \tag{A2}$$

$$d(p(t)) \equiv \int_{t_0}^t (r + \lambda_3 p(s))\, ds, \tag{A3}$$

where t_0 is the time when the shock arrives. Then, for any $X(t) > 0$ and $p(t) \in (0, \pi_0]$, we can rewrite (A1) as

$$\begin{aligned} e^{-d(p(t))}\left(G(X(t), p(t)) - D(X(t), p(t))\right) \\ = \sup\nolimits_{\tau\in\mathfrak{M}} \mathbb{E}\left[e^{-d(p(\tau))}\left(S(X(\tau), p(\tau)) - I - D(X(\tau), p(\tau))\right)|\mathcal{G}_t\right] \end{aligned} \tag{A4}$$

Because $e^{-d(p(\tau))}\left(S(X(\tau), p(\tau)) - I - D(X(\tau), p(\tau))\right)$ is C^1 everywhere and the dependence of $(X(t), p(t))$ on any initial point (X, p) is explicit and smooth, the problem satisfies the smooth-fit principle (Peskir and Shiryaev (2006)).[1] Therefore, at all points $\left(\bar{X}(p), p\right)$ the derivatives of $e^{-d(p)}(G(X,p) - D(X,p))$ and $e^{-d(p)}(S(X,p) - I - D(X,p))$

[1] On p.152 Peskir and Shiryaev (2006) prove this result for one-dimensional problems, but state that it also extends to higher dimensions (see p.150). See also p.144, where they state that the smooth-fit principle holds in multiple dimensions if the state process after starting at the boundary of the stopping region enters the interior of the stopping region immediately.

with respect to X and p must be the same. Taking the two derivatives, we obtain

$$G_X\left(\bar{X}(p), p\right) = S_X\left(\bar{X}(p), p\right), \tag{A5}$$

$$G_p\left(\bar{X}(p), p\right) = S_p\left(\bar{X}(p), p\right). \tag{A6}$$

Proof of the existence of a solution to the fixed-point problem in Eq. (39).

Note that a solution $\left(G(X,p), \bar{X}(p)\right)$ to Eqs. (35) - (38) will satisfy (39). From the above we know that if $\bar{X}(p)$ exists, then it satisfies (35) - (38). We thus need to demonstrate the existence of a boundary between an exercise region and a continuation region for any $p \in (0, \pi_0]$.

Consider state $(X(t), p(t))$, and suppose that the firm exogenously learns the type of the outstanding shock in a moment. Then, the value of the investment option just before the firm learns the type of the shock is equal to $(1-p(t))\,G(X(t), 0) + p(t)\,G(X(t), 1)$. Formally, let $(\mathcal{G}'_s, s \geq t)$ denote filtration generated by $(B(s), M(s), N(s))$, $s \geq t$, but under which the firm learns the identity of an outstanding shock in a moment. In other words, $\mathcal{G}'_s$ contains the identity of the shock for all $s > t$, but not for $s = t$. Notice that while information filtrations $(\mathcal{G}_s, s \geq t)$ and $(\mathcal{G}'_s, s \geq t)$ are different, the evolution of $X(s)$, $s \geq t$ under the full-information filtration is the same. Let $G'(X(t), p(t))$ denote the value of the investment option under $(\mathcal{G}'_s, s \geq t)$

$$\begin{aligned} G'(X(t), p(t)) &\equiv \sup\nolimits_{\tau \in \mathfrak{M}'} \mathbb{E}\left[\int_\tau^\infty e^{-rs} X(s)\,ds - e^{-r\tau} I | \mathcal{G}'_t\right] \\ &= (1-p(t))\,G(X(t), 0) + p(t)\,G(X(t), 1), \end{aligned} \tag{A7}$$

where $\mathfrak{M}'$ is the set of optimal stopping times adapted to $(\mathcal{G}'_s, s \geq t)$. Because $(\mathcal{G}_t)$ is a filtration generated by (B, M, N), for any $s > t$, $\mathcal{G}_s \subseteq \mathcal{G}'_s$, and for $s = t$, $\mathcal{G}_s$ and $\mathcal{G}'_s$ coincide. By definition of $G(X(t), p(t))$,

$$G(X(t), p(t)) = \mathbb{E}\left[\int_{\tau^*}^\infty e^{-rs} X(s)\,ds - e^{-r\tau} I | \mathcal{G}_t\right], \tag{1}$$

