Online Appendix to “Left Behind: Creative Destruction, Inequality, and the Stock Market”

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Outline

In Section A we discuss the identification of the model's structural parameters. In Section B, we shed light on the model mechanism by estimating restricted versions of the model. In Section C we include some additional results that complement the results in the main paper. In Section D, we discuss the construction of the estimation targets and our empirical methodology. Last, in Section E we provide proofs and detailed derivations.

A Identification

Here, we discuss the identification of the model's structural parameters. In Figures A.4 through A.8 we plot the Gentzkow and Shapiro (2014) measure of sensitivity of parameters to moments. We report the measure in elasticity form,

$$\hat{\lambda}_{i,j} = \lambda_{i,j} \frac{X^j(\theta)}{\theta^i}, \tag{A.1}$$

where $\lambda_{i,j}$ is the element of the sensitivity matrix Λ that corresponds to parameter i and moment j. The matrix Λ is computed as

$$\Lambda = -(G'WG)^{-1}G'W \tag{A.2}$$

where G is the numerical gradient of the sample moments $g(\theta) = X - \mathcal{X}$ and W is the optimal weighting matrix.

The parameters governing the volatility of the two technology shocks are of primary importance for the model's implications about the dynamics of aggregate quantities and asset prices. Accordingly, their estimated values $\hat{\sigma}_x = 8.2\%$ and $\hat{\sigma}_\xi = 11\%$ are estimated with considerable precision. Examining the GS sensitivity measure, we see that σ_x is primarily identified by the volatility of consumption growth, and the correlation between consumption and investment. Conversely, σ_ξ is mostly identified by the volatility of investment growth, and the correlation between investment and consumption. Recall from Figure 2 that both investment and consumption respond symmetrically to the disembodied shock x; by contrast, investment and consumption respond initially with opposite signs to ξ. Hence, the correlation between investment and consumption carries important information on the relative importance of these two shocks. In contrast, the parameters governing the mean growth rates of the two types of technology, μ_x and μ_ξ, are much less precisely estimated. To estimate μ_x, the model places considerable weight on the mean rate of consumption growth; by contrast, to estimate μ_ξ our estimation procedure leans on the the moments of the investment-to-output ratio; higher values of μ_ξ imply that the mean of the stationary distribution of ω is higher, which implies higher investment-to-output, but also a faster rate of mean reversion, and therefore a lower unconditional volatility for the investment-to-output ratio. However, since these moments also depend on other parameters (for instance, the mean growth rate of the economy depends on both μ_x and μ_ξ) these two parameters are less precisely estimated.

The parameters governing the dynamics of $\lambda_{f,t}$ are identified by both the aggregate investment-to-output ratio, as well as the dispersion and persistence of firm-level investment, innovation, and valuation ratios. Specifically, the parameter λ, which corresponds to the cross-sectional mean of $\lambda_{f,t}$ is primarily determined by the unconditional volatility in the investment-to-output ratio; the aggregate ratio of the value of new projects to total wealth (ν_t/M_t); and the persistence in innovation outcomes at the firm level. An increase in λ implies that the economy responds faster to a positive shock to x or ξ, which implies that the state variable ω exhibits a faster rate of mean reversion and hence lower unconditional volatility. In addition, increasing λ implies an increase in the frequency of investment at the firm level, which implies higher firm-level persistence. The parameter $\lambda_D = (\lambda_H - \lambda_L)/\lambda$, which controls the relative differences in firm growth rates between the low- and high-growth regimes, is primarily identified by the cross-sectional dispersion in firm investment. The estimated value assigns considerable difference in the rate at which firms acquire projects in the high- and low-growth regimes. The parameters governing the transition across the low- and high-growth regimes (8) are identified by the persistence of investment growth, innovation, and valuation ratios at the firm level, as well as cross-sectional differences in innovation outcomes across firms. In particular, increasing either μ_H or μ_L lowers the persistence of investment and valuation ratios at the firm level. Our estimates imply that the high-growth rate is pretty transitory, while the low-growth rate is highly persistent. Hence, at any given time, most firms in the model are in the low-growth regime.

The remaining parameters governing the production side of the model are also relatively precisely estimated. The share of capital in the production function ϕ is primarily identified by the capital share. Recall that the capital share in the model, defined as the share of output that accrues to the owners of capital (shareholders) is equal to ϕ minus the share of output that accrues to inventors. The rate of mean reversion of firm-specific shocks κ_u is identified by the persistence in profitability at the firm level. The volatility of firm-specific shocks σ_u is primarily identified by the persistence and dispersion in profitability across firms. The depreciation rate δ is identified by the average investment-to-output ratio; similar to deterministic models, a higher depreciation rate of capital implies a higher mean investment-to-output ratio. Last, the degree of adjustment costs α is primarily identified by the volatility of investment growth, the cross-sectional dispersion in Tobin's Q, and the volatility of the market portfolio. Recall that a higher value of α lowers adjustment costs; similar to standard models, doing so makes investment more volatile but lowers the volatility of prices.

The parameter that governs the share of households that participate in the stock market ψ is relatively well estimated, with a point estimate of 0.15 and a standard error of 0.05. Our estimate is largely in line with the facts reported in Poterba and Samwick (1995), who document that the households in the top 20% in terms of asset ownership consistently own more than 98% of all stocks. This parameter is primarily identified by the mean consumption share of stockholders.

The parameter that affects the division of surplus between shareholders and innovators is estimated at $\eta = 0.77$ with a standard error of 0.39. Hence, approximately one-fourth of the value of new investment opportunities in the economy accrues to the owners of the firm's public securities.[1] In general, this parameter has an ambiguous effect on the value premium. On the

[1] A quantitative evaluation of the plausibility of this parameter is challenging due to the lack of available data

one hand, increasing η increases the share of rents that go to innovators, and therefore increases the displacement risk that is faced by shareholders. On the other hand, however, it reduces the overall share of growth opportunities to firm value, which increases the volatility of the market portfolio—since the market is now more vulnerable to displacement. Consequently, this parameter is mainly identified by the average returns of the market portfolio and the value factor (which are noisily estimated), and to a lesser extent by the long-run volatility of consumption. A higher long-run volatility of consumption growth implies that the model needs a smaller value of η to match asset prices.

The remaining parameters correspond to household preferences, and therefore are not always well identified. For instance, the estimated utility curvature parameter is relatively high, ($\hat{\gamma} = 57$), but the large standard errors (41) suggest that the model output is not particularly sensitive to the precise value of γ. Examining the GS sensitivity measure, we see that the risk aversion coefficient is primarily identified by the mean of the market portfolio relative to its standard deviation and the mean of the value factor. Similarly, the elasticity of inter-temporal substitution has a point estimate of $\theta = 2.3$ but the large standard errors reveal that the quantitative implications of the model is robust to different values of θ. Examining the GS sensitivity measure reveals that θ is primarily identified through the volatility of the interest rate in the model and the long-run volatility of consumption growth. A higher willingness to substitute across time implies a higher serial correlation for consumption growth, and hence a higher long-run variability in consumption growth. The parameter governing the household's effective discount rate ρ is somewhat better identified, mainly through the mean of the investment-output ratio, as well as the mean interest rate relative to the growth rate of the economy.

Last, the parameter governing the share of relative consumption h is identified primarily by the differences in risk premia between the market and the value factor. Recalling the discussion in Figure 4, higher values of h imply that households are more averse to displacement risk—they resent being left behind—and hence increases the risk premium associated with displacement risk. However, higher values of h also lower the household's effective risk aversion towards aggregate—that is, non-displacive—shocks, that is, the parameter γ_1 in the expression for the stochastic discount factor (A.102). Since the equity premium compensates investors partly for this aggregate risk, increasing h lowers the equity premium while increasing the mean returns of the value factor.

on valuations of private firms. However, if we interpret the selling of ideas to firms as the inventors creating new startups that go public and are subsequently sold to large, publicly traded firms, this pattern is consistent with the empirical fact that most of the rents from acquisitions go to the owners of the target firm (Asquith and Kim, 1982). In addition, a high estimate of η is consistent with the idea that ideas are a scarcer resource than capital, and since the 'innovators' own the ideas, it is conceivable that they extract most of the rents from the creation of new projects. An illustrative example is the case of the British advertising agency Saatchi and Saatchi, described in Rajan and Zingales (2000). In 1994, Maurice Saatchi, the chairman of Saatchi and Saatchi, proposed for himself a generous compensation package. The U.S. fund managers, who controlled 30 percent of the shares, voted down the proposal at the general shareholders' meeting. The Saatchi brothers, along with several key senior executives, subsequently left the firm and started a rival agency (M&C Saatchi), which in a short period of time captured several of the most important accounts of the original Saatchi & Saatchi firm. The original firm was severely damaged and subsequently changed its name. Another real world example of an inventor is a venture capitalist (VC). A large value of η would be consistent with the evidence that VC firms add considerable value to startups, but investors in these funds effectively capture none of these rents (see, e.g. Korteweg and Nagel, 2016).

Our baseline estimates imply that households attach a weight approximately equal to 80% on relative consumption. The small standard error (0.06) implies that this parameter is quantitatively important for the implications of the model.

In sum, the two parameters that are key to our main mechanism, η and h, are estimated with differing levels of precision, with h much more precisely estimated that η. This suggests that the precise value of η is much less important for the model's implications than h. However, these are local measures of sensitivity. Our findings stress the importance of the three relatively non-standard features of the model: incomplete markets, preference for relative consumption, and embodied technology shocks. Eliminating any of these three assumptions compromises the model's ability to simultaneously fit both the level as well as the cross-section in risk premia.

B Sensitivity Analysis

Here, we further shed light on the model mechanism by estimating restricted versions of the model. Recall that our model has three relatively non-standard features. First, our model features technology shocks that are embodied in new capital. Second, markets are incomplete in that households cannot sell claims on their proceeds from innovation. Third, household preferences are affected by their consumption relative to the aggregate economy. In addition, when taking the model to the data, we assumed that a subset of households did not participate in financial markets. Here, we examine how important these features are for the quantitative performance of the model. Last, we also estimate a version of the model that imposes an upper bound on the risk aversion of the representative household.

We proceed by estimating the model imposing several parameter restrictions. In Table A.2, we only report the key estimated parameters of the model, along with targeted moments. Tables A.3 and A.4 present the full set of tables. To understand the impact of these parameter restrictions on asset prices, we also report the equilibrium risk prices for the two technology shocks x and ξ, along with the risk exposures of the market portfolio and the value factor.[2]

B.1 Full participation

We first estimate a version of the model in which all households participate in financial markets. This is equivalent to restricting $\psi = 1$ in (33). Comparing columns two and three in Table A.2, we see that the restricted model does almost as well as the full model. Both the level as well as the dispersion in risk premia are comparable to the baseline model and similar to the data; decomposing the risk premia to risk prices and risk exposures yields similar results as the baseline model. The overall measure of fit is also close (0.025 vs 0.021). However, the restricted model does require high

[2] In particular, risk premia can be written as $E[R_i] - r_f = \beta_{x,i}\lambda_x + \beta_{\xi,i}\lambda_\xi$, where β are the risk exposures to the two technology shocks, and $\lambda_x \equiv -cov\left(dx_t, d\log\Lambda_t\right)/dt$ and $\lambda_\xi \equiv -cov\left(d\xi_t, d\log\Lambda_t\right)/dt$ the risk prices. This equation is an approximation that is useful for exposition: it only holds conditionally. If risk prices and risk exposures are correlated, the unconditional expression is more complicated. However, it is a relatively accurate and useful approximation in that it helps decompose the role of the two technology shocks in affecting risk premia.

curvature of the utility function (γ of 105 vs 57), a higher surplus share to inventors (0.81 vs 0.77), and a larger preference weight on relative consumption (0.93 vs 0.84).

One reason why these estimated parameters are higher is that the restricted model underestimates the risks that financial markets participants face from innovation. As we saw in Figure 2, labor income rises in response to improvements in technology. This increase in labor income acts as a natural hedge for the displacement of households that occurs through financial markets. Restricting a fraction of the population to not participate effectively limits the share of the labor income that accrues to stock holders and therefore mitigates this hedging effect. Here, we should emphasize that this hedging benefit of labor income is an artifact of the stylized nature of our model. Specifically, we assume that technology has no displacive effect on labor, and that labor income is tradable without any frictions. A more realistic model that allows for endogenous displacement of human capital and possibly frictions, such as credit constraints, is outside the scope of this paper.

B.2 Restricting the surplus share to innovators

We next present the results when we restrict the surplus that accrues to innovators to be qual to zero, $\eta = 0$. Doing so effectively completes the markets for innovation outcomes, since in this case, all proceeds from new projects accrue to financial market participants. Examining column four of Table A.2, we see that this restricted model performs relatively worse in fitting risk premia. The value premium is equal to 3.4% (versus 6.3% in the baseline model), and the equity premium is equal to 5.3% per year (versus 6.7% in the baseline model). The estimated parameters imply that the preference weight on relative consumption is quite high (92%), while the share of households that participate is 5%.

Examining the risk prices, we see that the price of the embodied technology shock is quite a bit smaller than in the baseline case (-4.3% vs -7.8%) which attenuates the dispersion in risk premia. Interestingly, even though markets for innovation are complete, the risk price of ξ is still negative. This occurs due to an artifact of the stylized nature of our model. In particular, we assumed that households benchmark their consumption relative to the aggregate consumption in the economy, which includes the consumption of the non-participants. As we discussed in Section B.1 above, labor income acts as a hedge because it is homogenous and not subject to displacement. Because aggregate consumption and labor income respond with opposite signs to an embodied shock ξ, the marginal utility of market participants rises on impact, leading to a negative risk premium. Further, the restricted model with $\eta = 0$ generates a counterfactually low stock market participation rate, reflected in the low consumption share of shareholders (0.32 vs 0.43 in the data).

We conclude that incomplete markets is a crucial feature of our model, since the restricted version with $\eta = 0$ is inadequate in matching the data. However, as long as markets are incomplete, the model output is rather robust to imposing an upper bound of η. Specifically, in columns five and six of Table A.2, we present results when imposing an upper bound on η of 0.3 and 0.6, respectively. Imposing an upper bound on η has a minimal effect on the quantitative output of the model. In both cases, the model does a somewhat better job fitting both the level and the dispersion in risk premia—that is, the market portfolio and the value factor. The cost is somewhat higher utility curvature relative to the baseline model (γ of 83 and 72, respectively, vs 57 in the baseline model).

B.3 Restricting the weight on relative preferences

We next consider versions of the model in which we impose restrictions on the preference share of relative consumption h. In column seven, we show results from estimating a version without relative consumption preferences, $h = 0$. We see that this restricted version can only fit the equity premium, but not the value premium. To understand why, recall that the embodied shock ξ is primarily responsible in generating cross-sectional differences in risk premia across value and growth firms. The risk price on ξ largely depends on households' aversion to displacement risk, and hence h. Hence, we see that the risk price associated with ξ is about two-thirds smaller than in the baseline case.

In sum, preferences for relative consumption are an integral part of the model. However, the model can largely accommodate restrictions on the relative preference weight, h. To see this, in columns eight and nine, we present results using an upper bound of 0.3 and 0.6 on h. In this case, we see that imposing an upper bound on h has only a moderate impact on the model's ability to fit the value premium. Imposing an upper bound of 0.3 and 0.6 leads to a value premium of 4.1% and 5.2%, respectively, compared to 0.063 in the baseline model. The cost of imposing these restrictions on h is a somewhat higher equity premium than in the data (0.089 and 0.078, respectively).

B.4 Restricted model with no capital-embodied shocks

To further emphasize the role played by the embodied shock ξ, we next estimate a version of the model with only disembodied shocks. Column ten of Table A.2 shows that this restricted model can generate the equity premium, but not the value premium. This pattern reinforces the discussion in Section B, that the embodied shock is crucial in generating cross-sectional differences between value and growth firms.

B.5 Restricting the coefficient of relative risk aversion

Last, we also estimate a version of the model in which we impose an upper bound of 10 on the coefficient of relative risk aversion. As we see in the last column of Table A.2, the restricted model generates an equity premium of 3.9% and a value premium of 5.0%. Examining the risk prices of the two technology shocks, we see that the price of x is essentially zero; hence all risk premia are due to the embodied shock. To understand why this happens, notice that the parameter estimates imply that households place almost all of their preference weight on relative consumption, that is, $h = 0.99$. Doing so, magnifies the risk price of the embodied shock ξ, but at the cost of pushing the household's effective risk aversion towards purely aggregate shocks—the parameter γ_1 in the definition of the SDF in equation (A.102)— towards one. Hence, in this parametrization, households care almost exclusively about displacement risk, leading to a non-trivial risk price for ξ but a zero risk price for x.

B.6 Summary

In sum, these results highlight the importance of the three relatively non-standard features of the model: incomplete markets, preference for relative consumption, and embodied technology shocks. Eliminating any of these three assumptions compromises the model's ability to simultaneously fit both the level as well as the cross-section in risk premia. Specifically, cross-sectional differences in risk premia arise because of displacement. To fit the data, the model requires that there is sufficient displacement risk, and that households care enough about displacement. The model can largely accommodate a strong prior belief about an upper bound on η or h, as long as these features are not eliminated fully. Last, restricting the coefficient of relative risk aversion to lie below 10 still results in risk premia that are not too different than the data.

C Additional Results

This section is organized as follows. In Section C.1 we analyze the cashflow risk implied by the model. In Section C.2 we demonstrate by the market to book ratio is a good proxy for the firms' technology risk exposures. In Section C.3 we verify the results of the existing literature on income inequality, which concludes that most of the inequality is within cohorts (O'Rand and Henretta, 2000). In Section C.4 we show that, unlike many existing models, the model can generate a positive correlation between consumption corporate payout.

