Web-Appendix for "Competition among Sellers in Securities Auctions"

Alexander S. Gorbenko **Andrey Malenko**

Appendix A Proofs

Proof of Lemma 1. See the proof of Lemma 2 in the web-appendix for DeMarzo, Kremer, and Skrzypacz (2005).

Proof of Lemma 2. Suppose for a moment that k is a continuous variable. If we prove that the lemma holds for any k, this will automatically imply that it holds for k taking values $0, 1, 2, ...$ First, let us prove that $V(k)$ is an increasing and concave function of k. Differentiating $V(k)$,

$$V'(k) = \int_{v_L}^{v_H} \left(\frac{1}{k} + \log F(v) \right) v d \left(F(v)^k \right) > 0,$$

because $\int_{v_L}^{v_H} \left(\frac{1}{k} + \log F(v) \right) d \left(F(v)^k \right) = 0$ as the first derivative of $\int_{v_L}^{v_H} d \left(F(v)^k \right) = 1$. Differentiating again,

$$V''(k) = \int_{v_L}^{v_H} \log F(v) \left(\frac{2}{k} + \log F(v) \right) v d \left(F(v)^k \right).$$

Note that $\int_{v_L}^{v_H} \log F(v) \left(\frac{2}{k} + \log F(v) \right) d \left(F(v)^k \right) = 0$ as the second derivative of $\int_{v_L}^{v_H} d \left(F(v)^k \right) = 1$. Also, $\log F(v) \left(\frac{2}{k} + \log F(v) \right)$ is positive for $v < F^{-1} \left(e^{-2/k} \right)$ and negative for $v > F^{-1} \left(e^{-2/k} \right)$. Hence, $V''(k) < 0$.

In a similar way, we can prove that $U^b(v,k,S)$ is decreasing and convex in k. Write $U^b(v,k,S)$ as

$$\text{(A1)} \qquad U^b(v,k,S) = \int_{v_L}^{v} (v - ES(s(y,S),v))\, d\left(F(y)^{k-1}\right).$$

Differentiating,

$$\text{(A2)} \qquad U_k^b(v,k,S) = \int_{v_L}^{v} (v - ES(s(y,S),v))\left(\frac{1}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right).$$

Note that $\int_{v_L}^{v}\left(\frac{1}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) \leq 0$ for any v, because $\log F(y)$ is an increasing function of y and $\int_{v_L}^{v_H}\left(\frac{1}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) = 0$. Also, $s(y,S)$ is increasing in y by Lemma 1. Therefore, (A2) is negative. Differentiating (A1) again,

$$\text{(A3)} \qquad U_{kk}^b(v,k,S) = \int_{v_L}^{v} (v - ES(s(y,S),v)) \log F(y)\left(\frac{2}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right).$$

Note that $\log F(y)\left(\frac{2}{k-1} + \log F(y)\right)$ is positive for $y < F^{-1}\left(e^{-2/(k-1)}\right)$ and negative for $y > F^{-1}\left(e^{-2/(k-1)}\right)$. This and $\int_{v_L}^{v_H} \log F(y)\left(\frac{2}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) = 0$ imply that $\int_{v_L}^{v} \log F(y)\left(\frac{2}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) \geq 0$ for all v. Combining this with the fact that $s(y,S)$ is increasing in y, so $v - ES(s(y,S),v)$ is decreasing in y, yields $U_{kk}^b(v,k,S) > 0$.

Proof of Lemma 3. Equation (11), which determines $q\left(S,\tilde{S}\right)$, can be written in the following form:

(A4)
$$\sum_{k=1}^{n_b} \binom{n_b - 1}{k-1}\left[q^{k-1}(1-q)^{n_b-k}\, U^b(k,S) - \left(\frac{1-q}{n_s-1}\right)^{k-1}\left(1 - \frac{1-q}{n_s-1}\right)^{n_b-k} U^b\left(k,\tilde{S}\right)\right] = 0.$$

Denote the left-hand side of (A4) by $g\left(q,S,\tilde{S}\right)$. Note that

$$\text{(A5)}\quad g\left(0,S,\tilde{S}\right) = U^{b}\left(1,S\right)-\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}\left(\frac{1}{n_s-1}\right)^{k-1}\left(1-\frac{1}{n_s-1}\right)^{n_b-k}U^{b}\left(k,\tilde{S}\right)>0;$$

$$\text{(A6)}\quad g\left(1,S,\tilde{S}\right) = U^{b}\left(n_b,S\right)-U^{b}\left(1,\tilde{S}\right)<0,$$

because when there is only one bidder, he captures all surplus from the auction above v_L. Differentiating (A4),

(A7)

$$g_q\left(q,S,\tilde{S}\right) = \textstyle\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}q^{k-1}\left(1-q\right)^{n_b-k}\left(\frac{k-1}{q}-\frac{n_b-k}{1-q}\right)U^{b}\left(k,S\right)$$
$$+\tfrac{1}{n_s-1}\textstyle\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}\left(\frac{1-q}{n_s-1}\right)^{k-1}\left(1-\frac{1-q}{n_s-1}\right)^{n_b-k}\left(\frac{k-1}{\frac{1-q}{n_s-1}}-\frac{n_b-k}{1-\frac{1-q}{n_s-1}}\right)U^{b}\left(k,\tilde{S}\right).$$

Consider the first term of (A7). Since probabilities $\binom{n_b-1}{k-1}q^{k-1}\left(1-q\right)^{n_b-k}$ sum to one over k, taking the derivative of their sum means that for any q and n_b,

$$\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}q^{k-1}\left(1-q\right)^{n_b-k}\left(\frac{k-1}{q}-\frac{n_b-k}{1-q}\right)=0.$$

Moreover, the terms of this sum are negative for $k<q\left(n_b-1\right)+1$ and positive, otherwise. Because it immediately follows from Lemma 2 that $U^{b}\left(k,S\right)$ is a decreasing function of k as an expectation of $U^{b}\left(v,k,S\right)$ with respect to v, the first term of (A7) is negative. By the same argument, the second term of (A7) is also negative. Hence, $g\left(q,S,\tilde{S}\right)$ is a monotonically decreasing function of q. Combining this with (A5) and (A6), we conclude that there exists a unique solution to (11). This proves part (a) of the lemma.

If S_1 is a steeper set of securities than S_2, $U^{b}\left(k,S_1\right)<U^{b}\left(k,S_2\right)$ for any $k>1$ with equality for $k=1$. This is because the total surplus is unaffected by the choice of security design, and the seller's revenues are higher when the security design is steeper. Thus, an increase in the steepness of the firm's own security design decreases $g\left(q,S,\tilde{S}\right)$. Similarly, an increase in the steepness of the other firms' security design increases $g\left(q,S,\tilde{S}\right)$. Because $g\left(q,S,\tilde{S}\right)$ is decreasing in q for any S and $\tilde{S}$, the point $q\left(S, \tilde{S}\right)$ at which g crosses zero moves in the same direction as $g\left(q, S, \tilde{S}\right)$ if S or $\tilde{S}$ is altered. This implies parts (b) and (c) of the lemma. To prove part (d), notice that $q = \frac{1}{n_s}$ is the solution of (A4) when $S = \tilde{S}$.