Operations Research
http://dx.doi.org/10.1287/opre.1120.1053ec

e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—“A Sampling-Based Approach to Appointment Scheduling” by Mehmet A. Begen, Retsef Levi, and Maurice Queyranne, *Operations Research*, http://dx.doi.org/10.1287/opre.1120.1053.

5. Online Appendix - Proofs

Proof. **(Polynomial Time Algorithm Theorem 4)**

Theorem 7.1 of Begen and Queyranne (2011) implies that $F_{\widehat{\mathbf{p}}}(\cdot)$ can be minimized in $O(\sigma(n)\, EO\, n^2 \log(\lceil h/2n \rceil))$ where $\sigma(n)$ is the number of function evaluations required to minimize a submodular set function over an n-element ground set and EO is the time needed for an expected cost evaluation. We find the expected cost for $F_{\widehat{\mathbf{p}}}(\cdot)$ in $O(nN)$ by computing the total cost for each realization (takes $O(n)$ time) and then taking the average of N total cost realizations, i.e., sample average approximation. Finally, Theorem 4 of Orlin (2007) shows that $\sigma(n) = O(n^5)$. □

Proof. **(Lemma 6)**

The proof is by induction on $|\mathcal{F}|$. Let $1 \leq k \leq |\mathcal{F}|$ and

$$Y_k = \left\{ \left| Prob_{\mathbf{p}}\{O_k(\mathbf{p})\} - Prob_{\widehat{\mathbf{p}}}\{O_k(\mathbf{p})\} \right| \leq \epsilon' \right\}.$$

For $\mathcal{F} = 2$ the result holds since

$$\begin{aligned}
Prob\{Y_1 \cap Y_2\} &= 1 - Prob\{\overline{Y_1 \cap Y_2}\} \\
&= 1 - Prob\{\overline{Y_1} \cup \overline{Y_2}\} \\
&\geq 1 - (Prob\{\overline{Y_1}\} + Prob\{\overline{Y_2}\}) \quad (\text{since } Prob\{\overline{Y_1} \cup \overline{Y_2}\} \leq Prob\{\overline{Y_1}\} + Prob\{\overline{Y_2}\}) \\
&\geq 1 - 2\delta' \qquad\qquad (\text{since } Prob\{\overline{Y_1}\}, Prob\{\overline{Y_2}\} < \delta').
\end{aligned}$$

Suppose the result is true for $|\mathcal{F}| = k$, i.e., $Prob\left\{ \bigcap_{i=1}^{k} Y_i \right\} \geq 1 - k\delta'$. Let $Y = \bigcap_{i=1}^{k} Y_i$. Then,

$$\begin{aligned}
Prob\{Y \cap Y_{k+1}\} &= 1 - Prob\{\overline{Y \cap Y_{k+1}}\} \\
&= 1 - Prob\{\overline{Y} \cup \overline{Y_{k+1}}\} \\
&\geq 1 - (Prob\{\overline{Y}\} + Prob\{\overline{Y_{k+1}}\}) \; (\text{as } Prob\{\overline{Y} \cup \overline{Y_{k+1}}\} \leq Prob\{\overline{Y}\} + Prob\{\overline{Y_{k+1}}\}) \\
&\geq 1 - (k+1)\delta' \qquad\qquad (\text{as } Prob\{\overline{Y}\} < k\delta' \text{ and } Prob\{\overline{Y_{k+1}}\} < \delta').
\end{aligned}$$

Therefore the result is also true for $|\mathcal{F}| = k+1$, and hence the proof is complete. □

Proof. **(Lemma 7)**

Since $\widehat{A}$ is an optimal appointment vector for $F_{\widehat{\mathbf{p}}}$ there exists $\widehat{\mathbf{X}} \in \Theta$ such that $g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}} = 0$ for all $1 \le k \le n+1$. If $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}| < \epsilon' K'$ then this implies that $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - 0| < \epsilon' K'$, and hence there exists $g \in \partial F_{\mathbf{p}}(\widehat{\mathbf{A}})$ such that $|g_k| < \epsilon' K'$ for all $1 \le k \le n+1$. We now show that $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}| < \epsilon' K'$. We start by taking the difference $|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}|$ term by term and factoring out $\widehat{\mathbf{X}}$ terms by using equation (2).