where τ^* is the optimal stopping time, which is adapted to filtration $(\mathcal{G}_t)$. Because τ^* is adapted to $(\mathcal{G}_t)$, it is also adapted to $(\mathcal{G}'_t)$. Hence,

$$\begin{aligned} G'\left(X\left(t\right),p\left(t\right)\right) &= \sup\nolimits_{\tau\in\mathfrak{M}'} \mathbb{E}\left[\int_{\tau}^{\infty} e^{-rs}X\left(s\right)ds - e^{-r\tau}I|\mathcal{G}'_t\right] \geq \mathbb{E}\left[\int_{\tau^*}^{\infty} e^{-rs}X\left(s\right)ds - e^{-r\tau}I|\mathcal{G}'_t\right] \\ &= \mathbb{E}\left[\int_{\tau^*}^{\infty} e^{-rs}X\left(s\right)ds - e^{-r\tau}I|\mathcal{G}_t\right] = G\left(X\left(t\right),p\left(t\right)\right). \end{aligned} \tag{A9}$$

Combining (A7) with (A9), we obtain

$$\left(1-p\right)G\left(X,0\right) + pG\left(X,1\right) \geq G\left(X,p\right) \tag{A10}$$

for any (X, p).

Now, consider $G(X,0)$ and $G(X,1)$. There is no learning when $p \in \{0,1\}$, so the option pricing problems are standard. The exercise boundary for $p=0$ is given by $\bar{X}(0) = X^*$. To solve for $\bar{X}(1)$, note that in the range $p=1$, $X < (1+\varphi)X^*$, Eq. (35) is solved by

$$G\left(X,1\right) = \tilde{C}X^{\gamma} + \left(\frac{X}{\left(1+\varphi\right)X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right), \tag{A11}$$

where

$$\gamma = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2 + 2\left(r+\lambda_3\right)\sigma^2}\right] > \beta > 1. \tag{A12}$$

The unknown constant $\tilde{C}$ and the exercise boundary $\bar{X}(1)$ are given by the boundary conditions (36) - (37). Simplifying, we obtain an implicit solution for $\bar{X}(1)$:

$$\frac{\gamma-1}{\gamma}\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}\bar{X}\left(1\right) = I + \frac{\gamma-\beta}{\gamma}\left(\frac{\bar{X}\left(1\right)}{\left(1+\varphi\right)X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right). \tag{A13}$$

Define

$$Q(x) = I + \frac{\gamma-\beta}{\gamma}\left(\frac{x}{\left(1+\varphi\right)X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right) - \frac{\gamma-1}{\gamma}\left(\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}x\right). \tag{A14}$$

We have

$$Q(0) = I > 0,$$
$$Q((1+\varphi)X^*) = -\tfrac{\gamma-1}{\gamma}\tfrac{\varphi}{r-\alpha+\lambda_3}X^* < 0, \quad \text{(A15)}$$
$$Q''(x) = \beta(\beta-1)\tfrac{\gamma-\beta}{\gamma}\left(\tfrac{1}{(1+\varphi)X^*}\right)^{\beta}(\tfrac{X^*}{r-\alpha} - I)x^{\beta-1} > 0.$$

Hence, there exists a unique point between 0 and $(1+\varphi)X^*$, at which $Q(x) = 0$. It satisfies the sufficient conditions (Dixit and Pindyck (1994)), so it is equal to $\bar{X}(1)$.

Because both $\bar{X}(0)$ and $\bar{X}(1)$ are below $(1+\varphi)X^*$, for any $X > (1+\varphi)X^*$, $G(X,0) = \frac{1}{r-\alpha}X - I$ and $G(X,1) = \frac{1+\frac{\lambda_3}{(1+\varphi)(r-\alpha)}}{r+\lambda_3-\alpha}X - I$. Thus, for any $X > (1+\varphi)X^*$ and $p \in (0,\pi_0]$,

$$(1-p)G(X,0) + pG(X,1) = S(X,p) - I. \quad \text{(A16)}$$

By definition of $G(X,p)$,

$$G(X,p) \geq S(X,p) - I. \quad \text{(A17)}$$

Combining Eqs. (A10), (A16), and (A17), we find that for any $p \in (0,\pi_0]$ and any $X > (1+\varphi)X^*$, $G(X,p) = S(X,p) - I$, and thus it is always optimal to exercise the investment option. Also, for any $p \in (0,\pi_0]$ and any $X < (r-\alpha)I$, it is always optimal not to exercise the option, as the payoff from the exercise is negative in this range. Therefore, for any $p \in (0,\pi_0]$, there is a boundary between exercise and continuation regions.