C.1 Cashflow risk and risk prices across horizons

Our model generates a sizable equity premium due to joint movements in aggregate dividends and the stochastic discount factor. Here, we briefly examine how risk at different horizons contributes to asset prices. To do so, we follow Borovička, Hansen, and Scheinkman (2014) and construct shock exposure

$$\varepsilon_q(T-t, X_t) = \eta(X_t)\frac{E_t(D_T \mathcal{M}_t \log D_T | X_t)}{E_t(D_T|X_t)} \tag{A.3}$$

and shock-price elasticities,

$$\varepsilon_p(T-t, X_t) = \eta(X_t)\frac{E_t(D_T \mathcal{M}_t \log D_T | X_t)}{E_t(D_T|X_t)} - \eta(X_t)\frac{E_t(\pi_T D_T(\mathcal{M}_t \log D_T + \mathcal{M}_t \log \pi_T)|X_t)}{E_t(\pi_T D_T|X_t)}. \tag{A.4}$$

Here, $\mathcal{M}_t \log D_T$ is a Malliavin derivative – that is, it measures the contribution of a shock dB_t to the stochastic process D at time $T > t$. Here, $\eta(X)$ indexes the direction and size of the shock. Expression (A.3) is very similar to a non-linear impulse response function; it examines the effect of a shock today to future values of D_t. Expression (A.4) represents the sensitivity of the expected log return associated with cashflow equal to D_T to a marginal increase in the exposure of that cashflow to a time-t shock. These marginal risk prices potentially vary across horizons based on how the shock D propagates.

We focus on two cashflow processes, the total payout to holders of the market portfolio,

$$D_t = \phi\, Y_t - I_t - \eta\, \lambda\, \nu_t, \tag{A.5}$$

and the payout accruing from assets in place at time t,

$$D_s^{ap} = p_{Z,s}\, e^{-\delta(s-t)} K_t. \tag{A.6}$$

Even though aggregate dividends D can potentially become negative, this does not happen near the mean of the stationary distribution of ω. It only happens at extreme ranges of the state space that were not reached across 1,000 model simulations. We compute shock-exposure ε_q and shock-price ε_p elasticities for the two processes D and D^{ap} at the mean of the stationary distribution of ω using Monte Carlo simulations. We plot the estimated elasticities in Figure A.2.

The first two columns show the impulse response ε_q of the two cashflow processes to a technology shock. A positive disembodied shock (x) increases dividends, both for the overall market and for assets in place. By contrast, improvements in technology that are embodied in new vintages (ξ) lead to a decline in aggregate dividends in the short term – as firms fund new investments – and an increase in the long run. By contrast, the cash flows accruing from installed capital falls due to competition – the equilibrium price p_Z falls as the economy acquires more and better new capital. The last two columns show the shock-price elasticities ε_p. We see that the marginal risk prices are essentially flat across horizons.

We conclude that the model's implications about the term structure of risk premia stem mostly from the dynamics of cash flows—the impulse response of dividends in column four of Figure 2. Specifically, we saw in Panel A of Figure 2 that the contribution of the dividend dynamics induced by x to the equity premium rises modestly with the horizon. Panel B implies the opposite pattern; the contribution of the dividend dynamics induced by ξ to the equity premium is concentrated in the short and medium run, and the rise in long-run dividends contributes negatively to the equity premium. Thus, the equity premium in the model is concentrated at shorter maturities and the term structure of dividend strip risk premia is downward sloping.

We can perform a similar exercise for the value premium. Specifically, in terms of assets in place, the risk due to x is somewhat higher in the short run, while the exposure to ξ-shocks is negative and increases in magnitude with maturity. These patterns imply that, in the model, the value premium is most pronounced at longer maturities.

C.2 Firm risk exposures and the market-to-book ratio

We next examine the extent to which Q is a useful summary statistic for firm risk and risk premia in our model. According to Lemma 4 below, firm's log market-to-book ratio can be written as

$$\log Q_{f,t} - \log Q_t = \log \left[\frac{V_t}{V_t + G_t} \left(1 + \frac{v_1(\omega_t)}{v(\omega_t)}\, (\bar{u}_{ft} - 1)\right) + \frac{G_t}{V_t + G_t} \frac{1}{k_{f,t}} \left(1 + \frac{g_1(\omega_t)}{g(\omega_t)}\, \left(\frac{\lambda_{f,t}}{\lambda} - 1\right)\right)\right]. \tag{A.7}$$

Examining (A.7), we note that a firm's market to book ratio is increasing in the likelihood of future growth λ_f, decreasing in the firm's relative size z_f, and increasing in the firm's current productivity $\bar{u}_f$. This latter effect prevents Tobin's Q from being an ideal measure of growth opportunities, since it is contaminated with the profitability of existing assets.

To examine the extent to which Q is correlated with technology risk exposures, we examine how changes in the firm's current state $(\lambda_{f,t}, k_{f,t}, \bar{u}_{f,t})$ jointly affect both firm Q and risk exposures. We plot the results in Figure A.3. In each of the three columns, we vary one of the elements of the firm's current state ($\lambda_{f,t}$, $z_{f,t}$, or $\bar{u}_{f,t}$) and keep the other two constant at their steady-state mean. On the horizontal axis, we plot the change in the firm's Q (relative to the market). On the vertical axis we plot the firm's return exposure to x and ξ (panels A and B, respectively) and the firm's risk premium (panel C). We scale the horizontal axis so that it covers the 0.5% and 99.5% of the steady-state distribution of the each of the firm's state variables. The resulting cross-sectional distribution of Tobin's Q is highly skewed.

Examining Figure A.3, we see that regardless of the source of the cross-sectional dispersion in Q, the relation between Q and risk exposures is positive. The pattern in the first two columns is consistent with the impulse responses in the previous section – small firms with high probability of acquiring future projects have higher technology risk exposures than large firms with low growth potential. The last column shows that increasing the firm's current productivity $\bar{u}_f$ – holding λ_f and k_f constant increases both Q and risk exposures. This pattern might seem puzzling initially, since increasing productivity $\bar{u}_f$ while holding size k_f and investment opportunities λ_f constant will lower λ_f/z_f. However, altering $\bar{u}_f$ also has a cashflow duration effect: due to mean reversion in productivity, profitable firms have lower cashflow duration – their cashflows are expected to mean-revert to a lower level. This lower duration of high $\bar{u}_f$ firms implies that their valuations are less sensitive to the rise in discount rates following a positive technology shock– see the response of the interest rate in the paper. In our calibration, this duration effect overcomes the effect due to λ_f/z_f, implying a somewhat more positive stock price response for high-$\bar{u}_f$ firms. However, the magnitude of this effect is quantitatively minor.

The last row of Figure A.3 shows how the firms' risk premium (unlevered) is related to cross-sectional differences in Q. Recall that the two technology shocks carry risk premia of the opposite sign. The disembodied shock carries a positive risk premium; in the absence of other technology shocks, this would imply that firms' risk premia rise with their market-to-book ratio. However, the fact that the embodied shock carries a negative risk premium – coupled with its higher volatility – implies a lower risk premium for growth firms relative to value firms. Households are willing to accept lower average returns for investing in growth firms because doing so allows them to partially hedge the displacement arising from the embodied shock ξ – the decline in their continuation utility.

C.3 The importance of within cohort inequality

An important distinction between our paper and prior work is our focus in within- as opposed to between cohort inequality (Garleanu, Kogan, and Panageas, 2012). The literature on income inequality largely agrees that the degree of within-cohort inequality is an order of magnitude greater than between-cohort inequality (O'Rand and Henretta, 2000). Also, Song, Price, Guvenen, Bloom, and von Wachter (2015) document that most of the rise in income inequality over the 1978 to 2012 period is within-cohorts.

Here, we show that similar patterns hold if we measure inequality in terms of income, consumption or wealth. To illustrate this point, we compute inequality measures for consumption (using the CEX) and income or wealth (using the SCF). We first remove observation year effects from consumption, income and wealth. We then report the resulting inequality moments before and after removing cohort fixed effects. We also do so separately for the subsample of households that own stocks (since the limited participation model is now the baseline, following your point 6 below). Here, we define cohort effects as the birth year of the leading member of the household.

We deal with the age-cohort-period identification problem in two ways. The first set of columns reports results without any age effects, so assigns the maximum amount of variation to a cohort effect. However, this may be problematic for wealth inequality, since wealth accumulation mechanically grows with age. Hence, the second column reports results after removing a cubic polynomial in age similar to Garleanu, Kogan, and Panageas (2012). Even then, the linear age effect is not identified, so we set it to zero in order to maximize the explanatory power of cohort effects.

Table A.5 shows that cohort fixed effects explain a quantitatively minor amount of inequality in either consumption, wealth or income inequality—especially at the top of the distribution and among stock market participants. Standard variance decomposition methods assign substantially less than 1% of the overall variation in log income, or wealth to cohort effects, and at most 3% of the overall dispersion in consumption to cohort fixed effects.

C.4 Consumption and Corporate Payout

Here, we discuss the relation between consumption and net corporate payout in the model. In addition, we compute the risk and return characteristics of unlevered claims on both. We report the results in Table A.6. Examining the table, there are several points worth discussing.

First, the volatility of payout growth is considerable in the model, much higher than consumption growth. As a result, an (unlevered) claim on corporate payout is considerably more risky than a claim to aggregate consumption, and therefore carries a higher risk premium. Further, consumption and net corporate payout (dividends) are positively correlated. These features are in contrast to most existing general equilibrium models, in which payout is often either negatively correlated with consumption or substantially less risky (see, e.g. Rouwenhorst, 1995; Kaltenbrunner and Lochstoer, 2010; Croce, 2014). In our model, dividends and consumption are positively correlated because they respond with the same sign to both technology shocks in the model. Recalling Figure 2, we see that a positive disembodied shock x leads to an increase in both consumption and aggregate payout. By contrast, a positive embodied shock ξ leads to a short-run drop in consumption and dividends as resources are diverted to investment. Second, the volatility of net payout growth is considerably larger than the volatility of the returns to the corporate sector. In the model, this happens because corporate payout has a mean-reverting component—following a positive shock to ξ, net corporate payout drops to finance investment but then rises as investment is transformed into productive capital.

These features of the model are consistent with the data. Specifically, Larrain and Yogo (2008) show that net corporate payout is substantially more volatile than returns. Lustig, Van Nieuwerburgh, and Verdelhan (2013) argue that a claim to aggregate consumption has a substantially lower risk premium—approximately one-third of the risk premium of equities.

D Data and Estimation Methodology

Here, we briefly discuss the construction of the estimation targets and our empirical methodology.

D.1 Measuring the value of new blueprints

The market value of new blueprints ν_t plays a key role in the model's predictions, both for the dynamics of firm cashflows (32) and for the evolution of investors' wealth (27). To take the model to the data, we use data on patents and stock returns to construct an empirical proxy for ν_t. Specifically, we view patents as empirical equivalents to the blueprints in our model. The Kogan, Papanikolaou, Seru, and Stoffman (2017) methodology allows us to assign a dollar value to each patent, that is based on the change in the dollar value of the firm around a three-day window after the market learns that the firm's patent application has been successful. To replicate this construction in simulated data, we employ an approximation that does not require the estimation of new parameters:

In particular, we create $\hat{\omega}$ using a non-parametric variant of the Kogan et al. (2017) procedure. First, we create idiosyncratic stock returns for firm f around the day that patent j is granted to equal the 3-day return of the firm minus the return on the CRSP value-weighted index around the same window,

$$r^e_{f,j} = r_{f,j} - r_{m,j}. \tag{A.8}$$

Patents are issued every Tuesday. Hence, r_{fj} are the accumulated return over Tuesday, Wednesday and Thursday following the patent issue. Second, we compute an estimate of the value of patent j as the firm's market capitalization on the day prior the patent announcement V_{fj} times the idiosyncratic return to the firm truncated at zero,

$$\hat{\nu}_j = \frac{1}{N_j} \max(r^e_{f,j}, 0)\, V_{f,j}. \tag{A.9}$$

If multiple patents were granted in the same day to the same firm, we divide by the number of patents N. Relative to Kogan et al. (2017), we replace the filtered value of the patent $E[x_{f,d}|r_{f,d}]$ with $\max(r^e_{f,j}, 0)$. Our construction is an approximation to the measure in Kogan et al. (2017) that can be easily implemented in simulated data without the additional estimation of parameters. See the earlier working paper version of Kogan et al. (2017) for a comparison between the two measures. We follow a similar approach when constructing $\hat{\nu}_j$ in simulated data. We compute the excess return of the firm as in equation (A.8) around the times that the firm acquires a new project, and then construct $\hat{\nu}_j$ as in equation (A.9).

We construct an estimate of the aggregate value of new blueprints ν at time t as

$$\hat{\nu}_t = \sum_{j \in P_t} \hat{\nu}_j, \tag{A.10}$$

where P_t denotes the set of patents granted to firms in our sample in year t. Similarly, we measure the total dollar value of innovation produced by a given firm f in year t by summing the estimated values for all patents ν_j that were granted to the firm during that year t,

$$\hat{\nu}_{f,t} = \sum_{j \in P_{f,t}} \nu_j, \tag{A.11}$$

where now $P_{f,t}$ denotes the set of patents issued to firm f in year t. In the context of our model, (A.11) can be interpreted as the sum of the net present values of all projects acquired by firm f in the interval $s \in [t-1, t]$,

$$\nu_{f,t} = \int_{t-1}^{t} \nu_s \, dN_{f,s}. \tag{A.12}$$

To obtain stationary ratios, we scale both (A.10) and (A.11) by the market capitalization M of the market portfolio and firm f, respectively. We label the corresponding ratios ν_t/M_t and $\nu_{f,t}/M_{f,t}$ as estimates of the (relative) value of innovation at the aggregate and firm level, respectively.

Further, we note that the dollar reaction around the issue date is an understatement of the dollar value of a patent. The market value of the firm is expected to change by an amount equal to the NPV of the patent times the probability that the patent application is unsuccessful. This probability is not small; in the data less than half of the patent applications are successful. Consequently, when comparing the dispersion in firm innovation between the data and the model, we scale the firm innovation measures $\nu_{f,t}/M_{f,t}$ in both the data and the model such that they have the same mean (equal to 1) conditional on non-zero values.

In the model, the ratio of the value of new blueprints ν to the value of the stock market M is a monotone function of the state variable ω, as we can see in Panel A of Figure A.1. Hence, we view the ratio ν/M as a useful proxy for the state variable ω in the model and define

$$\hat{\omega}_t \equiv \log\left(\frac{\hat{\nu}_t}{M_t}\right). \tag{A.13}$$

In Panel B, we plot $\hat{\omega}_t$ in the data. Similar to Kogan et al. (2017), this time-series lines up well with the three major waves of technological innovation in the U.S. – the 1930s, 1960s and early 1970s, and 1990s and 2000s.

When examining the model's implications for displacement, we also construct a direct analogue of $\hat{\omega}$ at the industry level (excluding the given firm) as

$$\hat{\omega}_{I\setminus f} = \frac{\sum_{f' \in I\setminus f} \nu_{f,t}}{\sum_{f' \in I\setminus f} M_{f',t}}. \tag{A.14}$$

where industry I is defined by the 3-digit SIC code and $I \setminus f$ denotes the set of all firms in industry I excluding firm f.

D.2 Construction of Estimation Targets

Aggregate consumption: We use the Barro and Ursua (2008) consumption data for the United States, which covers the 1834-2008 period. We compute the estimate of long-run risk using the estimator in Dew-Becker (2014). We thank Ian Dew-Becker for sharing his code.

Volatility of shareholder consumption growth: The volatility of shareholder consumption growth is from the unpublished working paper version of Malloy, Moskowitz, and Vissing-Jorgensen (2009) and includes their adjustment for measurement error. Data period is 1980-2002. We are grateful to Annette Vissing-Jorgensen for suggesting this.

Aggregate investment and output: Investment is non-residential private domestic investment. Output is gross domestic product. Both series are deflated by population and the CPI. Data on the CPI are from the BLS. Population is from the Census Bureau. Data range is 1927-2010.

Firm Investment rate, Tobin's Q and profitability: Firm investment is defined as the change in log gross PPE. Tobin's Q equals the market value of equity (CRSP December market cap) plus book value of preferred shares plus long term debt minus inventories and deferred taxes over book assets. Firm profitability equals gross profitability (sales minus costs of goods sold) scaled by book capital (PPE). When computing correlation coefficients, we winsorize the data by year at the 1% level to minimize the effect of outliers. We simulate the model at a weekly frequency, $dt = 1/50$ and time aggregate the data at the annual level. In the model, we construct Tobin's Q as the ratio of the market value of the firm divided by the replacement cost of capital using end of year values. Replacement cost is defined as the current capital stock adjusted for quality, $\hat{K}_{ft} = e^{-\xi_t} K_{ft}$. The investment rate is computed as $I_{ft}/\hat{K}_{ft-1}$, where I_{ft} is the sum of firm investment expenditures in year t and $\hat{K}_{f,t-1}$ is capital at the end of year $t-1$. Similarly, we compute firm profitability as $p_{Z,t}Z_{f,t}/\hat{K}_{f,t-1}$, where $p_{Z,t}Z_{f,t}$ is the accumulated profits in year t and $\hat{K}_{f,t-1}$ is capital at the end of year $t-1$. Data range is 1950-2010. Following standard practice, we exclude financials and utilities.