$$
\begin{aligned}
|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}})| = \Bigg| & \sum_{j=k}^{n} \alpha_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{L}_{kj}(S) \Big(Prob_{\mathbf{p}}\{I_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \Big) \\
& - \alpha_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1} = S\} \Big) \\
& + \sum_{j=k}^{n} \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{T>}_{kj}(S) \Big(Prob_{\mathbf{p}}\{I^{>}_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{>}_j = S\} \Big) \\
& - \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I^{>}_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{>}_{k-1} = S\} \Big) \\
& + \sum_{j=k}^{n} \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{T=}_{kj}(S) \Big(Prob_{\mathbf{p}}\{I^{=}_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{=}_j = S\} \Big) \\
& - \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \sum_{i \in S} \widehat{X}^{T=}_{i\,k-1}(S) \Big(Prob_{\mathbf{p}}\{I^{=}_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{=}_{k-1} = S\} \Big) \\
& + \sum_{j=k}^{n} \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{T>}_{kj}(S) \Big(Prob_{\mathbf{p}}\{I^{>}_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{>}_j = S\} \Big) \\
& - \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I^{>}_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{>}_{k-1} = S\} \Big) \\
& + \sum_{j=k}^{n} \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{M=}_{kj}(S \cup \{j+1\}) \Big(Prob_{\mathbf{p}}\{I^{=}_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{=}_j = S\} \Big) \\
& - \gamma_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I^{=}_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I^{=}_{k-1} = S\} \Big) \Bigg| .
\end{aligned}
$$

The term $-\alpha_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I_{k-1} = S\} \Big)$ disappears since $\sum_{S \in \mathcal{P}^*([k-1])} Prob_{\mathbf{p}}\{I_{k-1} = S\} = 1 = \sum_{S \in \mathcal{P}^*([k-1])} Prob_{\widehat{\mathbf{p}}}\{I_{k-1} = S\}$. By using the triangular inequality we obtain

$$
\begin{aligned}
|g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}| \leq & \left|\sum_{j=k}^{n}\alpha_j \sum_{S\in\mathcal{P}^*([j])} \widehat{X}_{kj}^{L}(S)\Big(Prob_{\mathbf{p}}\{I_j=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j=S\}\Big)\right| \\
&+\left|\sum_{j=k}^{n}\beta_j \sum_{S\in\mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S)\Big(Prob_{\mathbf{p}}\{I_j^{>}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j^{>}=S\}\Big)\right| \\
&+\left|\beta_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I_{k-1}^{>}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^{>}=S\}\Big)\right| \\
&+\left|\sum_{j=k}^{n}\beta_j \sum_{S\in\mathcal{P}^*([j])} \widehat{X}_{kj}^{T=}(S)\Big(Prob_{\mathbf{p}}\{I_j^{=}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j^{=}=S\}\Big)\right| \\
&+\left|\beta_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \sum_{i\in S}\widehat{X}_{i\,k-1}^{T=}(S)\Big(Prob_{\mathbf{p}}\{I_{k-1}^{=}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^{=}=S\}\Big)\right| \\
&+\left|\sum_{j=k}^{n}\gamma_j \sum_{S\in\mathcal{P}^*([j])} \widehat{X}_{kj}^{T>}(S)\Big(Prob_{\mathbf{p}}\{I_j^{>}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j^{>}=S\}\Big)\right| \\
&+\left|\gamma_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I_{k-1}^{>}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^{>}=S\}\Big)\right| \\
&+\left|\sum_{j=k}^{n}\gamma_j \sum_{S\in\mathcal{P}^*([j])} \widehat{X}_{kj}^{M=}(S\cup\{j+1\})\Big(Prob_{\mathbf{p}}\{I_j^{=}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j^{=}=S\}\Big)\right| \\
&+\left|\gamma_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I_{k-1}^{=}=S\}-Prob_{\widehat{\mathbf{p}}}\{I_{k-1}^{=}=S\}\Big)\right|. \qquad (3)
\end{aligned}
$$