Closed form solutions for the model of Section 3.2 when $\alpha > 0$ and $\sigma = 0$.

While the exercise trigger $\bar{p}(X)$ is characterized by (39), it is not solvable in closed-form, since the value function $G(X,p)$ itself is not available in closed-form. However, for the special case in which $\sigma = 0$, $\alpha \geq 0$, the closed-form solution for the trigger is

$$\bar{p}(X)|_{\sigma=0} = \frac{X - rI}{\lambda_3\left[I + \left(\frac{X}{rI(1+\varphi)}\right)^{\frac{r}{\alpha}}\frac{\alpha I}{r-\alpha} - \frac{X}{(1+\varphi)(r-\alpha)}\right]}. \quad \text{(A18)}$$

The corresponding value of the investment option equals

$$G(X,p)|_{\sigma=0} = p\frac{\alpha I}{r-\alpha}\left(\frac{X}{(1+\varphi)\,rI}\right)^{\frac{r}{\alpha}} + (1-p)\,X^{\frac{r}{\alpha}}\Gamma\left(X\left(\frac{1}{p}-1\right)^{-\frac{\alpha}{\lambda_3}}\right), \tag{A19}$$

where

$$\begin{aligned}\Gamma(y) = \left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}} y\left(\tfrac{\lambda_1}{r-\alpha}e^{(\alpha-r)t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}}y\right)} + \lambda_2\tfrac{1+\varphi+\frac{\lambda_3}{r-\alpha}}{(r+\lambda_3-\alpha)(1+\varphi)}e^{(\alpha-r-\lambda_3)t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}}y\right)}\right) \\ -\left(\tfrac{y}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}}\tfrac{\alpha I}{r-\alpha}\left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{r}{\lambda_3}}\lambda_2 e^{-\lambda_3 t^*\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha}{\lambda_3}}y\right)},\end{aligned} \tag{A20}$$

where $t^*(z)$ is a function defined implicitly by

$$\frac{\lambda_2\lambda_3}{\lambda_1 e^{\lambda_3 t^*}+\lambda_2} = \frac{ze^{\alpha t^*}-rI}{\left(\frac{z}{(1+\varphi)rI}\right)^{\frac{r}{\alpha}}\frac{\alpha I e^{rt^*}}{r-\alpha}+I-\frac{ze^{\alpha t^*}}{(r-\alpha)(1+\varphi)}}. \tag{A21}$$

Notice that when $\alpha = 0$, investment does not occur when the jump reverts. Therefore, in this case, the trigger (A18) coincides with (21).[2]

Derivation of the investment trigger for the case of multiple shocks in Section 5.

Let $Y(X(t), n(t), p(t))$ be a continuously differentiable function, where $X(t)$ is the current value of the cash flow process, $n(t)$ is the current number of outstanding shocks, and $p(t) = (p_0(t), p_1(t), ..., p_n(t))$ is the vector of current beliefs. Applying Itô's lemma for semimartingales (see Theorem 33 in Protter (2004)), we get the dynamics of $Y(t)$

[2] Note that since in (21) X_0 denotes the level of $X(t)$ before the positive jump occurred, we need to use $\bar{p}(X_0(1+\varphi))$ to ensure equivalence.

under $(\mathcal{F}_t)$:

$$\begin{aligned} dY = & \left(\alpha X Y_X + \tfrac{1}{2}\sigma^2 X^2 Y_{XX} - \sum_{k=1}^{n} \tfrac{\partial Y}{\partial p_k} \lambda_3 p_k \left(k - \textstyle\sum_{i=1}^{n} p_i i\right) \right) dt \\ & + \sigma X Y_x dB(t) + \left[Y\left(X(1+\varphi), n+1, \hat{p}(p)\right) - Y\right] \left(dM_1(t) + dM_2(t)\right) \\ & + \left[Y\left(\tfrac{X}{1+\varphi}, n-1, \tilde{p}(p)\right) - Y\right] dN(t), \end{aligned} \tag{A22}$$