Market portfolio and risk-free rate moments: We use the reported estimate from the long sample of Barro and Ursua (2008) for the United States and cover the 1870-2008 sample (see Table 5 in their paper). In the data, the risk-free rate is the return on treasury bills of maturity of three months or less. The reported volatility of the interest rate in Barro and Ursua (2008), which equals 4.8%, is the volatility of the realized rate. Hence it is contaminated with unexpected inflation. We therefore target a risk-free rate volatility of 0.7% based on the standard deviation of the annualized yield of a 5-year Treasury Inflation Protected Security (the shortest maturity available) in the 2003-2010 sample. In the model, r_f is the instantaneous short rate; and R_M is the return on the value-weighted market portfolio.

Value factor, I/K, and E/P moments: We use the 10 value-weighted portfolios sorted on each of these characteristics from Kenneth French's Data Library. The value factor is the 10 minus 1 portfolio of firms sorted on book-to-market. The I/K and E/P spread are defined analogously. Data period for the value premium excludes data prior to the formation of the SEC (1936 to 2010); data period for the investment strategy (I/K) is 1964-2010; data period for the earnings-to-price strategy is 1952-2010. Standard errors for the empirical moments are included in parentheses. Standard errors for R^2 are computed using the delta method.

Consumption share of stockholders: Consumption share of stock holders is from Table 2 of Guvenen (2006). This number is also consistent with Heaton and Lucas (2000): using their data on Table AII we obtain an income share for stockholders of approximately 43%.

Consumption growth of shareholders: We use the series constructed in Malloy, Moskowitz, and Vissing-Jorgensen (2009), which covers the 1980-2002 period. We follow Jagannathan and Wang (2007) and construct annual consumption growth rates by using end-of-period consumption. In particular, we focus on the sample of households that are interviewed in December of every year, and use the average 8 quarter consumption growth rate of non-stockholders and stockholders, defined as in Malloy, Moskowitz, and Vissing-Jorgensen (2009).

Consumption measure of inequality: Mean/Median To estimate this measure in the data, we compute the difference between log aggregate per capita NIPA personal consumption expenditures and the logarithm of median consumption in the Consumer Expenditure Survey (CEX). To be consistent with the model, we estimate the median only using the set of households that are stockholders. The sample covers the 1982-2010 period.

Inequality Data Data sources are the Consumption Expenditure Survey (CEX), the Survey of Consumer Finances (SCF) and the data of Piketty and Saez (2003) and Saez and Zucman (2016). The top income and wealth shares are from Piketty and Saez (2003) and Saez and Zucman (2016). Top consumption shares are from CEX (1982-2010). Top shares are calculated relative to all households. The top percentile shares of income (total income) and wealth (net worth) are from the SCF (1989-2013); we report percentile ratios of the stock ownership sample (equity=1 in the SCF summary extracts) and after obtaining residuals from cohort and year dummies and cubic age effects. We use a similar procedure for the CEX data. The corresponding estimates in the model are computed from a long simulation of 10m households for 10,000 years. Percentile ratios are computed among the subset that participates in the stock market. In the model, income equals wages, payout and proceeds from innovations. In the data, we use the total income variable from the SCF, which includes salary, proceeds from owning a business and capital income. In the Piketty and Saez (2003) data, we use income shares inclusive of capital gains.

D.3 Estimation Methodology

The model has a total of 21 parameters. We calibrate two of these and estimate the rest. We choose the probability of household death as $\delta^h = 1/40$ to guarantee an average working life of 40 years. Second, we calibrate μ_I to match top income shares.We do so because the statistics reported in Table 1 in the paper are largely insensitive to the arrival rate of new blueprints μ_I. The reason why μ_I has a negligible impact on asset returns is risk aversion. As long as μ_I is small, and given moderate amounts of utility curvature, households make portfolio decisions by effectively ignoring the likelihood that they will themselves innovate—that is, they behave approximately as if $\mu_i = 0$.

To see this, note that the certainty equivalent of a bet that pays with probability μ_I an amount that is proportional to $1/\mu_I$ is negligible for small μ_I. We therefore calibrate the arrival rate to equal $\mu_I = 0.13\%$ so that, conditional on the other parameters, the model average top 1% income shares that are in line with the data. Including μ_I in the full estimation along with the inequality moment is computationally costly because of the number of simulations required to estimate inequality in the model with sufficient precision. Our algorithm therefore alternates between fixing μ_I and optimizing over all other parameters, while targeting all moments other than the income shares, and optimizing over μ_I with other parameters fixed, while targeting the top income shares.

Our estimation targets are reported in the first column of Table 1 in the paper. They include a combination of first and second moments of aggregate quantities, asset returns, and firm-specific moments. Due to data limitations, each of these statistics is available for different parts of the sample. We use the longest available sample to compute them. Many of the moments that we target are relatively standard in the literature. Others are less common, but they are revealing of the main mechanisms of our paper. We discuss them next.

First, the dynamic behavior of our model is summarized by the dynamics of the stationary variable ω_t. In the model, both the investment-to-output ratio (I_t/Y_t) as well as the relative value of new innovations (ν_t/M_t) are increasing functions of ω; therefore, the unconditional volatility of these ratios is informative about the model parameters. Similarly, fluctuations in ω lead to predictable movements in consumption growth. We therefore also include as a target an estimate of the long-run volatility of consumption growth using the methodology of Dew-Becker (2014) in addition to its short-term (annual) volatility. In contrast to short-term volatility, the long-run estimate takes into account the serial correlation in consumption growth, and is therefore revealing of the magnitudes of the fluctuations in ω.

Second, a large fraction of the parameters of the model affect the behavior of individual firms, specifically, the parameters governing the evolution of $\lambda_{f,t}$ and the persistence and dispersion of firm-specific shocks in (5). To estimate these parameters, we therefore include as targets the cross-sectional dispersion and persistence in firm investment, innovation, Tobin's Q, and profitability.

Third, our model connects embodied technology shocks to the return differential between value and growth firms. We thus include as estimation targets not only the first two moments of the market portfolio, but also the average value premium, defined as the difference in risk premia between firms in the bottom versus top decile in terms of their market-to-book ratios (Q), following Fama and French (1992). Given that the model has no debt, we create returns to equity by levering the value of corresponding dividend (payout) claims by a factor of 2.5.[3]

It is convenient to transform some of the parameters of the model. Specifically, we replace the volatility of the idiosyncratic shock with the variance of its ergodic distribution,

$$v \equiv \frac{\sigma_u^2}{2\,\kappa_u - \sigma_u^2}. \tag{A.15}$$

[3]This value lies between the estimates of the financial leverage of the corporate sector in Rauh and Sufi (2011) (which is equal to 2) and the values used in Abel (1999) and Bansal and Yaron (2004) (2.74-3).

Further, in place of the project arrival rate parameters $[\lambda_L, \lambda_H]$, we formulate the model in terms of the mean arrival rate,

$$\lambda \equiv \frac{\mu_L}{\mu_L + \mu_H}\lambda_L + \frac{\mu_H}{\mu_L + \mu_H}\lambda_H \tag{A.16}$$

and the relative difference,

$$\lambda_D \equiv \frac{\lambda_H - \lambda_L}{\lambda}. \tag{A.17}$$

We estimate the parameter vector p using the simulated minimum distance method (Ingram and Lee, 1991). Denote by X the vector of target statistics in the data and by $\mathcal{X}(p)$ the corresponding statistics generated by the model given parameters p, computed as

$$\mathcal{X}(p) = \frac{1}{S}\sum_{i=1}^{S} \hat{X}_i(p), \tag{A.18}$$

where $\hat{X}_i(p)$ is the 21×1 vector of statistics computed in one simulation of the model. We simulate the model at a weekly frequency, and time-aggregate the data to form annual observations. Each simulation has 1,000 firms. For each simulation i we first simulate 100 years of data as 'burn-in' to remove the dependence on initial values. We then use the remaining part of that sample, which is chosen to match the longest sample over which the target statistics are computed. Each of these statistic is computed using the same part of the sample as its empirical counterpart. In each iteration we simulate $S = 100$ samples, and simulate pseudo-random variables using the same seed in each iteration.

Our estimate of the parameter vector is given by

$$\hat{p} = \arg\min_{p\in\mathcal{P}}(X - \mathcal{X}(p))'W(X - \mathcal{X}(p)), \tag{A.19}$$

where $W = diag(XX')^{-1}$ is our choice of weighting matrix that ensures that the estimation method penalizes proportional deviations of the model statistics from their empirical counterparts.

We compute standard errors for the vector of parameter estimates $\hat{p}$ as

$$V(\hat{p}) = \left(1 + \frac{1}{S}\right)\left(\frac{\partial}{\partial p}\mathcal{X}(p)'W\frac{\partial}{\partial p}\mathcal{X}(p)\right)^{-1}\frac{\partial}{\partial p}\mathcal{X}(p)'W'V_X(\hat{p})W\frac{\partial}{\partial p}\mathcal{X}(p)\left(\frac{\partial}{\partial p}\mathcal{X}(p)'W\frac{\partial}{\partial p}\mathcal{X}(p)\right)^{-1}, \tag{A.20}$$

where

$$V_X(\hat{p}) = \frac{1}{S}\sum_{i=1}^{S}(\hat{X}_i(\hat{p}) - \mathcal{X}(\hat{p}))(\hat{X}_i(\hat{p}) - \mathcal{X}(\hat{p}))' \tag{A.21}$$

is the estimate of the sampling variation of the statistics in X computed across simulations.

The standard errors in (A.20) are computed using the sampling variation of the target statistics across simulations (A.21). We use (A.21), rather than the sample covariance matrix, because the statistics that we target are obtained from different datasets (e.g. cross-sectional versus time-series), and we often do not have access to the underlying data. Since not all of these statistics are moments, computing the covariance matrix of these estimates would be challenging even with access to the underlying data. Under the null of the model, the estimate in (A.21) would coincide with the empirical estimate. If the model is misspecified, (A.21) does not need to be a good estimate of the true covariance matrix of X. Partly for these reasons, we specify the weighting matrix as $W = diag(XX')^{-1}$, rather than scaling by the inverse of the sample covariance matrix of X. In principle we could weigh moments by the inverse of (A.21). However, doing so forces the model to match moments that are precisely estimated but economically less interesting, such as the dispersion in firm profitability or Tobin's Q.

Solving each iteration of the model is computationally costly, and thus computing the minimum (A.19) using standard methods is infeasible. We therefore use the Radial Basis Function (RBF) algorithm in Björkman and Holmström (2000). The Björkman and Holmström (2000) algorithm first fits a response surface to data by evaluating the objective function at a few points. Then, it searches for a minimum by balancing between local and global search in an iterative fashion. We use a commercial implementation of the RBF algorithm that is available through the TOMLAB optimization package. The RBF algorithm searches for an approximate minimum over a rectangular set. Table A.1 reports the bounds of this set. We confirm that the estimated parameters lie in the interior of the set.

E Analytic Appendix

To conserve space, we provide the solution for the extended model with limited participation. We begin by describing the differences between the extended model and our baseline model. The solution to the baseline model is a special case in which $q_S = 1$.

E.1 Limited Participation

There are two types of households in our economy, *workers* and *shareholders*. There is continuum of each type, with the total measure of households normalized to one. We denote the set of workers by $\mathcal{W}_t$ and the set of shareholders by $\mathcal{S}_t$

Workers

Workers in this economy are hand-to-mouth consumers. They do not participate in financial markets, supply labor inelastically and consume their labor income as it arrives. When a new household is born, it becomes a worker, independently of all other households and all other sources of randomness in the economy, with probability $1 - q_S$. Each worker receives a measure $\lambda(1-\psi)/\mu_I$ of blueprints upon innovating.

Shareholders

A newly born household becomes a shareholder with probability q_S. Workers that successfully innovate (see below) also become shareholders. Shareholders have access to financial markets, and optimize their life-time utility of consumption. Shareholders are not subject to liquidity constraints. In particular, shareholders sell their future labor income streams and invest the proceeds in financial claims. All shareholders have the same preferences, given by (13)-(14). Each shareholder i receives a measure of projects in proportion to her wealth $W_{i,t}$ relative to shareholders as a group, specifically $\lambda\,\psi\,\mu_I^{-1}\,W_{i,t}\,\left(\int_{j\in\mathcal{S}_t} W_{j,t}\,dj\right)^{-1}$. Here,

$$\psi = \frac{\mu_I + q_S\,\delta^h}{\mu_I + \delta^h} \tag{A.22}$$

is the steady-state fraction of households that participate in financial markets. The case $q_S = 1$ (or equivalently, $\psi = 1$) corresponds to our baseline model.

E.2 Proofs and Derivations

First, it is useful to establish the following lemma.

Lemma 1 (Stationary distribution for u) *The process u, defined as*

$$du_t = \kappa_u(1-u_t)\,dt + \sigma_u\,u_t\,dB_t^u \tag{A.23}$$

has a stationary distribution given by

$$f(u) = cu^{-2-\frac{2\,\kappa_u}{\sigma_u^2}}\exp\left(-\frac{2\theta}{u\sigma_u^2}\right), \tag{A.24}$$

where c is a constant that solves $\int_0^\infty f(u)du = 1$. Further, as long as $2\,\kappa_u \geq \sigma_u^2$, the cross-sectional variance of u is finite.

Proof. We follow (Karlin and Taylor, 1981, p. 221). The forward Kolmogorov equation for the stationary transition density $f(u)$ yields the ODE

$$0 = -\kappa_u\frac{\partial}{\partial u}\left[(1-u)f(u)\right] + \frac{1}{2}\sigma_u^2\,\frac{\partial^2}{\partial u^2}\left[u^2 f(u)\right] \tag{A.25}$$

Integrating the above with respect to u yields

$$k = -\kappa_u\left[(1-u)f(u)\right] + \frac{1}{2}\sigma_u^2\,\frac{\partial}{\partial u}\left[u^2 f(u)\right] \tag{A.26}$$

where k is a constant of integration. We set $k=0$ and find

$$f(u) = cu^{-2-\frac{2\,\kappa_u}{\sigma_u^2}}\exp\left(-\frac{2\theta}{u\sigma_u^2}\right), \tag{A.27}$$

where c is an unknown constant. By construction, the function f is positive. Further, setting the constant c to

$$\left(\int_0^\infty u^{-2-\frac{2\,\kappa_u}{\sigma_u^2}}\exp\left(-\frac{2\theta}{u\sigma_u^2}\right)du\right)^{-1} \tag{A.28}$$

(the above integral is finite as long as $\kappa_u > 0$) guarantees that $\int_0^\infty f(u)\,du = 1$, and therefore $f(u)$ is the stationary density of the diffusion process u.

The last part of the proof is to show that the variance of u is finite and positive as long as $2\,\kappa_u - \sigma_u^2 > 0$. Given the solution for c,

$$\int_0^\infty (u-1)^2\, c\, u^{-2-\frac{2\,\kappa_u}{\sigma_u^2}} \exp\left(-\frac{2\theta}{u\sigma_u^2}\right) du = \frac{\sigma_u^2}{2\,\kappa_u - \sigma_u^2}, \tag{A.29}$$

which is finite as long as $2\,\kappa_u - \sigma_u^2 > 0$. ■

Before proving the Propositions in the main text, we establish some preliminary results. First, we show how to relate the stochastic discount factor (SDF) to the value function of an investor. This is a straightforward application of the results in Duffie and Skiadas (1994) on the relation between the utility gradient and the equilibrium SDFs.

We focus on a single household and omit the household index. To simplify exposition, we present the result in a slightly more general form, not limiting it to the exact structure of our economy. As in our model, the household is solving a consumption-portfolio choice problem with one non-standard element: it receives a stochastic stream of gains from innovation in proportion to its financial wealth. Let W_t denote the household's wealth.

The market consists of I financial assets that pay no dividends. Let S_t denote the vector of prices of the financial assets. S_t is an Ito process

$$dS_t = \mu_t\, dt + \sigma_t\, dB_t. \tag{A.30}$$

The first asset is risk-free, its price growth at the equilibrium rate of interest r_t. Let $\mathcal{F}$ denote the natural filtration generated by the Brownian motion vector B_t.

The investor receives a flow of income from innovation projects according to an exogenous Poisson process N with the arrival rate λ. The process N is independent of the Brownian motion B. We assume that conditional on innovating, household's wealth increases by a factor of $\exp(\varrho_t)$, where the process ϱ is adapted to the filtration $\mathcal{F}$. The consumption process of the household, C, and its portfolio vector θ, are adapted to the filtration generated jointly by the exogenous processes N and B.

As in our model, the investor maximizes the stochastic differential utility function given by equations (13-14) in the main text, where we take the process $\overline{C}_t$ to be a general Ito process adapted to the filtration $\mathcal{F}$, subject to the dynamic budget constraint

$$dW_t = \delta W_t\, dt - C_t\, dt + (e^{\varrho_t} - 1)\, W_t\, dN_t + \theta_t dS_t, \quad W_t = \theta_t S_t, \tag{A.31}$$

and a credit constraint, which rules out doubling strategies and asymptotic Ponzi schemes:

$$W_t \geq 0. \tag{A.32}$$

Note that the first term in (A.31) captures the flow of income from annuities that the household collects conditional on its continued survival. The death process is a Poisson process with arrival rate δ, which is independent of N and B. We are now ready to define an SDF in relation to the value function of the household. In particular, we construct an SDF process that is adapted to the filtration $\mathcal{F}$, and hence does not depend on the household-specific innovation arrival process N.

