Stochastic Differential Games and Optimization Problems with Controlled Point Process Arrivals[1]

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Abstract

There is a very large literature on applications of stochastic control of jump diffusions and a smaller literature on such games. In many applications it is natural to assume that the arrival intensity is controlled, but except for two long-forgotten papers the literature instead assumes that it is the jump sizes that are controlled. The more natural assumption is typically avoided because a failed Lipschitz condition means that the classical existence and uniqueness proofs cannot be used. We here derive an asymptotic Markov equilibrium of the game with controlled jump intensities and show that it, at least in an example, is very similar to the Markov equilibrium of an analog game with controlled jump sizes. The paper thus makes two contributions: It supplies a way to solve some optimization problems and games with controlled jump intensities and it shows that the commonly used formulation with controlled jump sizes is quite defensible for at least some classes of games.

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Key Words: Stochastic control, Differential games with jump diffusions, Point processes with evolving intensities.

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I. INTRODUCTION

Suppose that a consumer is trying to decide how hard to search for lower prices or that the firms in an industry want to know how much each should invest in ongoing productivity improvement. Such problems, and many others, can be modeled as control problems or differential games in which the state variables follow Point processes as the consumer finds a lower price or the firms' R&D projects bear fruit. A key step in the model formulation is to decide exactly how the decision makers affect the state dynamics: In particular, whether they control the rate at which jumps (low prices or R&D successes) arrive or the size of those jumps. This choice depends on the application, but in many cases, including the price search and the productivity investment examples above, it seems natural to assume that players control the arrival intensity rather than the jump size. And yet, at least as far as we know, all published applications assume that it is the jump sizes, rather than the arrival intensities that are controlled.

A (the?) major reason for not looking at controlled arrival rates is that the resulting state dynamics do not satisfy Lipschitz conditions, such that one cannot apply the classical existence and uniqueness results. The fact that Jacod and Protter (1982) and Protter (1983) identified a set of conditions under which this class of problems can be solved has made no difference.[2] These results are simply not used: Protter himself does not cite them in his textbook on Stochastic Integration (2005) and Øksendal (2022) believes that the two assumptions – controlled jump

[2] More recently, Hernandez-Hernandez et al. have independently solved the problem in a 2019 working paper in which they derive Bellman style verification results through a martingale approach. However, as far as we know, also that paper has yet to have influence. We use both their results and those of Jacod-Protter in Section III. There are also economics models in which agents may search with higher or lower intensity, but these are simpler and not subject to the technical problems analyzed in this paper, because search ends with the first arrival.

sizes and controlled arrival intensities - give very similar results and thus the choice makes little difference.

In the present paper, we show how to find an asymptotic Markov equilibrium of a specific, but reasonably general, stochastic differential game with controlled jump intensities. The analysis shows that we can solve dynamic optimization problems and games with controlled jump intensities if we are willing to accept results that are subject to a couple of weak restrictions (theoretical bounds on the control variables and solutions that are unique in law rather than path wise unique). However, we also compare the equilibrium to that obtained for a similar problem with controlled jump sizes and find that the qualitative properties of the two equilibria are relatively close to each other. So we show how to overcome a barrier to the use of an often more natural formulation and suggest that its use might not make a major difference.

Literature

Since players in differential games find their strategies by solving control problems, the two literatures share a lot of foundations. Isaacs (1965) is generally considered to be the pioneering text in the study of (deterministic) differential games and Kushner and Chamberlain (1969) were probably the first to look at stochastic differential games. Since then, the literature has looked at the zero-sum case (e. g. Fleming and Souganitis, 1989), the cooperative case (e. g. Yeung and Petrosyan, 2006), and the non-cooperative variable-sum case. Most work on stochastic differential games fall in the latter case and represent the stochasticity as a Wiener process with controlled drift. It has, however, become clear that Point processes are more natural in a wide range of applications.

One of the first control theory models using Point processes is due to Merton (1971) and Øksendal and Sulem (2019) is a comprehensive textbook devoted to the topic. There are fewer game theory applications, but Wernerfelt (1988) gives conditions for existence of several equilibrium concepts, in particular Markov.