We now find an upper bound for $|g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\widehat{\mathbf{p}}}|$ by obtaining an upper bound for each $|\cdot|$ term in equation (3). We do so by using the fact that $\widehat{\mathbf{X}}\in\Theta$ and rewriting some of the probability terms. Note that we will show this for the first and the third terms as the remaining bounds are obtained in a similar manner to either of the first or the third. We start with the first term in equation (3). Let $\mathcal{P}^{\geq}([j])=\{S\in\mathcal{P}^*([j])\mid Prob_{\mathbf{p}}\{I_j=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j=S\}\geq 0\}$ for $j=1,\ldots,n$ and $\mathcal{P}^{<}([j])=\{S\in\mathcal{P}^*([j])\mid Prob_{\mathbf{p}}\{I_j=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j=S\}<0\}$ for $j=1,\ldots,n$. Then by the definition of $\mathcal{P}^{\geq}([\cdot])$ and $\mathcal{P}^{<}([\cdot])$ (equation (4)), the triangular inequality (equation (5) and equation (8)), and the fact that $\widehat{X}_{kj}^{L}(S)=0$ if $k\notin S$ (equation (6)) and $0\leq\widehat{X}_{kj}^{L}(S)\leq 1$ (equation (7)) we obtain

$$
\begin{aligned}
&\left|\sum_{j=k}^{n}\alpha_j \sum_{S\in\mathcal{P}^*([j])} \widehat{X}_{kj}^{L}(S)\Big(Prob_{\mathbf{p}}\{I_j=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j=S\}\Big)\right| \\
&=\Bigg|\sum_{j=k}^{n}\alpha_j \sum_{S\in\mathcal{P}^{\geq}([j])} \widehat{X}_{kj}^{L}(S)\Big(Prob_{\mathbf{p}}\{I_j=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j=S\}\Big) \\
&\qquad+\sum_{j=k}^{n}\alpha_j \sum_{S\in\mathcal{P}^{<}([j])} \widehat{X}_{kj}^{L}(S)\Big(Prob_{\mathbf{p}}\{I_j=S\}-Prob_{\widehat{\mathbf{p}}}\{I_j=S\}\Big)\Bigg| \qquad (4)
\end{aligned}
$$

$$\le \left| \sum_{j=k}^{n} \alpha_j \sum_{S \in \mathcal{P}^{\ge}([j])} \widehat{X}^L_{kj}(S) \Big(Prob_{\mathbf{p}}\{I_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \Big) \right|$$

$$+ \left| \sum_{j=k}^{n} \alpha_j \sum_{S \in \mathcal{P}^{<}([j])} \widehat{X}^L_{kj}(S) \Big(Prob_{\mathbf{p}}\{I_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \Big) \right| \quad (5)$$

$$= \left| \sum_{j=k}^{n} \alpha_j \sum_{S \in \mathcal{P}^{\ge}([j])} \widehat{X}^L_{kj}(S) \Big(\mathbb{1}\{k \in S\} Prob_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \Big) \right|$$

$$+ \left| \sum_{j=k}^{n} \alpha_j \sum_{S \in \mathcal{P}^{<}([j])} \widehat{X}^L_{kj}(S) \Big(\mathbb{1}\{k \in S\} Prob_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \Big) \right| \quad (6)$$

$$\le \left| \sum_{j=k}^{n} \alpha_j \sum_{S \in \mathcal{P}^{\ge}([j])} \Big(\mathbb{1}\{k \in S\} Prob_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \Big) \right|$$

$$+ \left| \sum_{j=k}^{n} \alpha_j \sum_{S \in \mathcal{P}^{<}([j])} \Big(\mathbb{1}\{k \in S\} Prob_{\mathbf{p}}\{I_j = S\} - \mathbb{1}\{k \in S\} Prob_{\widehat{\mathbf{p}}}\{I_j = S\} \Big) \right| \quad (7)$$

$$= \left| \sum_{j=k}^{n} \alpha_j \Big(Prob_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{\ge}([j])\} - Prob_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{\ge}([j])\} \Big) \right|$$

$$+ \left| \sum_{j=k}^{n} \alpha_j \Big(Prob_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} - Prob_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} \Big) \right|$$

$$\le \sum_{j=k}^{n} \alpha_j \left| \Big(Prob_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{\ge}([j])\} - Prob_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{\ge}([j])\} \right| \Big)$$

$$+ \sum_{j=k}^{n} \alpha_j \left| \Big(Prob_{\mathbf{p}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} - Prob_{\widehat{\mathbf{p}}}\{k \in I_j, I_j \in \mathcal{P}^{<}([j])\} \right| \Big) \quad (8)$$

$$\le \sum_{j=k}^{n} \alpha_j \epsilon' + \sum_{j=k}^{n} \alpha_j \epsilon' \le 2\epsilon' \alpha_{\max} n.$$