Lemma 2 (SDF) *Let $C_t^\star$, $\theta_t^\star$, and $W_t^\star$ denote the optimal consumption strategy, portfolio policy, and the wealth process of the household respectively. Let $J_t^\star$ denote the value function under the optimal policy. Define the process Λ_t as*

$$\Lambda_t = \exp\left(\int_0^t \delta + \frac{\partial\phi(C_s^\star, J_s^\star; \overline{C}_s)}{\partial J_s^\star} + \lambda\left(e^{(1-\gamma)\varrho_s} - 1\right) ds\right) A_t, \tag{A.33}$$

where

$$A_t = \frac{\partial\phi(C_t^\star, J_t^\star; \overline{C}_t)}{\partial C_t^\star} \exp\left(\int_0^t \gamma\varrho_s \, dN_s\right). \tag{A.34}$$

Then Λ_t is a stochastic discount factor consistent with the price process S and adapted to filtration $\mathcal{F}$.

Proof.

Let $\mathcal{M}$ denote the market under consideration, and define a fictitious market $\hat{\mathcal{M}}$ as follows. $\hat{\mathcal{M}}$ has the same information structure as $\mathcal{M}$, with modified price processes for financial assets. Specifically, let

$$R_t = \exp\left(\int_0^t \delta \, ds + \varrho_s \, dN_s\right) \tag{A.35}$$

and define price processes in the market $\hat{\mathcal{M}}$ as

$$\hat{S}_t = R_t S_t. \tag{A.36}$$

The budget constraint in the market $\hat{\mathcal{M}}$ is standard,

$$d\hat{W}_t = -C_t \, dt + \hat{\theta}_t d\hat{S}_t, \quad \hat{W}_t = \hat{\theta}_t \hat{S}_t. \tag{A.37}$$

If a consumption process $\{C\}$ can be financed by a portfolio policy θ in the original market $\mathcal{M}$, it can be financed by the policy $R^{-1}\theta$ in the fictitious market $\hat{\theta} = R^{-1}\theta$, and vice versa. Thus, the set of feasible consumption processes is the same in the two markets, and therefore the optimal consumption processes are also the same. Since the consumption-portfolio choice problem in the fictitious market is standard, according to (Duffie and Skiadas, 1994, Theorem 2), the utility gradient of the agent at the optimal consumption policy defines a valid SDF process $\hat{\Lambda}_t$,

$$\hat{\Lambda}_t = \exp\left(\int_0^t \frac{\partial\phi(C_s^\star, J_s^\star; \overline{C}_s)}{\partial J_s^\star} \, ds\right) \frac{\partial\phi(C_t^\star, J_t^\star; \overline{C}_t)}{\partial C_t^\star}. \tag{A.38}$$

Thus, for all $t < T$,

$$\hat{\Lambda}_t R_t S_t = \hat{\Lambda}_t \hat{S}_t = \mathrm{E}_t\left[\hat{\Lambda}_T \hat{S}_T\right] = \mathrm{E}_t\left[\hat{\Lambda}_T R_T S_T\right] \tag{A.39}$$

and therefore $\Lambda'_t = \hat{\Lambda}_t R_t$ is a valid SDF in the original market $\mathcal{M}$. Note that Λ'_t is not adapted to the filtration $\mathcal{F}$, since it depends on the agent's innovation process N. In other words, Λ'_t is an agent-specific SDF process.

The last remaining step is to show that the process Λ_t is adapted to the filtration $\mathcal{F}$ and a valid SDF. First, we show that the process $\exp\left(\int_0^t \partial\phi(C_s^\star, J_s^\star; \overline{C}_s)/\partial J_s^\star\right)$ is adapted to $\mathcal{F}$, and the process $\partial\phi(C_t^\star, J_t^\star; \overline{C}_t)/\partial C_t^\star$ can be decomposed as $A_t R_t^{-\gamma}$, where A_t is also adapted to $\mathcal{F}$. Given the homotheticity of the stochastic differential utility function and the budget constraint (A.31), standard arguments show that the agent's value function and the optimal consumption policy can be expressed as

$$J_t^\star = (W_t^\star)^{1-\gamma}\,\varrho_{J,t}, \tag{A.40}$$

where $\varrho_{J,t}$ is a stochastic process adapted to $\mathcal{F}$. The optimal wealth and consumption processes then take form

$$W_t^\star = R_t\varrho_{W,t}, \quad C_t^\star = R_t\varrho_{C,t}, \tag{A.41}$$

where $\varrho_{W,t}$ and $\varrho_{C,t}$ are adapted to $\mathcal{F}$, and therefore

$$J_t^\star = R_t^{1-\gamma}\varrho_{W,t}^{1-\gamma}\varrho_{J,t}. \tag{A.42}$$

We next use these expressions to evaluate the partial derivatives of the aggregator ϕ:

$$\frac{\partial\phi(C_t^\star, J_t^\star; \overline{C}_t)}{\partial J_t^\star} = -\frac{\rho(1-\gamma)}{1-\theta^{-1}} - \frac{\rho(1-\gamma)^{\frac{1-\theta^{-1}}{\gamma-1}}(\gamma-\theta^{-1})}{1-\theta^{-1}}\left(\varrho_{C,t}^{1-h}\left(\varrho_{C,t}/\overline{C}_t\right)^h\right)^{1-\theta^{-1}}\left(\varrho_{W,t}^{1-\gamma}\varrho_{J,t}\right)^{\frac{1-\theta^{-1}}{\gamma-1}}, \tag{A.43}$$

which is adapted to $\mathcal{F}$; and

$$\frac{\partial\phi(C_t^\star, J_t^\star; \overline{C}_t)}{\partial C_t^\star} = \rho(1-\gamma)^{\frac{1-\theta^{-1}}{\gamma-1}}\overline{C}_t^{h(1-\theta^{-1})}\varrho_{C,t}^{-\theta^{-1}}\left(\varrho_{W,t}^{1-\gamma}\varrho_{J,t}\right)^{\frac{1-\theta^{-1}}{\gamma-1}}R_t^{-\gamma}. \tag{A.44}$$

Thus, the process

$$A_t = \frac{\partial\phi(C_t^\star, J_t^\star; \overline{C}_t)}{\partial C_t^\star}e^{-\gamma\delta t}R_t^{\gamma} \tag{A.45}$$

is adapted to $\mathcal{F}$. Based on the above results, we express Λ'_t as

$$\Lambda'_t = \exp\left(\int_0^t \frac{\partial\phi(C_s, J_s; \overline{C}_s)}{\partial J_s} + \gamma\delta\, ds\right) A_t R_t^{1-\gamma} \tag{A.46}$$

Define $\Lambda_t = \mathrm{E}[\Lambda'_t|\mathcal{F}_t]$. Since all asset price processes in the original market are adapted to $\mathcal{F}$, Λ_t is also a valid SDF process. Using the equality (see below)

$$\mathrm{E}\left[R_t^{1-\gamma}|\mathcal{F}_t\right] = \exp\left(\int_0^t \delta(1-\gamma) + \lambda\left(e^{(1-\gamma)\varrho_s} - 1\right)\, ds\right), \tag{A.47}$$

we find

$$\Lambda_t = \exp\left(\int_0^t \frac{\partial\phi(C_s^\star, J_s^\star; \overline{C}_s)}{\partial J_s^\star} + \delta + \lambda\left(e^{(1-\gamma)\varrho_s} - 1\right)\, ds\right) A_t. \tag{A.48}$$

To complete the proof, we show that

$$\mathrm{E}\left[(R_t)^{1-\gamma}|\mathcal{F}_t\right] = \exp\left(\int_0^t \delta(1-\gamma) + \lambda\left(e^{(1-\gamma)\varrho_s} - 1\right)\, ds\right). \tag{A.49}$$

Fix the path of ϱ_s and consider only the uncertainty associated with Poisson process N. Define

$$M_t = \exp\left(\int_0^t \varrho_s(1-\gamma)\, dN_s - \int_0^t \lambda\left(e^{(1-\gamma)\varrho_s} - 1\right)\, ds\right). \tag{A.50}$$

Then

$$dM_t = -M_t\lambda\left(e^{(1-\gamma)\varrho_s} - 1\right)\, dt + \left(e^{(1-\gamma)\varrho_s} - 1\right) M_t\, dN_t, \tag{A.51}$$

and therefore

$$\mathrm{E}\left[dM_t|N_s,\, s \le t\right] = 0 \tag{A.52}$$

and M_t is a martingale. So, $\mathrm{E}\left[M_t|\mathcal{F}_t\right] = 1$. ■

Next, we consider some of the equilibrium relations in order to gain intuition for the overall structure of the solution. Define

$$\zeta_{j,t} = u_{j,t}\, e^{\xi_{\tau(j)}}\, K_{j,t}. \tag{A.53}$$

and

$$Z_t = \int_{J_t} e^{\xi_{s(j)}}\, u_{j,t}\, K_{j,t}\, dj. \tag{A.54}$$

The labor hiring decision is static. The firm managing project j chooses L_{jt} as the solution to

$$\Pi_{j,t} = \sup_{L_{j,t}} \left[\zeta_{j,t}^{\phi}\, \left(e^{x_t} L_{j,t}\right)^{1-\phi} - \mathrm{w}_t\, L_{j,t}\right] \tag{A.55}$$

The firm's choice

$$L^{\star}_{j,t} = \zeta_{j,t}\, \left(\frac{(1-\phi)\, e^{(1-\phi)x_t}}{\mathrm{w}_t}\right)^{\frac{1}{\phi}}. \tag{A.56}$$

After clearing the labor market, $\int_{\mathcal{J}_t} L_{j,t} dj = 1$, the equilibrium wage is given by

$$\mathrm{w}_t = (1-\phi)\, e^{(1-\phi)\, x_t}\, Z_t^{\phi}, \tag{A.57}$$

and the choice of labor allocated to project j is

$$L^{\star}_{j,t} = \zeta_{j,t}\, Z_t^{-1}. \tag{A.58}$$

Aggregate output of all projects equals

$$Y_t = \int_{J_t} \zeta_{j,t}\, e^{(1-\phi)\, x_t}\, Z_t^{\phi-1}\, dj = e^{(1-\phi)\, x_t}\, Z_t^{\phi}. \tag{A.59}$$

Here, note that the definition of aggregate output, along with other aggregate quantities in the model, requires aggregating firm-level quantities. Aggregation over the continuum of firms should satisfy a law of large numbers, canceling out firm-specific randomness. Several aggregation procedures with such property have been developed in the literature, and the exact choice of the aggregation procedure is not important for our purposes. Specifically, we follow Uhlig (1996) and define the aggregate as the Pettis integral. We denote the aggregate over firms by an integral over the set of firms, $\in_0^1 \cdot df$. For alternative constructions that deliver the law of large numbers in the cross-section, see for instance Sun (2006) and Podczeck (2010), as well as the discussion of similar issues in Constantinides and Duffie (1996).

The project's flow profits are

$$\Pi_{j,t} = \sup_{L_{j,t}} \left[\zeta_{j,t}^{\phi} \, (e^{x_t} L_{j,t})^{1-\phi} - \mathrm{w}_t \, l_{j,t} \right] = p_t \, \zeta_{j,t} \tag{A.60}$$

where

$$p_t = \phi \, Y_t \, Z_t^{-1} \tag{A.61}$$

Because firms' investment decisions do not affect its own future investment opportunities, each investment maximizes the net present value of cash flows from the new project. Thus, the optimal investment in a new project j at time t is the solution to

$$\sup_{K_{j,t}} \mathrm{E}_t \left[\int_t^{\infty} \frac{\Lambda_s}{\Lambda_t} \Pi_{j,s} \, ds \right] - K_{j,t}^{1/\alpha} = \sup_{K_{j,t}} \left[P_t \, K_{j,t} e^{\xi_t} - K_{j,t}^{1/\alpha} \right], \tag{A.62}$$

where P_t is the time-t price of the asset with the cash flow stream $\exp(-\delta(s-t))p_s$:

$$P_t = \mathrm{E}_t \left[\int_t^{\infty} \frac{\Lambda_s}{\Lambda_t} e^{-\delta(s-t)} \, p_s \, ds \right]. \tag{A.63}$$

The optimal scale of each new project is then given by

$$k_t^{\star} = \left(\alpha \, e^{\xi_t} \, P_t \right)^{\frac{\alpha}{1-\alpha}}. \tag{A.64}$$

Note that the solution does not depend on the identity of the firm, i.e., all firms, faced with an investment decision at time t, choose the same scale for the new projects. The optimal investment scale depends on the current market conditions, specifically, on the current level of the embodied productivity process ξ_t, and the current price level P_t.

We thus find that the aggregate stock of quality-adjusted installed capital in the intermediate good sector, defined by (7), evolves according to

$$dK_t = \left(-\delta \, K_t + \lambda \, e^{\xi_t} \, k_t^{\star} \right) \, dt = \left(-\delta \, K_t + \lambda \, e^{\xi_t} \, \left(\alpha \, e^{\xi_t} \, P_t \right)^{\frac{\alpha}{1-\alpha}} \right) \, dt. \tag{A.65}$$

An important aspect of (A.65) is that the growth rate of the capital stock K_t depends only on its current level, the productivity level ξ_t, and the price process P_t. Furthermore, as we show below, we can clear markets with the price process P_t expressed as a function of the state vector

$X_t = (x_t, \xi_t, K_t)$. Thus, X_t follows a Markov process in equilibrium.

We express equilibrium processes for aggregate quantities and prices as functions of X_t. For instance, the fact that investment decisions are independent of u implies that $Z_t = K_t$. Aggregate investment I_t is given by

$$I_t = \lambda \,(k_t^\star)^{1/\alpha}\,, \tag{A.66}$$

The aggregate consumption process satisfies

$$C_t = Y_t - I_t = K_t^\phi e^{(1-\phi)x_t} - \lambda\,(k_t^\star)^{1/\alpha}\,. \tag{A.67}$$

Prices of long-lived financial assets, such as the aggregate stock market, depend on the behavior of the stochastic discount factor. In equilibrium, the SDF is determined jointly with the value function of the households, as shown in Lemma 2. Below we fully characterize the equilibrium dynamics and express Λ_t as a function of X_t.

Define the two variables

$$\chi_t = \frac{1-\phi}{1-\alpha\,\phi}\,x_t + \frac{\phi}{1-\alpha\,\phi}\xi_t. \tag{A.68}$$

and

$$\omega_t = \Big(\xi_t + \alpha\,\chi_t - \,\log K_t\Big) \tag{A.69}$$

ω_t and χ_t are linear functions of the state vector X_t. In Lemma 3 below, we characterize the SDF and aggregate equilibrium quantities as functions of ω_t and χ_t.

In the formulation of the lemma, we characterize the value function of a household, as well as prices of financial assets, such as P_t in (A.63), using differential equations. Verification results, such as (Duffie and Lions, 1992, Sec. 4), show that a classical solution to the corresponding differential equation, subject to the suitable growth and integrability constraints, characterizes the value function. Similarly, the Feynman-Kac Theorem (Karatzas and Shreve, 1991, e.g, Theorem 7.6) provides an analogous result for the prices of various financial assets. Because we solve for equilibrium numerically, we cannot show that the classical solutions to our differential equations exist and satisfy the sufficient regularity conditions. With this caveat in mind, in the following lemma we characterize the equilibrium processes using the requisite differential equations.