We formulate and solve investment games with controlled jump sizes and controlled jump intensities in Sections II and III, respectively. Comparisons and concluding comments are made in Section IV.

II. GAME WITH CONTROLLED JUMP SIZES

In this Section we find the Markov equilibrium of a two-person stochastic differential game in which the state variables follow Poisson processes and players control the jump sizes. We assume that the game is symmetric because we want to show that the two formulations, that with controlled jump sizes and that with controlled jump intensities, share the property that initially identical players can be expected to differ more and more, thus suggesting that the two formulations at least in this case yield similar qualitative predictions. We analyze a linear–quadratic model for two reasons: First, we want to get an explicit solution, and second, we can look at it as a Taylor approximation to a much larger set of games. Since the Markov equilibrium is defined as a situation in which both players simultaneously follow their optimal control strategy, we can find the value functions simply by solving the Hamilton-Jacoby-Bellman (HJB) equation from Theorem 5.1 in Øksendal and Sulem (2019). To get an explicit solution, we guess that it is a low degree polynomial and find the specific coefficients from the terminal conditions and the HJB equation. Once we have them, the value - and policy functions allow us to characterize several interesting properties of the equilibrium.

Formally, two players, X and Y, compete in continuous time. The outcome at time t depends on both their stocks, the state variables $(x_t, y_t) \epsilon R^{+2}$, $x_0 = y_0 = 0$. At time $T < \infty$, the game terminates, and all stocks become worthless. Until then, the stocks of X and Y grow according to independent Poisson processes with intensity λ and jumps of size (u_t, v_t), respectively. The players choose the size of the jumps in their stocks as twice differentiable functions of the state variables and time[3] such that X 's strategy is $u(x_t, y_t, t)$ and Y's is $v(y_t, x_t, t)$.[4] We will represent the stochastic processes x_t, y_t, u_t, and v_t by their "right continuous with left limits" versions (commonly known as "cadlag" from French), such that the size of the jump at t is $x_t - x_{t-}$ etc..

The description above implies that the stocks grow over time according to

$$(1) \quad x_t = \int_0^t \boldsymbol{u}(x_{s-}, y_{s-}, s-) dN_x[\lambda] ds,$$

$$(2) \quad y_t = \int_0^t \boldsymbol{v}(y_{s-}, x_{s-}, s-) dN_Y[\lambda] ds$$

where $N_x[\lambda]$ and $N_Y[\lambda]$ are independent Poisson processes with intensity λ and the integrals sum the jumps at arrivals between 0 and t.[5] We are going to need *(1)* and *(2)* to satisfy an "at most linear growth" condition such that they have unique solutions (Theorem 1.19 in Øksendal and Sulem, 2019). This condition will hold if the jump sizes are linear or concave functions of the state and we will proceed as if that is the case and then come back and check after we have derived the equilibrium strategies.

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3 While the control (the jump size) chosen at t is a C^2 function of t and the state, a jump in the state will typically cause a discontinuous change in the chosen jump size. So the value of the control variable will not follow a continuous path over time.
4 We assume that both players observe the state at all times and that the payoff functions are common knowledge.
5 The problem formulation suggests a filtration induced by the states (x_s, y_s, s), and we will conduct the analysis in that context.

We assume that X and Y maximize the second order polynomials $\int_0^T e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t + \alpha x_t - \beta y_t + \sigma - \frac{1}{2}u_t^2 - \pi u_t]dt$ and $\int_0^T e^{-\rho t}[\gamma y_t^2 - \eta x_t^2 - \theta x_t y_t + \alpha y_t - \beta x_t + \sigma - \frac{1}{2}v_t^2 - \pi v_t]dt$, respectively. We can, for example, think of the game as describing a duopoly in which two firms invest in various assets that improve their competitive positions. To keep the expressions shorter and make the argument more transparent, we start by looking at the case in which $\alpha = \beta = \sigma = \pi = 0$, $(\rho, \eta, \theta) \epsilon R^{+3}$, and $\gamma > \eta$. This means that X and Y maximize $\int_0^T e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - \frac{1}{2}u_t^2]dt$ and $\int_0^T e^{-\rho t}[\gamma y_t^2 - \eta x_t^2 - \theta x_t y_t - \frac{1}{2}v_t^2]dt$, respectively. (The extension to the more general case is easy and we will return to it later.)