Similarly,

$$\left| \sum_{j=k}^{n} \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{T>}_{kj}(S) \Big(Prob_{\mathbf{p}}\{I^>_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^>_j = S\} \Big) \right| \le 2\epsilon' \beta_{\max} n,$$

$$\left| \sum_{j=k}^{n} \beta_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{T=}_{kj}(S) \Big(Prob_{\mathbf{p}}\{I^=_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^=_j = S\} \Big) \right| \le 2\epsilon' \beta_{\max} n,$$

$$\left| \sum_{j=k}^{n} \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{T>}_{kj}(S) \Big(Prob_{\mathbf{p}}\{I^>_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^>_j = S\} \Big) \right| \le 2\epsilon' \gamma_{\max} n,$$

$$\left| \sum_{j=k}^{n} \gamma_j \sum_{S \in \mathcal{P}^*([j])} \widehat{X}^{M=}_{kj}(S \cup \{j+1\}) \Big(Prob_{\mathbf{p}}\{I^=_j = S\} - Prob_{\widehat{\mathbf{p}}}\{I^=_j = S\} \Big) \right| \le 2\epsilon' \gamma_{\max} n.$$

We now find an upper bound for the third term in equation (3) by using the definition of $I^>_{k-1}$.

$$\left| \beta_{k-1} \sum_{S \in \mathcal{P}^*([k-1])} \Big(Prob_{\mathbf{p}}\{I^>_{k-1} = S\} - Prob_{\widehat{\mathbf{p}}}\{I^>_{k-1} = S\} \Big) \right|$$

$$
\begin{aligned}
&= \Bigg|\beta_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \Bigg(Prob_{\mathbf{p}}\{I_{k-1}=S \text{ and } P_{i,k-1} > A_k - A_i \,:\, i\in S\} \\
&\qquad\qquad - Prob_{\widehat{\mathbf{p}}}\{I_{k-1}=S \text{ and } P_{i,k-1} > A_k - A_i \,:\, i\in S\}\Bigg)\Bigg| \\
&= \Bigg|\beta_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \sum_{i=1}^{k-1} \Bigg(Prob_{\mathbf{p}}\{I_{k-1}=S \text{ and } P_{i,k-1} > A_k - A_i\}\mathbb{1}\{i\in S\} \\
&\qquad\qquad - Prob_{\widehat{\mathbf{p}}}\{I_{k-1}=S \text{ and } P_{i,k-1} > A_k - A_i\}\mathbb{1}\{i\in S\}\Bigg)\Bigg| \\
&= \Bigg|\beta_{k-1} \sum_{i=1}^{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \Bigg(Prob_{\mathbf{p}}\{I_{k-1}=S \text{ and } P_{i,k-1} > A_k - A_i\}\mathbb{1}\{i\in S\} \\
&\qquad\qquad - Prob_{\widehat{\mathbf{p}}}\{I_{k-1}=S \text{ and } P_{i,k-1} > A_k - A_i\}\mathbb{1}\{i\in S\}\Bigg)\Bigg| \\
&= \Bigg|\beta_{k-1} \sum_{i=1}^{k-1} \Bigg(Prob_{\mathbf{p}}\{i\in I_{k-1} \text{ and } P_{i,k-1} > A_k - A_i\} - Prob_{\widehat{\mathbf{p}}}\{i\in I_{k-1} \text{ and } P_{i,k-1} > A_k - A_i\}\Bigg)\Bigg| \\
&= \Bigg|\beta_{k-1} \sum_{i=1}^{k-1} \Bigg(Prob_{\mathbf{p}}\{i\in I^{>}_{k-1}\} - Prob_{\widehat{\mathbf{p}}}\{i\in I^{>}_{k-1}\}\Bigg)\Bigg| \\
&\le \beta_{k-1} \sum_{i=1}^{k-1} \Bigg| Prob_{\mathbf{p}}\{i\in I^{>}_{k-1}\} - Prob_{\widehat{\mathbf{p}}}\{i\in I^{>}_{k-1}\}\Bigg| \\
&\le \beta_{k-1} \sum_{i=1}^{k-1} \epsilon' \le \epsilon'\beta_{\max} n.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\Bigg|\beta_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \sum_{i\in S} \widehat{X}^{T=}_{i\,k-1}(S) \Bigg(Prob_{\mathbf{p}}\{I^{=}_{k-1}=S\} - Prob_{\widehat{\mathbf{p}}}\{I^{=}_{k-1}=S\}\Bigg)\Bigg| &\le \epsilon'\beta_{\max} n, \\
\Bigg|\gamma_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \Bigg(Prob_{\mathbf{p}}\{I^{>}_{k-1}=S\} - Prob_{\widehat{\mathbf{p}}}\{I^{>}_{k-1}=S\}\Bigg)\Bigg| &\le \epsilon'\gamma_{\max} n, \\
\gamma_{k-1} \sum_{S\in\mathcal{P}^*([k-1])} \Bigg(Prob_{\mathbf{p}}\{I^{=}_{k-1}=S\} - Prob_{\widehat{\mathbf{p}}}\{I^{=}_{k-1}=S\}\Bigg)\Bigg| &\le \epsilon'\gamma_{\max} n.
\end{aligned}
$$