Lemma 3 (Equilibrium) *Let the seven functions, $f(\omega)$, $s(\omega)$, $\kappa(\omega)$, $i(\omega)$, $v(\omega)$, $g(\omega)$, $h(\omega)$ solve the following system of four ordinary differential equations,*

$$\begin{aligned} 0 =& A_1(\omega) f(\omega)^{\frac{\gamma-\theta^{-1}}{\gamma-1}} + f(\omega)\left\{c_0^f - (1-\gamma)\,s(\omega) + \left[\left(1+\frac{\psi}{\mu_I}\,s(\omega)\right)^{1-\gamma} - 1\right]\mu_I\right\} \\ &+ f'(\omega)\left\{c_1^f - (1-\alpha\,\phi)\,\kappa(\omega)\right\} + f''(\omega)\,c_2^f, \\ 0 =& \,\phi\,e^{-\phi\omega}B(\omega) + v'(\omega)\left\{c_1^f - (1-\alpha\,\phi)\,\kappa(\omega)\right\} + v''(\omega)\,c_2^f, \end{aligned} \tag{A.70}$$

$$+ v(\omega)\Bigg\{c_0^f - \frac{\gamma-\theta^{-1}}{1-\gamma}A_1(\omega)f(\omega)^{\frac{1-\theta^{-1}}{\gamma-1}} + \mu_I\left(\left(1+\frac{\psi}{\mu_I}s(\omega)\right)^{1-\gamma} - 1\right) + \gamma\, s(\omega) - \kappa(\omega)\Bigg\}, \tag{A.71}$$

$$0 = (1-\eta)\,(1-\alpha)\,v(\omega)\,\kappa(\omega) + g'(\omega)\Bigg\{c_1^f - (1-\alpha\,\phi)\,\kappa(\omega)\Bigg\} + g''(\omega)\,c_2^f$$

$$+ g(\omega)\Bigg\{c_0^f - \frac{\gamma-\theta^{-1}}{1-\gamma}A_1(\omega)f(\omega)^{\frac{1-\theta^{-1}}{\gamma-1}} + \mu_I\left(\left(1+\frac{\psi}{\mu_I}s(\omega)\right)^{1-\gamma} - 1\right) + \gamma\, s(\omega)\Bigg\}, \tag{A.72}$$

$$0 = (1-\phi)\,e^{-\phi\omega}B(\omega) + h'(\omega)\Bigg\{c_1^f - (1-\alpha\,\phi)\,\kappa(\omega)\Bigg\} + h''(\omega)\,c_2^f$$

$$+ h(\omega)\Bigg\{c_0^f - \delta^h - \frac{\gamma-\theta^{-1}}{1-\gamma}A_1(\omega)f(\omega_t)^{\frac{1-\theta^{-1}}{\gamma-1}} + \mu_I\left(\left(1+\frac{\psi}{\mu_I}s(\omega_t)\right)^{1-\gamma} - 1\right) + \gamma\, s(\omega)\Bigg\}, \tag{A.73}$$

and three algebraic equations,

$$s(\omega) = \frac{\eta\,(1-\alpha)\,v(\omega)\,\kappa(\omega)}{v(\omega)+g(\omega)+\psi\,h(\omega)}, \tag{A.74}$$

$$\kappa(\omega) = \lambda^{1-\alpha}\,e^{(1-\alpha\phi)\,\omega}[i(\omega)]^{\alpha}, \tag{A.75}$$

$$\left(\frac{i(\omega)}{\lambda}\right)^{1-\alpha} = \alpha e^{(1-\alpha\phi+\phi(1-\theta_1^{-1}))\omega}\,v(\omega)\,f(\omega)^{\frac{\gamma-\theta^{-1}}{1-\gamma}}\,\left[(1-i(\omega))\right]^{\theta_1^{-1}}\left(1-\frac{(1-\psi)(1-\phi)}{1-i(\omega)}\right)^{1/\theta}. \tag{A.76}$$

The constants c_0^f, c_1^f, c_2^f and ϕ^d are

$$c_0^f = \Bigg\{\delta^h(1-\gamma) - \frac{\rho(1-\gamma)}{1-\theta^{-1}} + (1-\gamma_1)(1-\phi)\mu_x + \frac{1}{2}(1-\phi)^2\,\sigma_x^2\,(1-\gamma_1)^2$$

$$+ \frac{1}{2}\left(\frac{\phi(1-\gamma_1)}{1-\alpha\,\phi}\right)^2\left(\sigma_\xi^2+\alpha^2(1-\phi)^2\sigma_x^2\right)$$

$$+ \frac{\phi(1-\gamma_1)}{1-\alpha\,\phi}\left(\mu_\xi + \alpha\,(1-\phi)\,\mu_x + (1-\gamma_1)\,\alpha(1-\phi)^2\,\sigma_x^2\right)\Bigg\}, \tag{A.77}$$

$$c_1^f = \Bigg\{\mu_\xi + \alpha\,(1-\phi)\,\mu_x + (1-\alpha\,\phi)\delta + (1-\gamma_1)\,\alpha(1-\phi)^2\,\sigma_x^2 + \frac{\phi(1-\gamma_1)}{1-\alpha\,\phi}\left(\sigma_\xi^2+\alpha^2(1-\phi)^2\sigma_x^2\right)\Bigg\}, \tag{A.78}$$

$$c_2^f = \frac{1}{2}\left(\sigma_\xi^2+\alpha^2(1-\phi)^2\sigma_x^2\right) \tag{A.79}$$

and the functions $A_1(\omega)$ and $B(\omega)$ are defined as

$$A_1(\omega) = \frac{\rho(1-\gamma)}{1-\theta^{-1}}\,\left[(1-i(\omega))\right]^{1-\theta_1^{-1}}\left(1-\frac{(1-\psi)(1-\phi)}{1-i(\omega)}\right)^{1-\theta^{-1}}\,e^{-\phi(1-\theta_1^{-1})\,\omega}, \tag{A.80}$$

$$B(\omega) = [(1 - i(\omega))]^{-\theta_1^{-1}} \left(1 - \frac{(1-\psi)(1-\phi)}{1 - i(\omega)}\right)^{-1/\theta} f(\omega)^{\frac{\gamma-\theta^{-1}}{\gamma-1}} e^{\phi\,\theta_1^{-1}\,\omega}. \tag{A.81}$$

Then we can construct price processes and individual policies that satisfy the definition 1, so that the value function of a shareholder household n with relative wealth $W_n/W = w_n$ is given by

$$J(w_n, \chi, \omega) = \frac{1}{1-\gamma} w_n^{(1-\gamma)} e^{(1-\gamma_1)\chi} f(\omega), \tag{A.82}$$

where $\gamma_1 = 1 - (1-\gamma)(1-h)$, and K_t follows

$$\frac{dK_t}{K_t} = -\delta\, dt + \kappa(\omega_t)\, dt. \tag{A.83}$$

Proof. We start with a conjecture, which we confirm below, that the equilibrium price process P_t satisfies

$$P_t = K_t^{-1} e^{\chi_t}\, v(\omega_t) B(\omega_t)^{-1}, \tag{A.84}$$

the equilibrium aggregate value of assets in place is

$$V_t = e^{\chi_t}\, v(\omega_t) B(\omega_t)^{-1}, \tag{A.85}$$

the value of growth opportunities for the average firm ($\lambda_f = \lambda$) is

$$G_t = e^{\chi_t} g(\omega_t) \left(B(\omega_t)\right)^{-1}, \tag{A.86}$$

and the aggregate value of human capital is

$$H_t = e^{\chi_t} h(\omega_t) \left(B(\omega_t)\right)^{-1}. \tag{A.87}$$

We then characterize the equilibrium SDF and the optimal policies of the firms and households, and show that all markets clear and the above conjectures are consistent with the equilibrium processes for cash flows and the SDF.

We denote the time-t net present value of the new projects (the maximum value in (A.62)) by ν_t. According to equations (A.64, A.66) above,

$$\nu_t = \left(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\right) \left(P_t e^{\xi_t}\right)^{1/(1-\alpha)} = \alpha^{\frac{1}{1-\alpha}} \left(\frac{1}{\alpha} - 1\right) \left(e^{\xi_t}\, P_t\right)^{\frac{1}{1-\alpha}}. \tag{A.88}$$

The aggregate investment process, according to (A.64, A.66), is given by

$$I_t = \lambda \left(\alpha\, e^{\xi_t}\, P_t\right)^{\frac{1}{1-\alpha}} = \lambda \frac{\alpha}{1-\alpha} \nu_t. \tag{A.89}$$

Using (A.89) and market clearing (A.66), K_t follows

$$\frac{dK_t}{K_t} = \left(-\delta\, K_t + \lambda\, e^{\xi_t}\, \left(\alpha\, e^{\xi_t}\, P_t\right)^{\frac{\alpha}{1-\alpha}}\right)\, dt = -\delta\, dt + \kappa(\omega_t)\, dt,$$

where we have used (A.84), (A.76), and (A.81) for the last equality. The equilibrium dynamics of the aggregate quality-adjusted capital stock thus agrees with (A.83).

Next, we establish the dynamics of the SDF Λ. Consider the evolution of household's wealth share. All shareholders solve the same consumption-portfolio choice problem, different only in the level of household wealth, and households have homothetic preferences. Thus, the evolution of a shareholder household's wealth share (defined as the ratio of household wealth to the total wealth of all shareholders) is given by an equation similar to (27) but taking into account the presence of households that do not participate in financial markets:

$$\frac{dw_{n,t}}{w_{n,t}} = \delta^h\, dt - \frac{\lambda\eta\nu_t}{V_t + G_t + \psi\, H_t}\, dt + \psi\, \frac{\lambda\eta\nu_t}{V_t + G_t + \psi\, H_t}\, \mu_I^{-1} dN^I_{n,t}. \tag{A.90}$$

Equation (A.90) takes into account that the total measure of new blueprints that accrue to shareholders is equal to ψ. The benchmark model in the paper corresponds to the case where $\psi = 1$. Based on the asset prices in (A.85–A.87) and (A.74), we find that

$$\frac{\lambda\eta\nu_t}{V_t + G_t + \psi\, H_t} = s(\omega_t), \tag{A.91}$$

and therefore wealth shares follow

$$\frac{dw_{n,t}}{w_{n,t}} = (\delta^h - s(\omega_t))\, dt + \psi\mu_I^{-1} s(\omega_t)\, dN^I_{n,t}. \tag{A.92}$$

Next, we derive the the consumption process of households from the market clearing conditions. Then, optimality of this process follows from asset prices being consistent the SDF implied by this process. Based on the aggregate consumption process (A.67) and equilibrium wage process (A.57), along with the definition of ω and χ, the consumption of shareholders as a group is

$$C^S_t \equiv \int_{n\in\mathcal{S}_t} C_{n,t}\, dn = C_t - (1-\psi)\, \mathrm{w}_t = e^{\chi_t} e^{-\phi\,\omega_t} \left(1 - i(\omega_t) - (1-\psi)(1-\phi)\right). \tag{A.93}$$

Preference homotheticity implies that the consumption of a each shareholder household is proportional to its wealth share, so

$$C_{n,t} = w_{n,t}\, C^S_t. \tag{A.94}$$

Optimality of household consumption and portfolio choices implies that the SDF in Lemma 2 above, defined using a shareholder households' consumption process, is a valid equilibrium SDF in this economy. In particular, we obtain

$$\Lambda_t = \Lambda_0 \, \exp\left(\int_0^t \delta^h + \frac{\partial \phi(C_{n,s}, J_{n,s}; \overline{C}_s)}{\partial J_{n,s}} + \mu_I \left(\left(1 + \frac{\psi}{\mu_I} s(\omega_s) \right)^{1-\gamma} - 1 \right) ds \right) A_t, \tag{A.95}$$

where Λ_0 is a constant and

$$A_t \equiv \frac{\partial \phi(C_{n,t}, J_{n,t}; \overline{C}_t)}{\partial C_{n,t}} \exp\left(\int_0^t \gamma \log\left(1 + \frac{\psi}{\mu_I} s(\omega_s) \right) dN_{n,s} \right). \tag{A.96}$$

In the formulation of Lemma 2, we set the gain from innovation to its equilibrium value,

$$\varrho_t \equiv \log\left(1 + \frac{\psi}{\mu_I} s(\omega_t) \right). \tag{A.97}$$

Note also that the SDF is defined only up to a multiplicative positive constant.

Equation (A.70) is the Hamilton-Jacobi-Bellman equation for the value function, if the latter is expressed in the form of (A.103). Using this expression for the value function and the process for shareholder consumption (A.94), we find that

$$A_t = A_0 \, e^{-\gamma_1 \, \chi_t} \hat{w}_t^{-\gamma} \, B(\omega_t), \tag{A.98}$$

where A_0 is a constant and $B(\omega)$ satisfies (A.81) and

$$d\hat{w}_t = \left(\delta^h - s(\omega_t) \right) dt. \tag{A.99}$$

The process $\hat{w}_t$ is the same as the process for the household wealth shares, conditional on no innovation shocks, i.e., setting $N_{n,t}$ to be constant. Further,

$$\frac{\partial \phi(C_t^\star, J_t^\star; \overline{C}_t)}{\partial J_t^\star} = -\frac{\rho}{1-\theta^{-1}} \left((\gamma - \theta^{-1}) \, l(\omega)^{1-\theta^{-1}} \, \left[f(\omega) \right]^{\frac{1-\theta^{-1}}{\gamma-1}} + (1-\gamma) \right) \tag{A.100}$$

where

$$l(\omega) \equiv (1 - i(\omega) - (1-\psi)(1-\phi)) \, ((1 - i(\omega)))^{-h}. \tag{A.101}$$

We are now in a position to complete the proof by verify that the conjectured price processes in (A.84–A.87) are consistent with the equilibrium SDF above. Note that equations (A.71–A.73) are the valuation equations for V_t, G_t, and H_t respectively, based on the Feynman-Kac Theorem (Karatzas and Shreve, 1991, e.g, Theorem 7.6), given the equilibrium SDF above and the conjectured expressions in (A.85–A.87). By definition of V_t and K_t, $P_t = K_t^{-1} V_t$, which establishes the consistency of A.84. ■

The following proposition illustrates how to construct a valid SDF in our economy.

Proposition 1 (Stochastic Discount Factor) *The process Λ_t, given by the following equation,* *is adapted to the market filtration $\mathcal{F}$ and is a valid SDF:*

$$\log \Lambda_t = \int_0^t b(\omega_s)\, ds - \gamma_1 \chi_t - \frac{1}{\theta_1}\left(\log C_t - \chi_t\right) - \frac{1-\kappa}{\kappa} \log f(\omega_t). \tag{A.102}$$

In the above equation, $\kappa \equiv \frac{1-\gamma}{1-\theta^{-1}}$, $\gamma_1 \equiv 1-(1-\gamma)(1-h)$, and $\theta_1 \equiv \big(1-(1-\theta^{-1})(1-h)\big)^{-1}$. In the first term, the function $b(\omega)$ is defined in the proof of the proposition in the Appendix. In the last term, the function $f(\omega)$ is related to the value function J of an investor with relative wealth w_{it},

$$f(\omega_t) = (1-\gamma)\, J(w_{i,t}, \chi_t, \omega_t) \left(w_{i,t}^{1-\gamma} e^{(1-\gamma_1)\chi_t}\right)^{-1}. \tag{A.103}$$

Proof of Proposition 1. Proposition follows directly from (A.95) and (A.98) in the proof of the Lemma 3 above. The process $b(\omega_t)$ is given by

$$b(\omega) = (1-\gamma)\delta^h - \rho\,\kappa - \rho\,(1-\kappa)\,\left(1-i(\omega_t)\right)^{(1-\theta_1^{-1})}\,\left(f(\omega)\right)^{-\kappa^{-1}} + \gamma\, s(\omega) + \mu_I \left(\left(1+\frac{\psi}{\mu_I} s(\omega)\right)^{1-\gamma} - 1\right). \tag{A.104}$$

where the functions $i(\omega)$, $f(\omega)$, and $s(\omega)$ are defined in Lemma 3 above. ■

Lemma 4 (Market value of a firm) *The market value of a firm equals*

$$S_{f,t} = e^{\chi_t}\, B(\omega_t)^{-1}\left[v(\omega_t) k_{f,t} + v_1(\omega_t)\, k_{f,t}\,(\bar{u}_{f,t}-1) + g(\omega_t) + \left(\frac{\lambda_{f,t}}{\lambda} - 1\right) g_1(\omega_t)\right] \tag{A.105}$$

where

$$Z_{f,t} \equiv \sum_{j \in \mathcal{J}_{f,t}} e^{\xi_{\tau(j)}}\, u_{j,t}\, K_{j,t} \tag{A.106}$$

and $v(\omega)$ and $g(\omega)$ are defined in Lemma 3 above and the functions v_1 and g_1 solve the ODEs

$$\begin{aligned} 0 &= \phi\, e^{-\phi\omega}\, B(\omega) + v_1'(\omega)\left\{c_1^f - (1-\alpha\,\phi)\,\kappa(\omega)\right\} + v_1''(\omega)\, c_2^f \\ &\quad + v_1(\omega)\left\{c_0^f - \kappa_u - \frac{\gamma-\theta^{-1}}{1-\gamma} A_1(\omega) f(\omega)^{\frac{1-\theta^{-1}}{\gamma-1}} + \mu_I\left(\left(1+\frac{\psi}{\mu_I}s(\omega)\right)^{1-\gamma} - 1\right) + \gamma\, s(\omega) - \kappa(\omega)\right\}, \end{aligned} \tag{A.107}$$

$$\begin{aligned} 0 &= (1-\eta)\,(1-\alpha)\, v(\omega)\,\kappa(\omega) + g_1'(\omega)\left\{c_1^f - (1-\alpha\,\phi)\,\kappa(\omega)\right\} + g_1''(\omega)\, c_2^f \\ &\quad + g_1(\omega)\left\{c_0^f - \mu_L - \mu_H - \frac{\gamma-\theta^{-1}}{1-\gamma} A_1(\omega) f(\omega)^{\frac{1-\theta^{-1}}{\gamma-1}} + \mu_I\left(\left(1+\frac{\psi}{\mu_I}s(\omega)\right)^{1-\gamma} - 1\right) + \gamma\, s(\omega)\right\}. \end{aligned} \tag{A.108}$$

Proof. The proof follows closely the derivations of equations (A.84) and (A.86) above. We have that the value of assets in place for an existing firm with capital stock $K_{f,t}$ and cash flow $Z_{f,t}$ are given by

$$VAP_{f,t} = P_t K_{f,t} + P_{1,t}\left(Z_{f,t} - K_{f,t}\right), \tag{A.109}$$

where

$$P_{1,t} \equiv \mathrm{E}_t\left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t} e^{-(\delta+\kappa_u)(s-t)}\, p_s\, ds\right] = K_t^{-1} e^{\chi_t}\, v_1(\omega_t) B(\omega_t)^{-1} \tag{A.110}$$

and $v_1(\omega)$ satisfies the ODE (A.118). As above, we have used the SDF (A.95), equation (A.61), the definition of χ and ω and the Feynman-Kac theorem. Similarly, the present value of growth opportunities for a firm equals

$$\begin{aligned} PVGO_{f,t} &\equiv (1-\eta)\,\mathrm{E}_t\left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t}\lambda_{f,s}\,\nu_s\, ds\right] \\ &= PVGO_t + \lambda\,(1-\eta)\,\left(\frac{\lambda_{f,t}}{\lambda} - 1\right) E_t \int_t^\infty \frac{\Lambda_s}{\Lambda_t} e^{(\mu_L+\mu_H)(s-t)}\,\nu_s\, ds \qquad \text{(A.111)} \\ &= e^{\chi_t} g(\omega_t)\,(B(\omega_t))^{-1} + e^{\chi_t}\,\left(\frac{\lambda_{f,t}}{\lambda} - 1\right) g_1(\omega_t)\,(B(\omega_t))^{-1} \qquad \text{(A.112)} \end{aligned}$$

where $g_1(\omega)$ satisfies the ODE (A.118). As above, we have used the SDF (A.95), the definition of z and ω and the Feynman-Kac theorem , and the fact that

$$\mathrm{E}[\lambda_{f,s}|\lambda_{f,t}] = \lambda + \lambda\,\left(\frac{\lambda_{f,t}}{\lambda} - 1\right)\, e^{(\mu_L+\mu_H)(s-t)}. \tag{A.113}$$

■

Lemma 5 (Bond prices) *The price at time t of a zero coupon bond of maturity T, is given by*

$$P_b(\omega_t, T-t) = \frac{b(\omega_t, T-t)}{B(\omega_t)} \tag{A.114}$$

where $b(\omega_t, T-t)$ solves the following PDE

$$\begin{aligned} 0 = &\frac{\partial}{\partial t} b(\omega, T-t) + \frac{\partial}{\partial \omega} b(\omega, T-t)\left\{c_1^f - (1-\alpha\,\phi)\,\kappa(\omega)\right\} + \frac{\partial^2}{\partial \omega^2} b(\omega, T-t)\, c_2^f \\ &+ b(\omega, T-t)\left\{c_0^b - \frac{\gamma - \theta^{-1}}{1-\gamma} A_1(\omega) f(\omega)^{\frac{1-\theta^{-1}}{\gamma-1}} + \mu_I\left(\left(1 + \frac{\psi}{\mu_I} s(\omega)\right)^{1-\gamma} - 1\right) + \gamma\, s(\omega)\right\} \end{aligned} \tag{A.115}$$

where c_1^f and c_2^f are constants defined above,

$$\begin{aligned} c_0^b = &\left\{\delta^h(1-\gamma) - \frac{\rho(1-\gamma)}{1-\theta^{-1}} - \gamma_1\,(1-\phi)\mu_x + \frac{1}{2}(1-\phi)^2\,\sigma_x^2\,\gamma_1^2 + \frac{1}{2}\left(\frac{\phi\,\gamma_1}{1-\alpha\,\phi}\right)^2\left(\sigma_\xi^2 + \alpha^2(1-\phi)^2\sigma_x^2\right)\right. \\ &\left.- \frac{\phi\,\gamma_1}{1-\alpha\,\phi}\left(\mu_\xi + \alpha\,(1-\phi)\,\mu_x - \gamma_1\,\alpha(1-\phi)^2\,\sigma_x^2\right)\right\}, \end{aligned} \tag{A.116}$$

along with the terminal condition $b(\omega, 0) = B(\omega)$.