where X, p_k, and Y denote $X(t)$, $p_k(t)$, and $Y(t)$, respectively, and $\hat{p}(p)$ and $\tilde{p}(p)$ are the updated vectors of beliefs defined by (8) - (9). In (A22), the intensities of $M_1(t)$ and $M_2(t)$ are λ_1 and λ_2 if $n(t) < \bar{N}$ and zero if $n(t) = \bar{N}$. By the law of iterated expectations,

$$\mathbb{E}\left[dN(t)|\mathcal{G}_t\right] = \mathbb{E}\left[\mathbb{E}\left[dN(t)|\mathcal{F}_t\right]\mathcal{G}_t\right] = \mathbb{E}\left[k(t)\lambda_3 dt|\mathcal{G}_t\right] = \sum_{k=1}^{n} \lambda_3 k p_k dt. \tag{A23}$$

Therefore, the instantaneous conditional expected change in $Y(X, n, p)$ is equal to

$$\begin{aligned} \mathbb{E}\left[\tfrac{dY(X,n,p)}{dt}|\mathcal{G}_t\right] = & \ \alpha X Y_X + \tfrac{1}{2}\sigma^2 X^2 Y_{XX} - \sum_{k=1}^{n} \tfrac{\partial Y}{\partial p_k} \lambda_3 p_k \left(k - \textstyle\sum_{i=1}^{n} p_i i\right) \\ & + \mathbf{1}_{n<\bar{N}} \left[Y\left(X(1+\varphi), n+1, \hat{p}(p)\right) - Y\right] (\lambda_1 + \lambda_2) \\ & + \left[Y\left(\tfrac{X}{1+\varphi}, n-1, \tilde{p}(p)\right) - Y\right] \sum_{k=1}^{n} \lambda_3 k p_k, \end{aligned} \tag{A24}$$

where $\mathbf{1}_{n<\bar{N}}$ is an indicator function taking the value 1 if $n < \bar{N}$ and 0 otherwise.

If $Y(X, n, p)$ denotes the value of a contingent claim that continuously pays a cash flow of $y(X, n, p)$ and the discount rate is equal to r, then it must be the case that

$\mathbb{E}\left[\frac{dY(X,n,p)+y(X,n,p)dt}{dt}|\mathcal{G}_t\right] = rY(X,n,p)$. Hence,

$$\begin{aligned} rY = \alpha XY_X + \tfrac{1}{2}\sigma^2 X^2 Y_{XX} - \sum_{k=1}^{n} \tfrac{\partial Y}{\partial p_k}\lambda_3 p_k \left(k - \textstyle\sum_{i=1}^{n} p_i i\right) \\ +\mathbf{1}_{n<\bar{N}}\left[Y\left(X(1+\varphi), n+1, \hat{p}(p)\right) - Y\right](\lambda_1+\lambda_2) \\ \left[Y\left(\tfrac{X}{1+\varphi}, n-1, \tilde{p}(p)\right) - Y\right]\sum_{k=1}^{n}\lambda_3 k p_k + y(X,n,p). \end{aligned} \quad \text{(A25)}$$

Simplifying:

$$\begin{aligned} \left(r+\lambda_1+\lambda_2+\lambda_3\sum_{k=1}^{n} p_k k\right) Y = \alpha XY_X + \tfrac{1}{2}\sigma^2 X^2 Y_{XX} \\ -\lambda_3\sum_{k=1}^{n}\tfrac{\partial Y}{\partial p_k} p_k\left(k-\textstyle\sum_{i=1}^{n} p_i i\right) + \mathbf{1}_{n<\bar{N}}(\lambda_1+\lambda_2)\, Y\left(X(1+\varphi), n+1, \hat{p}(p)\right) \\ +\left(\lambda_3\sum_{k=1}^{n} p_k k\right) Y\left(\tfrac{X}{1+\varphi}, n-1, \tilde{p}(p)\right) + y(X,n,p). \end{aligned} \quad \text{(A26)}$$