Therefore we can bound $|g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\widehat{\mathbf{p}}})|$ from above:

$$
|g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}},\widehat{\mathbf{A}})_{\widehat{\mathbf{p}}})| \le \epsilon' n(2\alpha_{\max} + 6\beta_{\max} + 6\gamma_{\max}) \;\; (1 \le k \le n+1).
$$

Since the cost coefficients $(\mathbf{u},\mathbf{o})$ are assumed to be α-monotone we have $0 \le \alpha_i \le o_{\max}$, $\beta_i \le o_{\max}$ and $\gamma_i \le u_{\max} + o_{\max}$. Therefore $2\alpha_{\max} + 6\beta_{\max} + 6\gamma_{\max} \le (14o_{\max} + 6u_{\max})$ so we can take $K' = n(14o_{\max} + 6u_{\max})$. We also determine $|\mathcal{F}|$, the maximum number of events we need to compute

$|g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\mathbf{p}} - g_k(\widehat{\mathbf{X}}, \widehat{\mathbf{A}})_{\widehat{\mathbf{p}}})|$ for all k. For each k, we have $\big(5(n-k+1)+4(k-1)\big)$ therefore at most $5n$ events. Since $k \le n+1$ we have $|\mathcal{F}| = (n+1)(5n) = 5n^2+5$. This completes the proof. □

Proof. **(Lemma 9)**

Fix $\mathbf{A}$. Let $h(\mathbf{p}) = F(\mathbf{A}|\mathbf{p}) = \sum_{i=1}^{n}[o_i(C_i - A_{i+1})^+ + u_i(A_{i+1} - C_i)^+]$. We claim that h is convex. Recall that by Lemma 3.3.1 of Begen (2010) , we can rewrite $F(\mathbf{A}|\mathbf{p})$ and hence $h(\mathbf{p})$ as

$$h(\mathbf{p}) = F(\mathbf{A}|\mathbf{p}) = \sum_{i=1}^{n}\left[\alpha_i(C_i - A_{i+1}) + \beta_i(C_i - A_{i+1})^+ + \gamma_i(\max\{C_i, A_{i+1}\} - \sum_{k=1}^{i} p_k)\right]$$

for any $\alpha_i \in \mathbb{R}$ $(1 \le i \le n)$ where $\beta_i = (o_i - \alpha_i)$ and $\gamma_i = [(u_i + \alpha_i) - (u_{i+1} + \alpha_{i+1})]$. Recall that $C_i = \max_{k \le i}\{A_k + \sum_{t=k}^{i} p_t\}$ (by the Critical Path Lemma 4.1 of Begen and Queyranne (2011)) so C_i is convex in $\mathbf{p}$. By α-monotonicity $\alpha_i, \beta_i \ge 0$. Hence the terms $\alpha_i(C_i - A_{i+1})$ and $\beta_i(C_i - A_{i+1})^+$ are convex in $\mathbf{p}$. Furthermore, the term $\gamma_i(\max\{C_i, A_{i+1}\} - \sum_{k=1}^{i} p_k)$ is convex in $\mathbf{p}$. Therefore $h(\mathbf{p})$ is convex.