Proof. Proof follows from the definition of the zero coupon bond and the Feynman-Kac theorem. ■

Lemma 6 (Dividend Strip prices) *The price at time t of a dividend strip of maturity T, is given by*

$$P_d(\omega_t, T-t) = e^{\chi_t} \frac{d(\omega_t, T-t)}{B(\omega_t)} \tag{A.117}$$

where $d(\omega_t, T-t)$ solves the following PDE

$$0 = \frac{\partial}{\partial t} d(\omega, T-t) + \frac{\partial}{\partial \omega} d(\omega, T-t) \left\{ c_1^f - (1-\alpha\,\phi)\,\kappa(\omega) \right\} + \frac{\partial^2}{\partial \omega^2} d(\omega, T-t)\, c_2^f$$
$$+ d(\omega, T-t) \left\{ c_0^f - \frac{\gamma - \theta^{-1}}{1-\gamma} A_1(\omega) f(\omega_t)^{\frac{1-\theta^{-1}}{\gamma-1}} + \mu_I \left(\left(1 + \frac{\psi}{\mu_I} s(\omega_t) \right)^{1-\gamma} - 1 \right) + \gamma\, s(\omega) \right\} \tag{A.118}$$

where c_1^f and c_2^f are constants defined above, along with the terminal condition

$$d(\omega, 0) = B(\omega) e^{-\phi\omega} \left(1 - i(\omega) - \eta\, i(\omega) \frac{1-\alpha}{\alpha} \right).$$

Proof. Proof follows from the definition of the dividend strip and the Feynman-Kac theorem. ■

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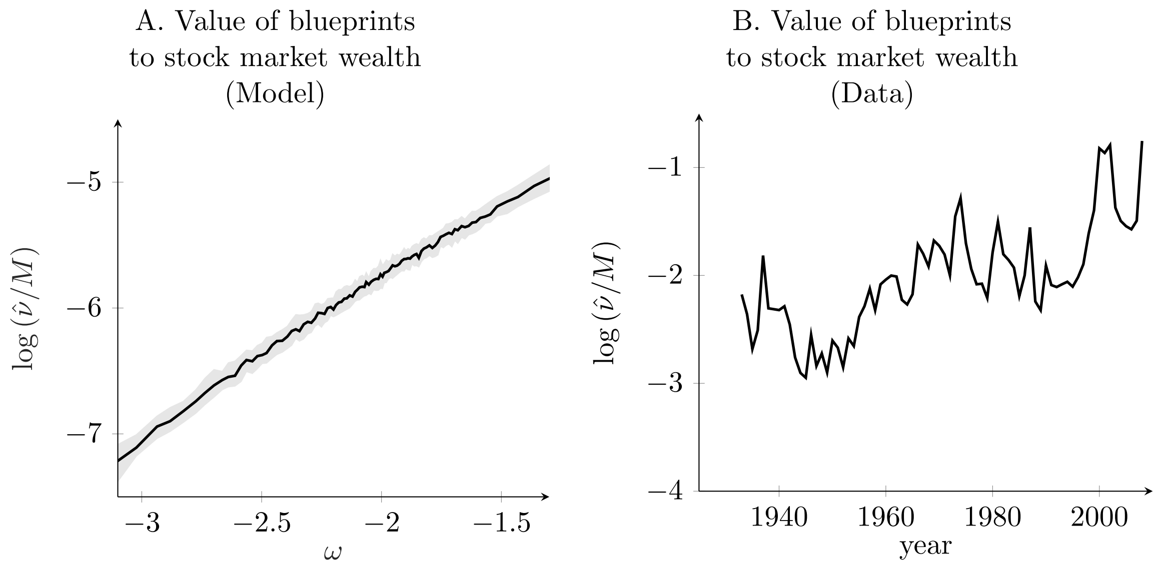
Figure A.1: Value of blueprints



Panel A plots the value of blueprints (A.10) to total stock market wealth M relative to the value of the state variable ω in the model, using a long panel of 3,000 years with 1,000 firms. The black line plots the average value of ω and the shaded line plots the standard deviation across simulations. Panel B plots the time series of the ratio of the estimated value of new blueprints $\hat{\nu}_t$ to the value of the stock market M_t in the data. See the appendix and Kogan et al. (2017) for more details. The estimated value of blueprints is constructed in the same way in both simulated as well as actual data. The total value of innovation in year t is scaled by end of year t market capitalization. Data period is 1933 to 2008.

Figure A.2: Shock-exposure and shock-price elasticities

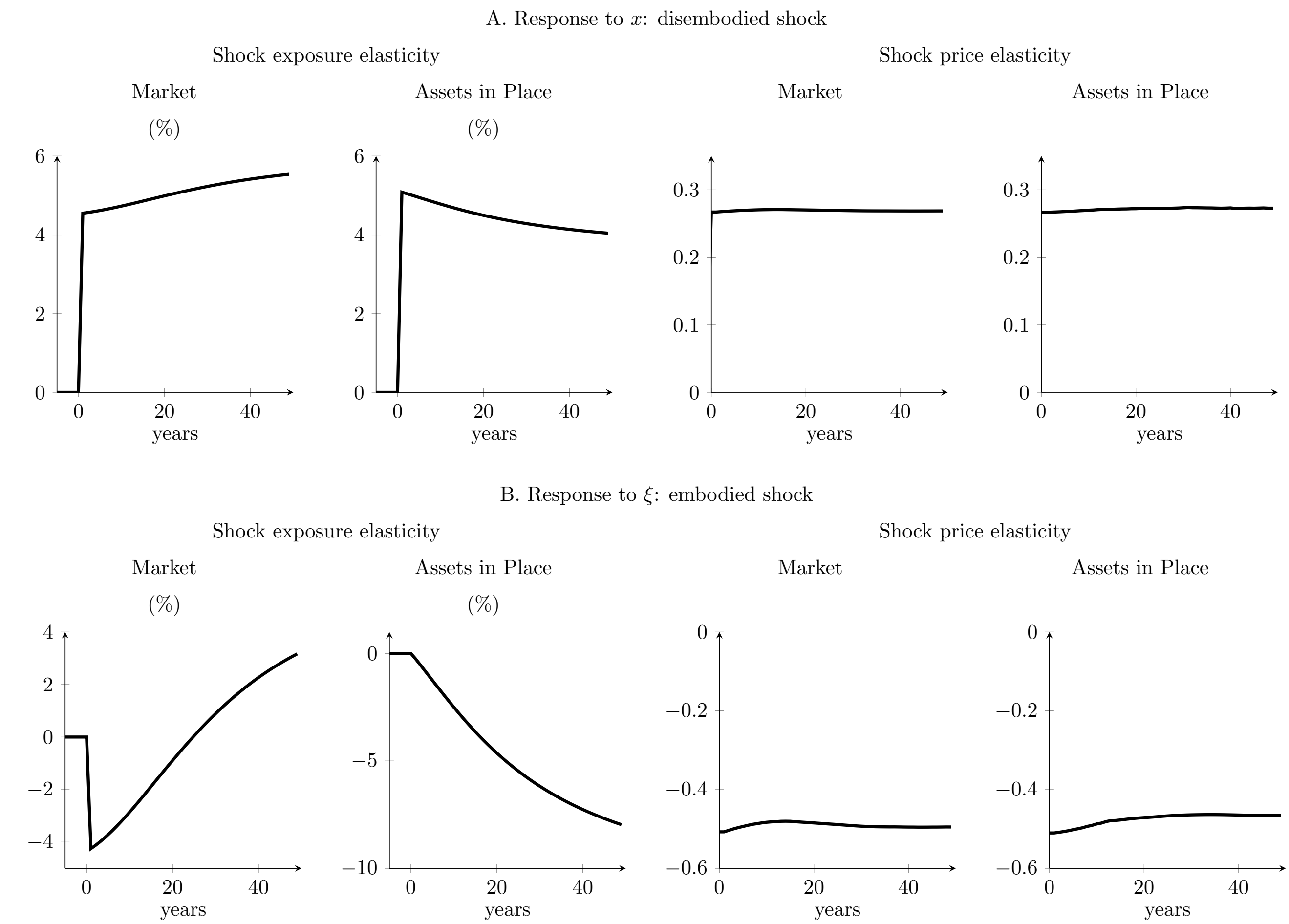


Figure plots shock-exposure and shock-price elasticities of the aggregate dividend process D, and the dividends from asset in place D^{ap} to the two technology shocks in the model. We construct the shock exposures taking into account the nonlinear nature of equilibrium dynamics: we introduce an additional shock of magnitude $\sigma\sqrt{dt}$ at time $0 + dt$ without altering the realizations of all future shocks. We then scale the resulting impulse responses by $1/\sqrt{dt}$. We compute these elasticities at the mean of the stationary distribution of ω.

Figure A.3: Firm risk exposures and risk premia

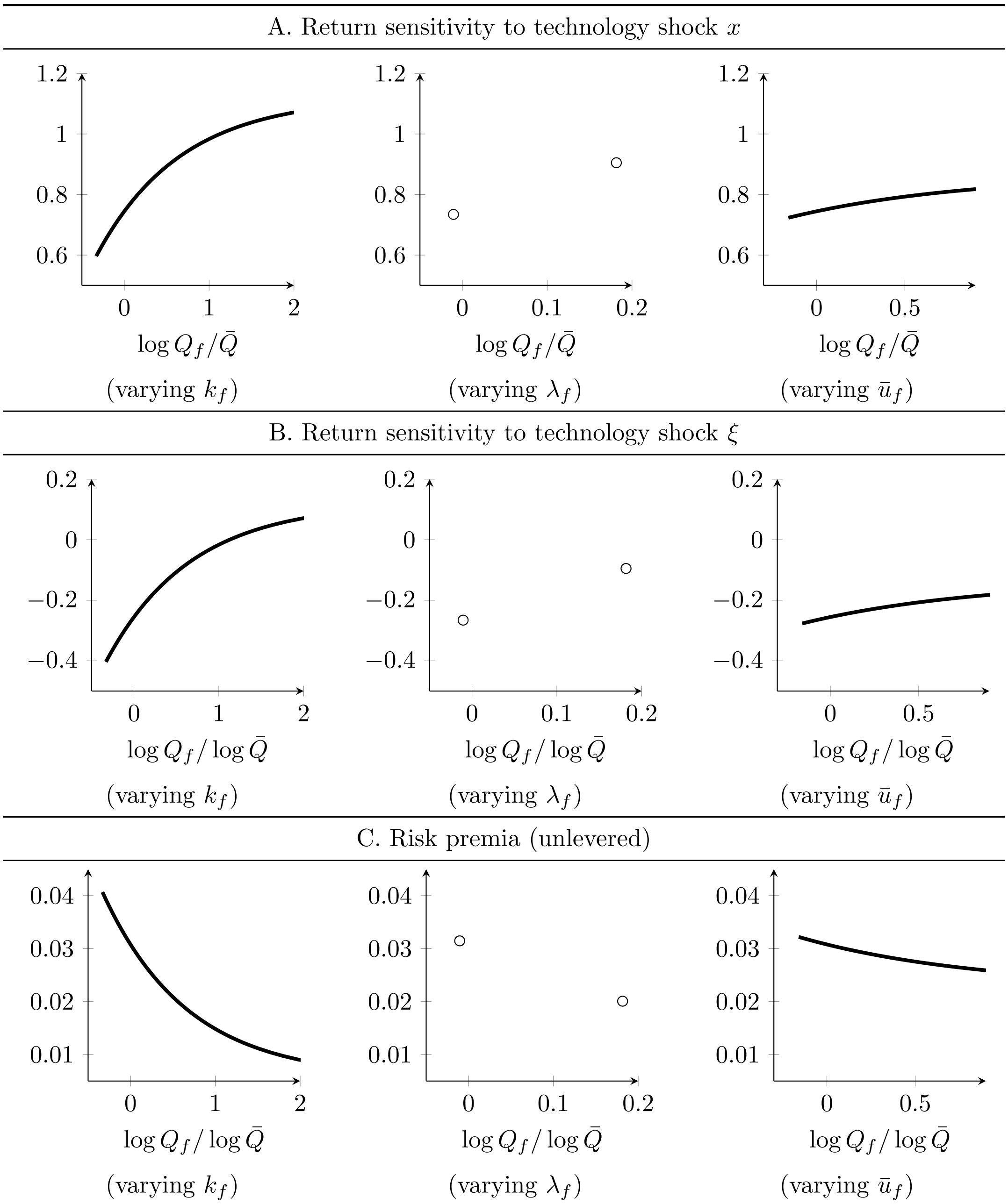


Figure illustrates how technology risk exposures (Panels A and B) and risk premia (Panel C) vary with the firm's market-to-book ratio (Q). A firm's market-to-book ratio is a function of the firm's relative size k_f, likelihood of future growth λ_f, and its current productivity $\bar{u}_f$. In each of the three columns, we examine how variation in Q due to each of these three state variables translates into variation in risk premia – while holding the other two at their average values, i.e. $\lambda_f = \lambda$, $k_f = 1$ and $\bar{u}_f = 1$. The range in the x-axis corresponds to the 0.5% and 99.5% of the range of each these three variables in simulated data. The value of the state variable ω is set to its unconditional mean, $\omega = \mathrm{E}[\omega_t]$.

Table A.1: Parameter boundaries for the RBF optimizer

Parameter	Symbol	Estimate	Lower Bound	Upper Bound
Preferences				
Risk aversion	γ	56.734	5	165
Elasticity of intertemporal substitution	θ	2.341	0.1	3
Effective discount rate	ρ	0.044	0.025	0.07
Preference weight on relative consumption	h	0.836	0	1
Technology				
Disembodied technology growth, mean	μ_x	0.016	0	0.05
Disembodied technology growth, volatility	σ_x	0.082	0.02	0.2
Embodied technology growth, mean	μ_ξ	0.004	0	0.05
Embodied technology growth, volatility	σ_ξ	0.110	0.02	0.2
Project-specific productivity, long-run volatility	υ_u	1.981	0.1	2.5
Project-specific productivity, mean reversion	κ_u	0.210	0.1	0.5
Production and Investment				
Cobb-Douglas capital share	ϕ	0.427		
Decreasing returns to investment	α	0.446	0.1	0.9
Depreciation rate	δ	0.029	0.01	0.1
Transition rate to low-growth state	μ_L	0.364	0.005	0.75
Transition rate to high-growth state	μ_H	0.021	0.005	0.75
Project mean arrival rate, mean	λ	0.812	0.1	3
Project mean arrival rate, rel. difference high-low growth states	λ_D	15.674	0.1	100
Incomplete Markets				
Fraction of project NPV that goes to inventors	η	0.767	0	1
Fraction of households that is a shareholder	ψ	0.148	0.04	1

Table reports the parameter bounds that we used in our implementation of the RBF search algorithm.