If *(1)* and *(2)* satisfy the above-mentioned "at most linear growth" condition, we can use Theorem 5.1[6] in Øksendal and Sulem (2019)[7], according to which the players want to find C^2 value functions $W_x(x_t, y_t, t)$, $W_y(y_t, x_t, t)$ from $R^{+2} \times [0, T] \rightarrow R$ that satisfy the HJB equations:[8]

$$(3)\ Max_{u(x,y,t)}\{e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - \tfrac{1}{2}u_t^2] + \partial W_x(x_t, y_t, t)/\partial t + \lambda[W_x(x_{t-} + u_{t-}, y_{t-}, t-) - W_x(x_t, y_{t-}, t-)] + \lambda[W_x(x_{t-}, y_{t-} + v_{t-}^*, t-) - W_x(x_{t-}, y_{t-}, t-)]\} = 0,\ W_x(x_T, y_T, T) = 0, \text{ for player } X,$$

$$(4)\ Max_{v(y,x,t)}\{e^{-\rho t}[\gamma y_t^2 - \eta x_t^2 - \theta x_t y_t - \tfrac{1}{2}v_t^2] + \partial Wy(y_t, x_t, t)/\partial t + \lambda[W_y(y_{t-} + v_{t-}, x_{t-}, t-) - W_y(y_{t-}, x_{t-}, t-)] + \lambda[W_y(y_{t-}, x_{t-} + u_{t-}^*, t-) - W_y(y_{t-}, x_{t-}, t-)]\} = 0,\ W_y(y_T, x_T, T) = 0, \text{ for player } Y.$$

The first terms in *(3)* and *(4)* are the payoffs at t and the remaining terms make up a first order Taylor approximation to the dynamics of W_x and W_y. The Markov equilibrium controls u_{t-}^* and v_{t-}^* are therefore given by the first order conditions:

[6] Theorem 5. 1 is about a control problem but the extension to differential games is immediate since we get a Markov equilibrium if both players follow their optimal control strategy. The interested reader can see examples in Chapter 6 of Øksendal and Sulem (2019) and in Mataramvura and Øksendal (2008).
[7] Chapter 1 of this book presents the underlying theory of Levy processes.
[8] Since the value functions are C^2, we can use the infinitesimal generators of x_t and y_t to describe the movements of the value functions in infinitesimal time-intervals.

$$(5)\ u_{t-}^{*}(x_{t-}, y_{t-}, t-) = e^{\rho t-}\lambda \partial W_x(x_{t-}, y_{t-}, t-)/\partial x,$$

$$(6)\ v_{t-}^{*}(y_{t-}, x_{t-}, t-) = e^{\rho t-}\lambda \partial W_y(y_{t-}, x_{t-}, t-)/\partial y$$

Since the players face symmetric problems, $W_x(\,) = W_y(\,)$ and we can drop the x and y subscripts, use $W(\,)$ for the value function, and henceforth focus on X. Substitution of *(5)* and *(6)* into *(3)* allows us to rewrite the HJB equation as:

$$(7)\ e^{-\rho t}(\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t) - e^{\rho t}\lambda^2[\partial W(x_{t-}, y_{t-}, t-)/\partial x]^2/2 + \partial W(x_t, y_t, t)/\partial t +$$
$$\lambda[W(x_{t-} + e^{\rho t}\lambda \partial W(x_{t-}, y_{t-}, t-)/\partial x, y_{t-}, t-) - W(x_{t-}, y_{t-}, t-)] + \lambda[W(x_{t-}, y_{t-} + e^{\rho t}\lambda \partial W(y_{t-}, x_{t-}, t-)/\partial y, t-) -$$
$$W(x_{t-}, y_{t-}, t-)] = 0,\ \ W(x_T, y_T, T) = 0$$ [9]