Recall that $\nu = \min\{u_1, u_2, ..., u_n, o_1, o_2, ..., o_n\}$ and $\widetilde{C}_i$'s are the completion times, but they are deterministic since we are using expected values, $\widetilde{p}_i$'s, for the processing times. We next show that $F_{\mathbf{p}}(\mathbf{A}) \ge \widetilde{f}(\mathbf{A})$ by applying Jensen's inequality to $h(\mathbf{p})$ and applying Lemma 3.3.1 of Begen (2010) to $F(\mathbf{A}|\widetilde{\mathbf{p}})$:

$$\begin{aligned}
F_{\mathbf{p}}(\mathbf{A}) &= E_{\mathbf{p}}[h(\mathbf{p})] \\
&\ge F(\mathbf{A}|\mathbf{E}\mathbf{p}) = F(\mathbf{A}|\widetilde{\mathbf{p}}) = \sum_{i=1}^{n}\left[\alpha_i(\widetilde{C}_i - A_{i+1}) + \beta_i(\widetilde{C}_i - A_{i+1})^+ + \gamma_i(\max\{\widetilde{C}_i, A_{i+1}\} - \sum_{k=1}^{i}\widetilde{p}_k)\right] \\
&= \sum_{i=1}^{n}[o_i(\widetilde{C}_i - A_{i+1})^+ + u_i(A_{i+1} - \widetilde{C}_i)^+] \\
&\ge \sum_{i=1}^{n}\nu[(\widetilde{C}_i - A_{i+1})^+ + (A_{i+1} - \widetilde{C}_i)^+] = \widetilde{f}(\mathbf{A}).
\end{aligned}$$

Next we obtain $\widetilde{\mathbf{A}} \in \arg\min_{\mathbf{A}} \widetilde{f}(\mathbf{A})$. Note that $\widetilde{f}(\mathbf{A}) \ge 0$ for all $\mathbf{A}$. Set $\widetilde{A}_1 = 0, ..., \widetilde{A}_{i+1} = \sum_{j=1}^{i}\widetilde{p}_j, ..., \widetilde{A}_{n+1} = \sum_{j=1}^{n}\widetilde{p}_j$, i.e., $\widetilde{A}_{i+1} - \widetilde{A}_i = \widetilde{p}_i$ and $\widetilde{A}_{i+1} = \sum_{k=1}^{i}\widetilde{p}_k$ for $1 \le i \le n$. Then $\widetilde{A}_{i+1} = \widetilde{C}_i$ for all $i = 1, ..., n$ and $\widetilde{f}(\widetilde{\mathbf{A}}) = 0$. Therefore $\widetilde{\mathbf{A}} = (0, \widetilde{p}_1, ..., \sum_{j=1}^{n}\widetilde{p}_j)$ is indeed optimal for $\widetilde{f}$.

We next show $F_{\mathbf{p}}(\mathbf{A}) \geq \frac{\nu}{n}||\mathbf{A}-\widetilde{\mathbf{A}}||_1$ by showing $\widetilde{f}(\mathbf{A}) \geq \frac{\nu}{n}||\mathbf{A}-\widetilde{\mathbf{A}}||_1$. Note that $\sum_{i=1}^{n}[(\widetilde{C}_i - A_{i+1})^+ + (A_{i+1}-\widetilde{C}_i)^+] = \sum_{i=1}^{n}|(\widetilde{C}_i - A_{i+1})|$, and the result would follow if we show $\sum_{i=1}^{j}|(\widetilde{C}_i - A_{i+1})| \geq |A_{j+1}-\widetilde{A}_{j+1}| = |A_{j+1} - \sum_{t=1}^{j}\widetilde{p}_t|$ for all $j=1,2,..,n$. We now show $\sum_{i=1}^{j}|(\widetilde{C}_i - A_{i+1})| \geq |A_{j+1} - \sum_{t=1}^{j}\widetilde{p}_t|$ for all $j=1,2,..,n$. We distinguish two cases. First, suppose that $A_{j+1} \leq \sum_{t=1}^{j}\widetilde{p}_t$. Since $\sum_{t=1}^{j}\widetilde{p}_t \leq \widetilde{C}_j$, we have $A_{j+1} \leq \widetilde{C}_j$. Therefore

$$\begin{aligned}\sum_{i=1}^{j}|(\widetilde{C}_i - A_{i+1})| &\geq |(\widetilde{C}_j - A_{j+1})| \\ &\geq |\sum_{t=1}^{j}\widetilde{p}_t - A_{j+1}| = |\widetilde{A}_{j+1} - A_{j+1}|\end{aligned}$$