Table A.2: Robustness: Comparison across restricted models

Parameter	Symbol	BASE	FullPart	Restrict η			Restrict h			No ξ	Restrict γ
			$\psi = 1$	$\eta = 0$	$\eta \leq 0.3$	$\eta \leq 0.6$	$h = 0$	$h \leq 0.3$	$h \leq 0.6$		$\gamma \leq 10$
Risk aversion	γ	56.73	104.57	92.80	83.35	72.49	15.74	30.85	34.34	73.37	10.00
Elasticity of intertemporal substitution	θ	2.34	2.30	2.15	1.96	1.87	0.82	1.50	1.81	1.73	2.75
Fraction of population that is a shareholder	ψ	0.15	-	0.05	0.06	0.13	0.05	0.05	0.05	0.16	0.05
Preference weight on relative consumption	h	0.84	0.93	0.92	0.89	0.85	0.00	0.30	0.60	0.85	0.99
Fraction of project NPV that goes to inventors	η	0.77	0.81	0.00	0.30	0.60	0.86	0.84	0.75	0.27	0.95
Disembodied technology growth, volatility	σ_x	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.08	0.08	0.08
Embodied technology growth, volatility	σ_ξ	0.11	0.12	0.11	0.11	0.11	0.12	0.12	0.11	-	0.13

Moment	DATA	BASE	FullPart	Restrict η			Restrict h			No ξ	Restrict γ
			$\psi = 1$	$\eta = 0$	$\eta \leq 0.3$	$\eta \leq 0.6$	$h = 0$	$h \leq 0.3$	$h \leq 0.6$		$\gamma \leq 10$
Shareholder consumption share, mean	0.429	0.464	-	0.320	0.406	0.449	0.408	0.409	0.397	0.443	0.399
Shareholder consumption growth, volatility	0.037	0.039	-	0.044	0.039	0.039	0.039	0.036	0.039	0.036	0.042
Investment-to-output ratio (log), volatility	0.305	0.288	0.282	0.325	0.305	0.291	0.312	0.315	0.304	0.025	0.312
Investment growth, volatility	0.130	0.105	0.101	0.119	0.108	0.107	0.112	0.110	0.109	0.048	0.108
Investment and consumption growth, correlation	0.472	0.373	0.378	0.401	0.390	0.386	0.275	0.182	0.314	0.998	0.315
Market portfolio, excess returns, mean	0.063	0.067	0.068	0.053	0.066	0.071	0.081	0.089	0.078	0.053	0.039
Market portfolio, excess returns, volatility	0.185	0.131	0.138	0.132	0.125	0.129	0.139	0.119	0.128	0.116	0.161
Value factor, mean	0.065	0.063	0.051	0.034	0.056	0.059	0.014	0.041	0.052	-0.010	0.050
Distance (mean relative deviation)		0.014	0.020	0.024	0.015	0.015	0.061	0.050	0.022	0.240	0.028
Additional Statistics											
Risk price for x		0.027	0.026	0.021	0.031	0.032	0.053	0.053	0.039	0.041	0.000
Market exposure to x		1.313	1.347	1.375	1.245	1.293	1.324	1.265	1.221	1.313	1.263
Value-minus-growth risk exposure to x		-0.184	-0.154	-0.303	-0.237	-0.194	-0.096	-0.194	-0.252	-0.327	-0.200
Risk price for ξ		-0.078	-0.065	-0.043	-0.068	-0.079	-0.023	-0.050	-0.057		-0.049
Market exposure to ξ		-0.653	-0.687	-0.657	-0.468	-0.464	-0.743	-0.610	-0.647		-0.922
Value-minus-growth risk exposure to ξ		-0.875	-0.885	-0.877	-0.966	-0.862	-0.736	-0.986	-1.104		-1.066

This table results from estimating restricted versions of the model. To conserve space, we only report the key estimated parameters of the model, along with targeted moments; the full set of results are in the Online Appendix. In addition, we present the risk prices for the two technology shocks, defined as $\lambda_x \equiv -cov\left(dx_t, d\log\Lambda_t\right)/dt$ and $\lambda_\xi \equiv -cov\left(d\xi_t, d\log\Lambda_t\right)/dt$, as well as the risk exposures (beta) of the market portfolio and the value factor with respect to the two shocks.

Table A.3: Robustness across restricted models: Goodness of Fit

	DATA	BASE	FullPart	Restrict η			Restrict h			No ξ	Restrict γ
			$\psi = 1$	$\eta = 0$	$\eta \leq 0.3$	$\eta \leq 0.6$	$h = 0$	$h \leq 0.3$	$h \leq 0.6$		$\gamma \leq 10$
Aggregate quantities											
Consumption growth, mean	0.015	0.014	0.013	0.012	0.013	0.014	0.015	0.016	0.014	0.010	0.013
Consumption growth, volatility (short-run)	0.036	0.039	0.039	0.041	0.039	0.039	0.037	0.033	0.037	0.038	0.039
Consumption growth, volatility (long-run)	0.041	0.053	0.053	0.055	0.052	0.054	0.052	0.047	0.051	0.048	0.054
Shareholder consumption share, mean	0.429	0.464	-	0.320	0.406	0.449	0.408	0.409	0.397	0.443	0.399
Shareholder consumption growth, volatility	0.037	0.039	-	0.044	0.039	0.039	0.039	0.036	0.039	0.036	0.042
Investment-to-output ratio, mean	0.089	0.083	0.070	0.094	0.075	0.080	0.082	0.081	0.084	0.100	0.090
Investment-to-output ratio (log), volatility	0.305	0.288	0.282	0.325	0.305	0.291	0.312	0.315	0.304	0.025	0.312
Investment growth, volatility	0.130	0.105	0.101	0.119	0.108	0.107	0.112	0.110	0.109	0.048	0.108
Investment and consumption growth, correlation	0.472	0.373	0.378	0.401	0.390	0.386	0.275	0.182	0.314	0.998	0.315
Aggregate Innovation, volatility	0.370	0.369	0.371	0.397	0.362	0.362	0.397	0.383	0.380	0.123	0.430
Capital share, mean	0.356	0.354	0.323	0.373	0.394	0.363	0.315	0.328	0.342	0.353	0.303
Asset Prices											
Market portfolio, excess returns, mean	0.063	0.067	0.068	0.053	0.066	0.071	0.081	0.089	0.078	0.053	0.039
Market portfolio, excess returns, volatility	0.185	0.131	0.138	0.132	0.125	0.129	0.139	0.119	0.128	0.116	0.161
Risk-free rate, mean	0.020	0.020	0.022	0.020	0.022	0.022	0.024	0.022	0.022	0.026	0.023
Risk-free rate, volatility	0.007	0.007	0.007	0.008	0.007	0.007	0.006	0.005	0.006	0.002	0.007
Value factor, mean	0.065	0.063	0.051	0.034	0.056	0.059	0.014	0.041	0.052	-0.010	0.050
Value factor, volatility*	0.243	0.152	0.184	0.122	0.138	0.143	0.274	0.263	0.196	0.088	0.223
Value factor, CAPM alpha*	0.050	0.046	0.031	0.034	0.044	0.045	-0.004	0.016	0.026	-0.003	0.030
Cross-sectional moments											
Investment rate, IQR	0.175	0.163	0.174	0.155	0.159	0.163	0.174	0.172	0.164	0.101	0.188
Investment rate, persistence	0.223	0.228	0.204	0.230	0.213	0.226	0.171	0.154	0.200	0.179	0.209
Tobin Q, IQR	1.139	0.882	0.816	1.046	1.125	0.978	0.562	0.874	0.874	0.663	0.926
Tobin Q, persistence	0.889	0.948	0.940	0.951	0.951	0.952	0.899	0.933	0.942	0.947	0.946
Firm innovation, 90-50 range	0.581	0.542	0.602	0.633	0.575	0.530	0.667	0.701	0.621	0.683	0.640
Firm innovation, persistence	0.551	0.567	0.552	0.560	0.585	0.595	0.502	0.550	0.601	0.489	0.633
Profitability, IQR	0.902	0.936	0.948	0.917	0.925	0.927	0.923	0.944	0.911	0.675	0.966
Profitability, persistence	0.818	0.815	0.772	0.844	0.825	0.857	0.765	0.849	0.779	0.861	0.792
		0.014	0.020	0.024	0.015	0.015	0.061	0.050	0.022	0.240	0.028

Starred moments are not part of the estimation targets. They are included here for comparison.

Table A.4: Robustness across restricted models: Parameters

Parameter	symbol	BASE	$\psi = 1$	$\eta = 0$	$\eta \leq 0.3$	$\eta \leq 0.6$	$h = 0$	$h \leq 0.3$	$h \leq 0.6$	No ξ	Restrict γ
Preference											
Risk aversion	γ	56.73	104.57	92.80	83.35	72.49	15.74	30.85	34.34	73.37	10.00
Elasticity of intertemporal substitution	θ	2.34	2.30	2.15	1.96	1.87	0.82	1.50	1.81	1.73	2.75
Effective discount rate	ρ	0.04	0.04	0.04	0.05	0.05	0.06	0.06	0.05	0.05	0.04
Preference weight on relative consumption	h	0.84	0.93	0.92	0.89	0.85	0.00	0.30	0.60	0.85	0.99
Technology Shocks											
Disembodied technology growth, mean	μ_x	0.02	0.02	0.01	0.02	0.02	0.01	0.02	0.01	0.01	0.02
Disembodied technology growth, volatility	σ_x	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.08	0.08	0.08
Embodied technology growth, mean	μ_ξ	0.00	0.00	0.00	0.00	0.00	0.01	0.00	0.01	-	0.00
Embodied technology growth, volatility	σ_ξ	0.11	0.12	0.11	0.11	0.11	0.12	0.12	0.11	-	0.13
Project-specific productivity, mean reversion	κ_u	0.21	0.28	0.17	0.20	0.15	0.29	0.16	0.27	0.11	0.25
Project-specific productivity, volatility	σ_u	0.53	0.61	0.47	0.51	0.44	0.61	0.46	0.58	0.32	0.58
Production and Investment											
Decreasing returns to investment	α	0.45	0.36	0.58	0.46	0.45	0.39	0.40	0.45	0.35	0.41
Depreciation rate	δ	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.08	0.03
Cobb Douglas Capital Share	ϕ	0.43	0.42	0.37	0.41	0.42	0.43	0.43	0.42	0.40	0.43
Transition rate to low-growth state	μ_L	0.36	0.26	0.19	0.19	0.35	0.23	0.13	0.23	0.41	0.12
Transition rate to high-growth state	μ_H	0.02	0.02	0.01	0.01	0.02	0.02	0.01	0.02	0.01	0.01
Project mean arrival rate, mean	λ	0.81	0.70	0.60	0.68	0.91	0.86	1.00	0.97	3.61	1.05
—, diff. high-low growth states	λ_D	15.67	13.98	15.32	16.55	13.41	11.44	12.32	13.14	29.75	12.06
Incomplete Markets											
Fraction of population that is a shareholder	ψ	0.14	-	0.05	0.06	0.13	0.05	0.05	0.05	0.16	0.05
Fraction of project NPV that goes to inventors	η	0.77	0.81	0.00	0.30	0.60	0.86	0.84	0.75	0.27	0.95

Table A.5: Inequality measures: overall, vs within-cohort

Consumption Inequality (CEX)	no age effects				cubic age effects			
	all HH		stockholders		all HH		stockholders	
	overall	within cohort	overall	within cohort	overall	within cohort	overall	within cohort
99-90 ratio	1.82	1.82	1.77	1.74	1.81	1.82	1.77	1.75
99-95	1.50	1.50	1.49	1.46	1.50	1.50	1.48	1.46
95-90	1.21	1.21	1.19	1.19	1.21	1.21	1.19	1.20
90-10	4.17	4.12	3.16	3.14	4.20	4.11	3.17	3.13
90-50	1.96	1.96	1.79	1.78	1.97	1.96	1.79	1.78
50-10	2.12	2.11	1.76	1.76	2.13	2.10	1.77	1.75
Wealth Inequality (SCF)	no age effects				cubic age effects			
	all HH		stockholders		all HH		stockholders	
	overall	within cohort	overall	within cohort	overall	within cohort	overall	within cohort
99-90 ratio	7.55	6.59	6.49	5.98	6.52	6.51	5.83	5.81
99-95	4.03	3.66	3.48	3.33	3.61	3.58	3.20	3.21
95-90	1.87	1.80	1.87	1.80	1.81	1.82	1.82	1.81
90-10	154.70	94.92	50.91	29.35	94.22	91.21	29.16	28.52
90-50	7.14	6.47	5.96	5.16	6.47	6.23	5.17	5.13
50-10	21.66	14.67	8.54	5.69	14.57	14.65	5.64	5.56
Income Inequality (SCF)	no age effects				cubic age effects			
	all HH		stockholders		all HH		stockholders	
	overall	within cohort	overall	within cohort	overall	within cohort	overall	within cohort
99-90 ratio	3.86	3.82	4.42	4.29	3.71	3.77	4.31	4.27
99-95	2.71	2.66	2.96	2.89	2.61	2.65	2.87	2.88
95-90	1.42	1.44	1.49	1.49	1.42	1.42	1.50	1.48
90-10	11.36	9.68	6.76	6.30	9.84	9.07	6.28	6.03
90-50	2.96	2.79	2.62	2.56	2.76	2.70	2.52	2.50
50-10	3.84	3.47	2.58	2.46	3.57	3.35	2.49	2.41

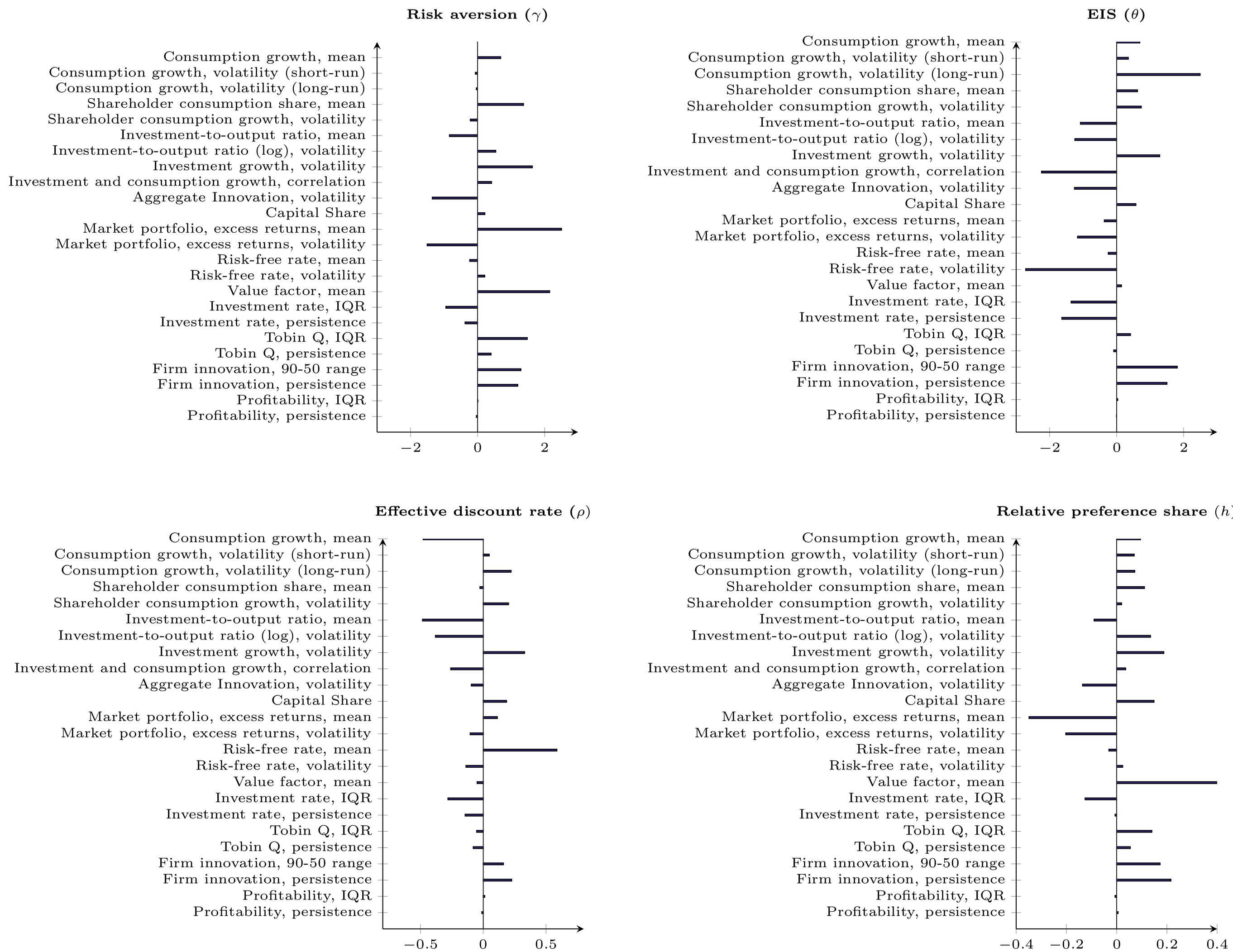
Table reports inequality measures for consumption (using the CEX) and income or wealth (using the SCF). We first remove observation year effects from consumption, income and wealth. We then report the resulting inequality moments before and after removing cohort fixed effects. We also do so separately for the subsample of households that own stocks. Here, we define cohort effects as the birth year of the leading member of the household. We deal with the age-cohort-period identification problem in two ways. The first set of columns reports results without any age effects, so assigns the maximum amount of variation to a cohort effect. However, this may be problematic for wealth inequality, since wealth accumulation mechanically grows with age. Hence, the second column reports results after removing a cubic polynomial in age similar to Garleanu, Kogan, and Panageas (2012). Even then, the linear age effect is not identified, so we set it to zero in order to maximize the explanatory power of cohort effects.