We will guess that the solution to *(7)* has the form:[10]

$$(8)\ W^g(x_t, y_t, t) \equiv e^{-\rho t}[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t + f(t)x_t y_t],$$
$$a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = 0.$$

Using that the equilibrium strategies are symmetric, this gives:

$$(9)\ u_{t-}^{g} = \lambda[2a(t)x_{t-} + b(t) + f(t)y_{t-}]$$

$$(10)\ v_{t-}^{g} = \lambda[2a(t)y_{t-} + b(t) + f(t)x_{t-}]$$

$$(11)\ \lambda[W^g(x_{t-} + \lambda 2a(t)x_{t-} + \lambda b(t) + \lambda f(t)y_{t-}, y_{t-}, t-) - W^g(x_{t-}, y_{t-}, t-)] =$$
$$e^{-\rho t}\lambda^2[a(t)\lambda + 1]\{4a(t)^2x_{t-}^2 + 4a(t)b(t)x_{t-} + b(t)^2 + f(t)^2y_{t-}^2 + 2b(t)f(t)y_{t-} + 4a(t)f(t)x_{t-}y_{t-}\}.$$

$$(12)\ \lambda[W^g(x_{t-}, y_{t-} + \lambda 2a(t)y_{t-} + \lambda b(t) + \lambda f(t)x_{t-}, t-) - W^g(x_{t-}, y_{t-}, t-)] =$$
$$e^{-\rho t}\lambda^2\{\lambda d(t)f(t)^2x_{t-}^2 + [\lambda b(t)d(t) + e(t)]f(t)x_{t-} + b(t)[\lambda b(t)d(t) + e(t)] + 4a(t)d(t)[\lambda a(t) + 1]y_{t-}^2 +$$
$$2[b(t)d(t) + 2\lambda a(t)b(t)d(t) + a(t)e(t)]y_{t-} + 2d(t)f(t)[2\lambda a(t) + 1]x_{t-}y_{t-}\}.$$

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[9] Since v_t^* is in *(3)* and u_t^* is in *(4)*, two HJB equations are coupled. We can, however, write v_t^* in terms of W_x as $e^{\rho t-}\lambda \partial W_x(y_{t-}, x_{t-}, t-)/\partial y$ and u_t^* in terms of W_y as $e^{\rho t-}\lambda \partial W_y(x_{t-}, y_{t-}, t-)/\partial x$ therefore decouple the two equations.

[10] The superscript g indicates that $W^g(\,)$ is a guess.

So if our guess is correct, *(9)* and *(10)* show that the jump sizes are linear in the state and therefore that *(1)* and *(2)* have unique solutions. Rewriting the HJB equation in terms of the coefficients in $W^g(x_t, y_t, t)$ gives:

$$(13)\ \gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - \lambda^2[2a(t)x_{t-} + b(t) + f(t)y_{t-}]^2/2 + [a'(t) - \rho a(t)]x_t^2 + [b'(t) - \rho b(t)]x_t + [c'(t) - \rho c(t)] + [d'(t) - \rho d(t)]y_t^2 + [e'(t) - \rho e(t)]y_t + [f'(t) - \rho f(t)]x_{t-}y_t + \lambda^2[a(t)\lambda + 1]\{4a(t)^2x_{t-}^2 + 4a(t)b(t)x_{t-} + b(t)^2 + f(t)^2y_{t-}^2 + 2b(t)f(t)y_{t-} + 4a(t)f(t)x_{t-}y_{t-}\} + \lambda^2\{\lambda d(t)f(t)^2x_{t-}^2 + [\lambda b(t)d(t) + e(t)]f(t)x_{t-} + b(t)[\lambda b(t)d(t) + e(t)] + 4a(t)d(t)[\lambda a(t) + 1]y_{t-}^2 + 2[b(t)d(t) + 2\lambda a(t)b(t)d(t) + a(t)e(t)]y_{t-} + 2d(t)f(t)[2\lambda a(t) + 1]x_{t-}y_{t-}\} = 0.$$