Let $S(X,n,p)$ denote the value of the underlying project. It is the expected discounted value of cash flows that the firm gets if it immediately exercises the investment option. $S(X,n,p)$ is thus a special case of $Y(X,n,p)$, with $y(X,n,p) = X$. Thus $S(X,n,p)$ must satisfy:

$$\begin{aligned} \left(r+\lambda_1+\lambda_2+\lambda_3\sum_{k=1}^{n} p_k k\right) S = \alpha XS_X + \tfrac{1}{2}\sigma^2 X^2 S_{XX} \\ -\lambda_3\sum_{k=1}^{n}\tfrac{\partial S}{\partial p_k} p_k\left(k-\textstyle\sum_{i=1}^{n} p_i i\right) + \mathbf{1}_{n<\bar{N}}(\lambda_1+\lambda_2)\, S\left(X(1+\varphi), n+1, \hat{p}(p)\right) \\ +\left(\lambda_3\sum_{k=1}^{n} p_k k\right) S\left(\tfrac{X}{1+\varphi}, n-1, \tilde{p}(p)\right) + X. \end{aligned} \quad \text{(A27)}$$

The solution can be written as

$$S(X,n,p) = a_0^n X + \sum_{k=1}^{n} p_k\left(a_k^n - a_0^n\right) X, \quad \text{(A28)}$$

where constants a_k^n, $k = 0, 1, ..., n$, $n = 0, 1, ..., \bar{N}$ are defined later.

Let $G(X, n, p)$ denote the value of the investment option. Before the investment occurs, $G(X, n, p)$ is a special case of $Y(X, n, p)$, with $y(X, n, p) = 0$. Thus $G(X, n, p)$ must satisfy:

$$\begin{aligned}\left(r + \lambda_1 + \lambda_2 + \lambda_3 \sum_{k=1}^{n} p_k k\right) G = \alpha X G_X + \tfrac{1}{2}\sigma^2 X^2 G_{XX} \\ -\lambda_3 \sum_{k=1}^{n} \tfrac{\partial G}{\partial p_k} p_k \left(k - \textstyle\sum_{i=1}^{n} p_i i\right) + \mathbf{1}_{n<\bar{N}} (\lambda_1 + \lambda_2) G\left(X(1+\varphi), n+1, \hat{p}(p)\right) \\ + \left(\lambda_3 \sum_{k=1}^{n} p_k k\right) G\left(\tfrac{X}{1+\varphi}, n-1, \tilde{p}(p)\right).\end{aligned} \tag{A29}$$

The optimal investment decision can be described by a trigger function $\bar{X}(n, p)$. Eq. (A29) is solved subject to the following value-matching and smooth-pasting conditions:

$$\begin{aligned} G\left(\bar{X}(n,p), n, p\right) = S\left(\bar{X}(n,p), n, p\right) - I, \\ G_X\left(\bar{X}(n,p), n, p\right) = S_X\left(\bar{X}(n,p), n, p\right), \\ \left(\lambda_3 \textstyle\sum_{k=1}^{n} \left(\frac{\partial G\left(\bar{X}(n,p),n,p\right)}{\partial p_k} - \frac{\partial S\left(\bar{X}(n,p),n,p\right)}{\partial p_k}\right)\right) p_k \left(k - \textstyle\sum_{i=1}^{n} p_i i\right) = 0. \end{aligned} \tag{A30}$$

The first equation is the value-matching condition. The second and third equations are the smooth-pasting conditions (with respect to X and t, respectively).

Combining (A27), (A29) and (A30) gives us:[3]

$$\begin{aligned} \bar{X}(n,p) - rI = \tfrac{\sigma^2}{2} \bar{X}(n,p)^2 G_{XX}\left(\bar{X}(n,p), n, p\right) + \mathbf{1}_{n<\bar{N}} (\lambda_1 + \lambda_2) \\ \times \left[G\left(\bar{X}(n,p)(1+\varphi), n+1, \hat{p}(p)\right) + I - S\left(\bar{X}(n,p)(1+\varphi), n+1, \hat{p}(p)\right)\right] \\ + \left(\lambda_3 \textstyle\sum_{k=1}^{n} p_k k\right) \left[G\left(\tfrac{\bar{X}(n,p)}{1+\varphi}, n-1, \tilde{p}(p)\right) + I - S\left(\tfrac{\bar{X}(n,p)}{1+\varphi}, n-1, \tilde{p}(p)\right)\right]. \end{aligned} \tag{A31}$$

When X is very close to the trigger $\bar{X}(n, p)$, the arrival of a new positive shock will result

[3] Note that since $S(X, n, p)$ is linear in X, $S_{XX} = 0$.

in immediate investment. This implies:

$$G\left(\bar{X}(n,p)(1+\varphi), n+1, \hat{p}(p)\right) = S\left(\bar{X}(n,p)(1+\varphi), n+1, \hat{p}(p)\right) - I. \quad \text{(A32)}$$