Table A.6: Model: Consumption and Corporate Payout

Model moments	Value
Consumption growth, volatility	0.038
Claim to aggregate consumption, return volatility	0.047
Claim to aggregate consumption, risk premium	0.017
Net corporate payout (‘dividend’) growth, volatility	0.122
Claim to aggregate corporate payout (financial wealth), return volatility	0.061
Claim to aggregate corporate payout (financial wealth), risk premium	0.027
Correlation, net payout and consumption growth	0.367

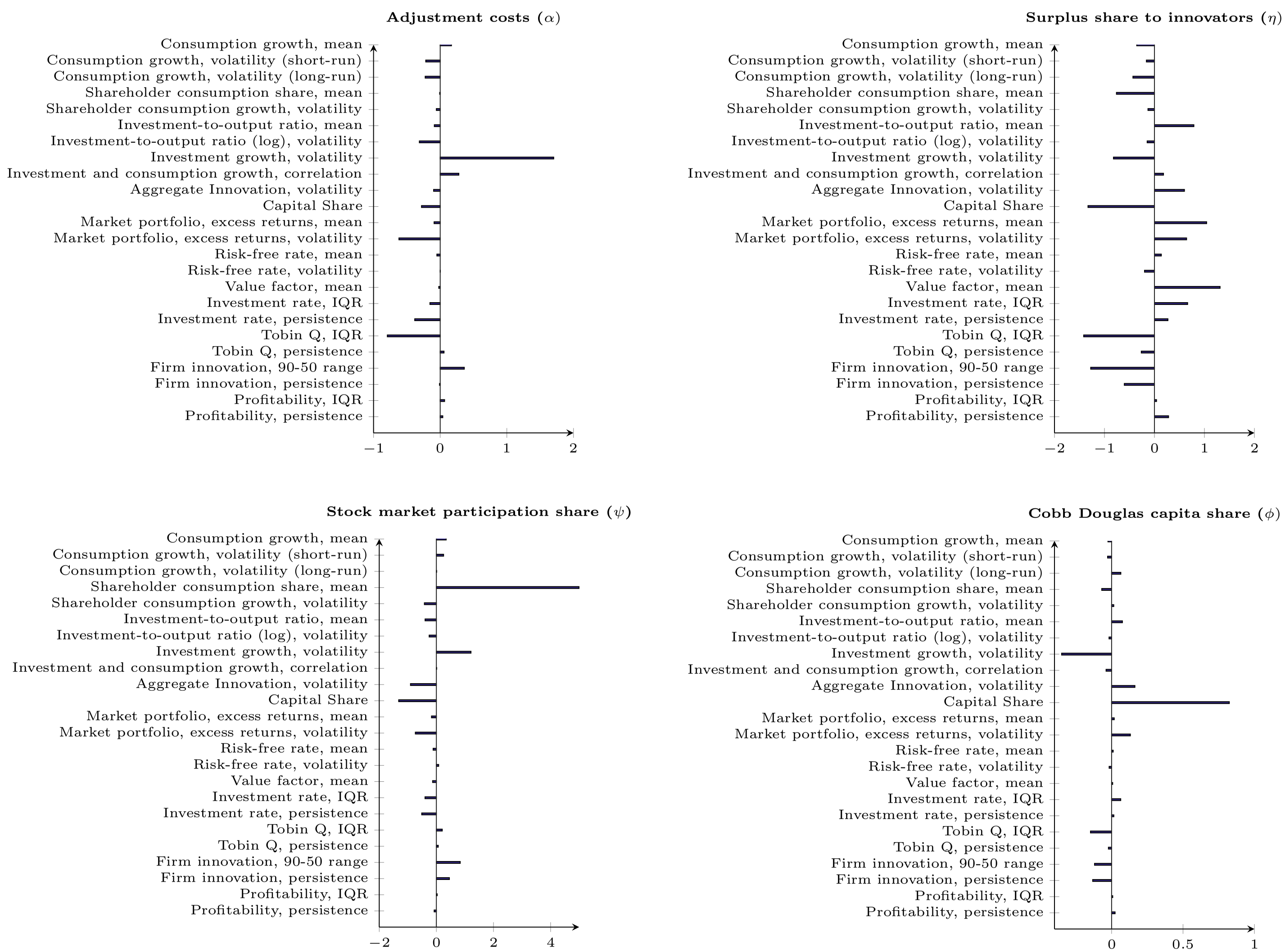
The table reports the risk and return characteristics of aggregate consumption and corporate payout, defined in equation (A.5). In addition to the volatility of the underlying cashflows, we compute the volatility and risk premium on unlevered claims on these two components.

Figure A.4: Sensitivity of parameters to moments (GS)



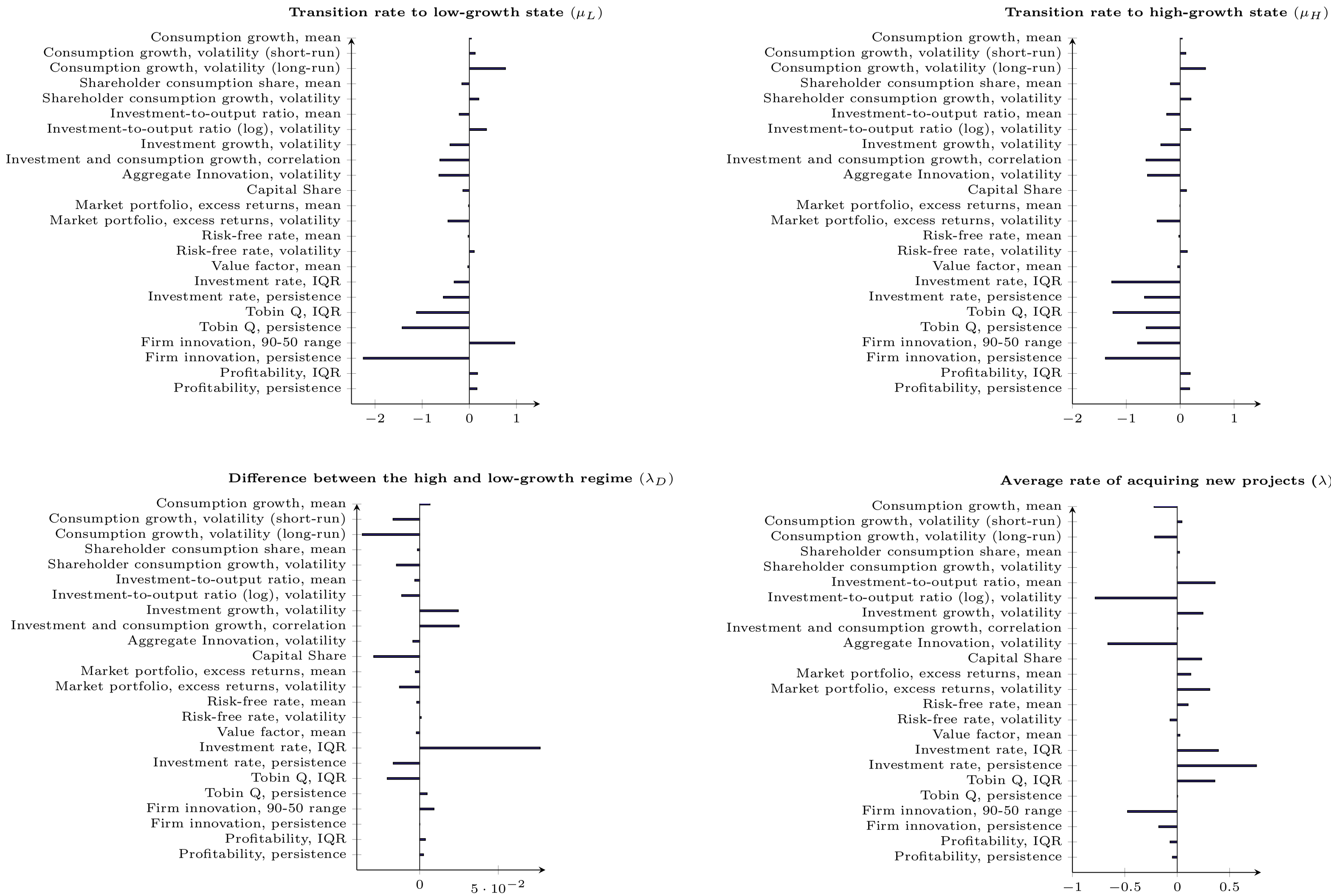
We report the Gentzkow and Shapiro (2014) sensitivity measure of the estimated parameters to moments. We report the measure in elasticity form, $\lambda_{i,j}\,\frac{X^j}{\theta^i}$, where $\lambda_{i,j}$ is the element of the sensitivity matrix Λ that corresponds to parameter i and moment j.

Figure A.5: Sensitivity of parameters to moments (GS)



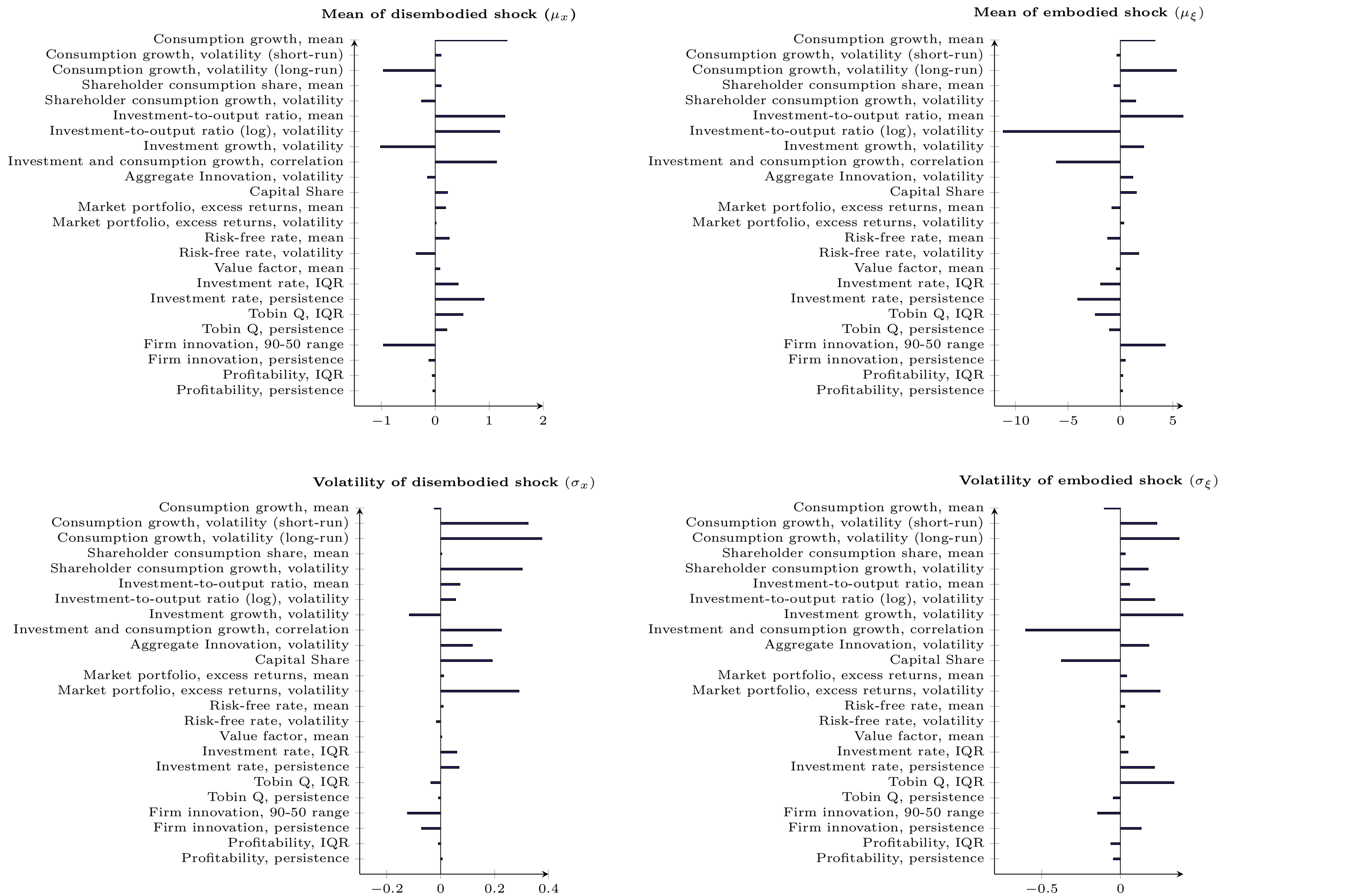
We report the Gentzkow and Shapiro (2014) sensitivity measure of the estimated parameters to moments. We report the measure in elasticity form, $\lambda_{i,j} \frac{X^j}{\theta^i}$, where $\lambda_{i,j}$ is the element of the sensitivity matrix Λ that corresponds to parameter i and moment j.

Figure A.6: Sensitivity of parameters to moments (GS)



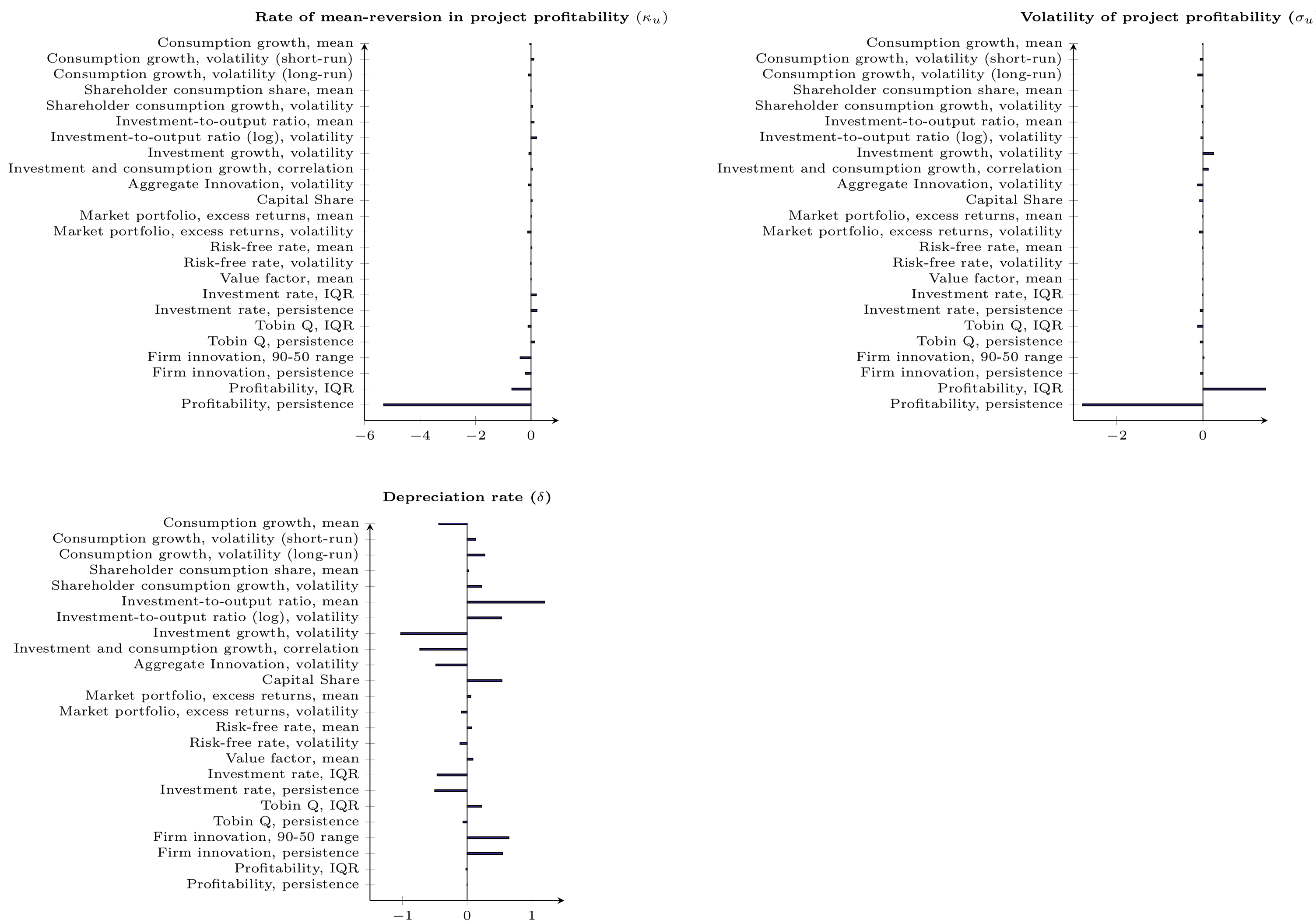
We report the Gentzkow and Shapiro (2014) sensitivity measure of the estimated parameters to moments. We report the measure in elasticity form, $\lambda_{i,j} \frac{X^j}{\theta^i}$, where $\lambda_{i,j}$ is the element of the sensitivity matrix Λ that corresponds to parameter i and moment j.

Figure A.7: Sensitivity of parameters to moments (GS)



We report the Gentzkow and Shapiro (2014) sensitivity measure of the estimated parameters to moments. We report the measure in elasticity form, $\lambda_{i,j}\frac{X^j}{\theta^i}$, where $\lambda_{i,j}$ is the element of the sensitivity matrix Λ that corresponds to parameter i and moment j.

Figure A.8: Sensitivity of parameters to moments (GS)



We report the Gentzkow and Shapiro (2014) sensitivity measure of the estimated parameters to moments. We report the measure in elasticity form, $\lambda_{i,j}\,\frac{X^j}{\theta^i}$, where $\lambda_{i,j}$ is the element of the sensitivity matrix Λ that corresponds to parameter i and moment j.