$$a(T) = b(T) = c(T) = d(T) = e(T) = 0.$$

$W^g(x_t, y_t, t)$ solves *(7)* if the constant term and the coefficients on x_t^2, x_t, y_t^2, y_t and x_ty_t all are zero at T. This means that the coefficients in $W^g(x_t, y_t, t)$ solve the following differential equations:

$$(14)\ a'(t) = -\gamma - 2a(t)^2\lambda^2[1 + 2a(t)\lambda] - \lambda^3 d(t)f(t)^2 + \rho a(t),\ a(T) = 0$$

$$(15)\ b'(t) = [-4\lambda^3 a(t)^2 - 2\lambda^2 a(t) + \lambda^3 d(t)f(t) + \rho]b(t) + \lambda^3 e(t)f(t),\ b(T) = 0$$

$$(16)\ c'(t) = -\lambda^2 b(t)[\lambda a(t)b(t) + b(t)/2 + \lambda b(t)d(t) + e(t)] + \rho c(t),\ c(T) = 0$$

$$(17)\ d'(t) = \eta - 4\lambda^2 a(t)d(t)[a(t)\lambda + 1] - \lambda^2 f(t)^2[a(t)\lambda + ½] + \rho d(t),\ d(T) = 0$$

$$(18)\ e'(t) = -b(t)[2\lambda a(t) + 1][2d(t) + f(t)] + 2a(t)e(t)] + \rho e(t),\ e(T) = 0$$

$$(19)\ f'(t) = \theta - 2\lambda^2 f(t)[2a(t)\lambda + 1][a(t) + d(t)] + \rho f(t),\ f(T) = 0$$

Since *(14)* - *(19)* is a system of ODEs with Lipschitz continuous right-hand sides, a unique solution exists (see e. g. Cid, 2003, and further references cited there), and we can conclude that:

Proposition 1: *Suppose that all jumps arrive with intensity* λ *and that these are controlled by players X and Y, respectively, then:*

(1. 1) The value functions $W^*(x_t, y_t, t)$ *and* $W^*(y_t, x_t, t)$ *are* $e^{-\rho t}[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t + f(t)x_ty_t]$ *and* $e^{-\rho t}[a(t)y_t^2 + b(t)y_t + c(t) + d(t)x_t^2 + e(t)x_t + f(t)x_ty_t]$*, respectively.*

(1. 2) The Markov equilibrium strategies are $u_{t-}{*}= \lambda[2a(t)x_{t-} + b(t) + f(t)y_{t-}]$ *and* $v_{t-}{*}= \lambda[2a(t)y_{t-} + b(t) + f(t)x_{t-}]$.

So the value functions are second order polynomials, and the policy functions are proportional to the arrival intensity and linear in both players' stocks.

Since Proposition 1 gives an explicit solution to the game, we can quite easily derive several interesting corollaries.

Corollary 1. 1: $a(t) > 0$ *and* $f(t) < 0$.

Proof: From *(14):* First, since $a'(t) < 0$ for t close to T, $a(t) > 0$ in that neighborhood. Second, because $a'(s) < 0$ if $a(s) = 0$, $a(s)$ cannot change sign and is therefore positive for all $t < T$. Similarly from *(19):* Since $f'(t) > 0$ for t close to T, $f(t) < 0$ in that neighborhood and because $f'(t) > 0$ if $f(t) = 0$, $f(t)$ cannot change sign and is therefore negative for all $t < T$.

QED

Corollary 1. 2: *If* $x_0 > y_0$*, and* $E_{t=k}(z_s)$ *denotes the expectation, taken at time* $k < s$*, of the period* s *value of* z*, then* $E_{t=k}(x_s/y_s)$ *grows with* s.