The second case is where $A_{j+1} > \sum_{t=1}^{j}\widetilde{p}_t$. Then

$$\begin{aligned}\sum_{i=1}^{j}|(\widetilde{C}_i - A_{i+1})| &\geq \sum_{i=1}^{j}(A_{i+1}-\widetilde{C}_i)^+ \\ &= \max(\widetilde{C}_j, A_{j+1}) - \sum_{t=1}^{j}\widetilde{p}_t \\ &\geq A_{j+1} - \sum_{t=1}^{j}\widetilde{p}_t = |A_{j+1} - \widetilde{A}_{j+1}|\end{aligned}$$

where the first equality follows from a property shown in the proof of Lemma 3.3.1 of Begen (2010). Hence we obtain

$$\sum_{i=1}^{j}|(\widetilde{C}_i - A_{i+1})| \geq |A_{j+1}-\widetilde{A}_{j+1}| \text{ for all } 1 \leq j \leq n.$$

Therefore for every $j=1,...,n$

$$\sum_{i=1}^{n}|(\widetilde{C}_i - A_{i+1})| \geq \sum_{i=1}^{j}|(\widetilde{C}_i - A_{i+1})| \geq |A_{j+1}-\widetilde{A}_{j+1}|$$

and hence

$$\begin{aligned}n\widetilde{f}(\mathbf{A}) &= n\nu\Big(\sum_{i=1}^{n}[(\widetilde{C}_i - A_{i+1})^+ + (A_{i+1}-\widetilde{C}_i)^+]\Big) \\ &= n\nu\Big(\sum_{i=1}^{n}|\widetilde{C}_i - A_{i+1}|\Big) \\ &\geq \ \nu\,||\mathbf{A}-\widetilde{\mathbf{A}}||_1\end{aligned}$$

as desired. Therefore $F_{\mathbf{p}}(\mathbf{A}) \geq \frac{\nu}{n}||\mathbf{A}-\widetilde{\mathbf{A}}||_1$. This completes the proof. □

Proof. **(Lemma 12)**

Let $L = f(y^*)/\lambda$. Consider the norm l_1 ball $B = B(\widetilde{y}, L)$, then $y^* \in B(\widetilde{y}, L) = \{y : \lambda||y^* - \widetilde{y}|| \leq f(y^*)\}$. The subgradient inequality at $\widehat{y}$ combined with Cauchy-Schwartz inequality yields $f(\hat{y}) - f(y^*) \leq \alpha||\widehat{y} - y^*||_1$ (since Cauchy-Schwartz inequality also holds for l_1 norm). We also have

$$\begin{aligned} ||\hat{y} - y^*||_1 &\leq ||\widehat{y} - \widetilde{y}||_1 + ||\widetilde{y} - y^*||_1 \\ &\leq f(\hat{y})/\lambda + L \\ &= f(\hat{y})/\lambda + f(y^*)/\lambda. \end{aligned}$$

So we obtain $f(\widehat{y}) - f(y^*) \leq \alpha(f(\widehat{y})/\lambda + f(y^*)/\lambda)$ and hence $f(\widehat{y}) \leq f(y^*)(\lambda + \alpha)/(\lambda - \alpha)$. If we choose $\alpha \leq \lambda\epsilon/3$ the result follows. □

Proof. **(Lemma 13)**

If $|g_k| < \epsilon\nu/(3(n+1)n)$ for all $1 \leq k \leq n+1$ then $||g||_1 \leq \epsilon\nu/(3n)$. We then directly apply Lemma 12 with $\overline{f}(\mathbf{A}) = \frac{\nu}{n}||\mathbf{A} - \tilde{\mathbf{A}}||_1$ $\big(F_{\mathbf{p}}(\mathbf{A}) \geq \frac{\nu}{n}||\mathbf{A} - \widetilde{\mathbf{A}}||_1$ by Lemma 9$\big)$ and $\alpha = \epsilon\nu/(3n)$ obtain the desired result. □