Thus, the second term on the right-hand side of (A31) is zero, so $\bar{X}(n,p)$ satisfies

$$\begin{aligned}\bar{X}(n,p) = \left(\lambda_3 \textstyle\sum_{k=1}^{n} p_k k\right)\left[G\left(\frac{\bar{X}(n,p)}{1+\varphi}, n-1, \tilde{p}(p)\right) + I - S\left(\frac{\bar{X}(n,p)}{1+\varphi}, n-1, \tilde{p}(p)\right)\right] \\ +rI + \frac{\sigma^2}{2}\bar{X}(n,p)^2 G_{XX}\left(\bar{X}(n,p), n, p\right),\end{aligned} \quad \text{(A33)}$$

Derivation of constants a_k^n in Eq. (A28).

Plugging (A28) in (A27), using the definitions of $\hat{p}(p)$ and $\tilde{p}(p)$, and simplifying, we obtain

$$\begin{aligned}&\left(r-\alpha+\lambda_1+\lambda_2+\lambda_3 \textstyle\sum_{k=1}^{n} p_k k\right) a_0^n + \left(r-\alpha+\lambda_1+\lambda_2+\lambda_3\right)\textstyle\sum_{k=1}^{n} p_k\left(a_k^n - a_0^n\right) \\ &= (1+\varphi)\left(\lambda_1 a_0^{n+1} + \lambda_2 a_1^{n+1} + \textstyle\sum_{k=1}^{n} p_k\left(\lambda_1\left(a_k^{n+1} - a_0^{n+1}\right) + \lambda_2\left(a_{k+1}^{n+1} - a_0^{n+1}\right)\right)\right) \\ &\quad + \frac{\lambda_3}{1+\varphi}\textstyle\sum_{k=1}^{n} p_k k a_{k-1}^{n-1} + 1,\end{aligned} \quad \text{(A34)}$$

for $n = 0, 1, ..., \bar{N}-1$. For $n = \bar{N}$, $S(X, n, p)$ satisfies (A27) without the terms with λ_1 and λ_2. Hence,

$$\begin{aligned}\left(r-\alpha+\lambda_3 \textstyle\sum_{k=1}^{\bar{N}} p_k k\right) a_0^{\bar{N}} + \\ \left(r-\alpha+\lambda_3\right)\textstyle\sum_{k=1}^{\bar{N}} p_k\left(a_k^{\bar{N}} - a_0^{\bar{N}}\right) = \frac{\lambda_3}{1+\varphi}\textstyle\sum_{k=1}^{\bar{N}} p_k k a_{k-1}^{\bar{N}-1} + 1.\end{aligned} \quad \text{(A35)}$$

Eqs. (A34) and (A35) must hold for all p. Hence, the coefficients in constant terms and p_k, $k = 1, ..., n$ on both sides of each equation must be equal. Thus, we get the following equations:

- for $n = 0, 1, ..., \bar{N}-1$:

$$\left(r-\alpha+\lambda_1+\lambda_2\right) a_0^n = (1+\varphi)\left(\lambda_1 a_0^{n+1} + \lambda_2 a_1^{n+1}\right) + 1; \quad \text{(A36)}$$

- for $k = 1, ..., n$ and $n = 1, ..., \bar{N} - 1$:

$$\begin{aligned} & k\lambda_3 a_0^n + (r - \alpha + \lambda_1 + \lambda_2 + \lambda_3)\left(a_k^n - a_0^n\right) \\ = \ & (1+\varphi)\left(\lambda_1\left(a_k^{n+1} - a_0^{n+1}\right) + \lambda_2\left(a_{k+1}^{n+1} - a_1^{n+1}\right)\right) + \frac{\lambda_3}{1+\varphi} a_{k-1}^{n-1}; \end{aligned} \tag{A37}$$

- for $k = 1, ..., \bar{N}$:

$$k\lambda_3 a_0^{\bar{N}} + (r - \alpha + \lambda_3)\left(a_k^{\bar{N}} - a_0^{\bar{N}}\right) = \frac{\lambda_3}{1+\varphi} a_{k-1}^{\bar{N}-1}. \tag{A38}$$

- finally, matching the constant term in the equation for $n = \bar{N}$:

$$(r - \alpha)\, a_0^{\bar{N}} = 1. \tag{A39}$$

This gives us $\left(\frac{\bar{N}}{2} + 1\right)\left(\bar{N} + 1\right)$ linear equations that fully determine $\left(\frac{\bar{N}}{2} + 1\right)\left(\bar{N} + 1\right)$ constants a_k^n, $k = 0, ..., n$, $n = 0, ..., \bar{N}$.

2 Numerical Procedures

Numerical procedure for computing $\bar{X}(p)$ in Section 3.

To compute the trigger functions we use a variation of the least-squares method developed by Longstaff and Schwartz (2003). Note that when $p = 0$, the model becomes standard, so $\bar{X}(0) = X^*$. Also, note that $p(T) \to 0$ as $T \to \infty$, where T is the time that passes after the arrival of the shock. Because of that, we can approximate $\bar{X}(p(T))$ for a large T by $\bar{X}(0)$. After that, we take a small Δ and compute $p(T - \Delta)$ from (11). Then, we use the least squares method of Longstaff and Schwartz (2003) to estimate the second derivative of the conditional expected payoff from waiting at time $T - \Delta$ until time T. Then, we use this estimate and (39) to compute $\bar{X}(p(T - \Delta))$. We repeat this N times for a sufficiently large N such that $p(T - N\Delta) > \pi_0$. More specifically, at any step n:

1. Use $\bar{X}(p(T-k\Delta))$, $k=0,1,...,n-1$ and the least squares method to estimate the second derivative of the conditional expected payoff from waiting at time $T-n\Delta$.

2. Use this estimate as input in (39) to compute $\bar{X}(p(T-n\Delta))$.

Numerical procedure for computing $\bar{X}(n,p)$ in Section 5.

We consider the following set of parameter values: $\bar{N}=2$, $r=0.04$, $\alpha=\sigma=0$, $\varphi=0.1$, $\lambda_1=0.33$, $\lambda_2=1$, $\lambda_3=3$, $I=25$. First, we compute $S(X,n,p)$. From (A36) - (A39), when $\bar{N}=2$, we get 6 linear equations that determine 6 unknowns a_0^0, a_0^1, a_1^1, a_0^2, a_1^2, and a_2^2. Solving this system of equations, we obtain $a_0^0=29.492$, $a_0^1=27.239$, $a_1^1=26.833$, $a_0^2=25$, $a_1^2=24.766$, $a_2^2=24.398$.

Second, we compute $\bar{X}(n,p)$. Because the option value is a relatively flat function of the exercise threshold around the optimal threshold, computing the option values by simulations yields relatively precise option values but relatively imprecise exercise threshold. To overcome this problem, we compute the option values by simulations and use them as inputs in Eq. (52) to compute the exercise thresholds:

$$\bar{X}(0,p)=rI, \tag{A40}$$

$$\bar{X}(1,p)=\lambda_3 p_1\left[G\left(\frac{\bar{X}(1,p)}{1+\varphi},0,1\right)+I-S\left(\frac{\bar{X}(1,p)}{1+\varphi},0,1\right)\right]+rI, \tag{A41}$$

$$\bar{X}(2,p)=\lambda_3(p_1+2p_2)\left[G\left(\frac{\bar{X}(2,p)}{1+\varphi},1,\tilde{p}(p)\right)+I-S\left(\frac{\bar{X}(2,p)}{1+\varphi},1,\tilde{p}(p)\right)\right]+rI. \tag{A42}$$

Specifically, our numerical procedure is set up in the following way:

1. For given thresholds $\bar{X}(1,1-p_1,p_1)$ and $\bar{X}(2,1-p_1-p_2,p_1,p_2)$, expressed as polynomials of p_1 and (p_1,p_2), respectively, calculate the option value $G(X,0,1)$ by simulations. We use the fourth-order polynomials and $20,000$ simulations.

2. Maximize the option value $G(X,0,1)$ with respect to coefficients of polynomials.

3. Using the computed thresholds, calculate option values $G(X,0,1)$ and $G(X,1,1-p_1,p_1)$.

4. Use option values $G(X,0,1)$ and $G(X,1,1-p_1,p_1)$ from step 3 to obtain the exercise thresholds $\bar{X}(1,p)$ and $\bar{X}(2,p)$ from (A41) and (A42).

The order of the polynomials used in step 1 can be expanded, and steps 3 and 4 can be iterated until convergence with very little effect on the values of the exercise thresholds.