Proof: Suppose again that X is ahead at time h. The players are equally likely to get the next arrival at time $h + i$, but the size of a player's arrival and post arrival stocks are proportional to his or her stock. So if X gets the arrival at $h + i$, we can write x_{h+i} as rx_h whereas Y's $h + i$ stock would be ry_h if it gets the first arrival. Therefore $E_{t=h}(x_{h+i}/y_{h+i}) = ½ [rx_h/y_h + x_h/(ry_h)] = ½(x_h/y_h)[(r^2 + 1)/r]$ which is larger than x_h/y_h for all $r \neq 1$.

QED

Corollary 1. 3: $E_{t=0}|x_s - y_s|$ *grows with s.*

Proof: Suppose that X is ahead at time h in the sense that $x_h > y_h$. In that case (since $\lambda[2a(t)x_h + b(t) + f(t)y_h] > \lambda[2a(t)y_h + b(t) + f(t)x_h]$ when $x_h > y_h$), $u_{t-}^* > v_{t-}^*$ and X invests more. The players are equally likely to get the next arrival at time $h + i$, but since X will invest more, $|x_h - y_h|$ will grow more if it gets the arrival than if Y does. The expected value of $|x_{h+i} - y_{h+i}|$ is therefore larger than $|x_h - y_h|$.The same mechanism applies for all later arrivals, and if Y gets the first arrival. So while $E_{t=0}(x_s - y_s) = 0$, $E_{t=0}|x_s - y_s|$ is strictly positive and grows with s.

QED

Corollary 1. 4: *The probability that $x_s - y_s$ changes sign decreases with s.*

Proof: Suppose that $x_s > y_s$. The players are equally likely to get the next arrival but X's will be larger and by Corollary 1. 3, the expected difference in sizes grows with $x_s – y_s$ and thus with time.

QED

Corollaries 1.2 – 1.4 suggest that a player who is ahead is more likely to increase their lead, that the players are expected to grow more different over time, and that the probability of a lead change decreases with time.

III. GAME WITH CONTROLLED JUMP INTENSITIES

This Section contains the main theoretical result of the paper. We formulate a two-player stochastic differential game in which the state dynamics are governed by Point processes and players control their jump intensities. We then use Corollary 31 from Jacod and Protter (1982) and Corollary 3.11 from Protter (1983) to characterize an asymptotic Markov equilibrium. To facilitate comparison with the case studied in Section II, we keep the models and the notation as similar as possible. The game is also here linear-quadratic, we guess a polynomial value function and find the coefficients as solutions to ordinary differential equations. Given this, we get value - and policy functions that are very similar to those in Section II and can show that close analogues of the four Corollaries hold. There are only three technical differences between the models. First, and most importantly, the players' stocks grow in fixed jump sizes but with controlled intensities. Second, the equilibria in this model are unique in law, a subtly weaker form of uniqueness than the pathwise uniqueness used in Section II. Third, the controls are bounded for a reason to be described below.

Formally, the game again starts at $x_0 = y_0 = 0$ and terminates when all stocks become worthless at $T < \infty$. The players' stocks grow in jumps of size $\varphi > 0$ that arrive according to non-homogeneous Point processes with intensities $u(x_{t-}, y_{t-}, t-)$ for X and $v(y_{t-}, x_{t-}, t-)$ for Y. The players choose the intensities as C^2 Markov controls bounded by the positive constant U; X 's strategy is $u(x_t, y_t, t) \leq U$ and Y's is $v(y_t, x_t, t) \leq U$. The stocks therefore develop over time according to:

(20) $x_t = \int_0^t dM_x[u(x_{s-}, y_{s-}, s-)]ds$

(21) $y_t = \int_0^t dM_y[v(y_{s-}, x_{s-}, s-)]ds$

where $M_x[u(x_{s-}, y_{s-}, s-)]$ and $M_y[v(y_{s-}, x_{s-}, s-)]$ are Point processes with jumps of size φ and intensities $u(x_{s-}, y_{s-}, s-)$ and $v(y_{s-}, x_{s-}, s-)$. X and Y maximize $\int_0^T e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - ½u_t^2]dt$

and $\int_0^T e^{-\rho t}[\gamma y_t^2 - \eta x_t^2 -- \theta x_t y_{t-} - \frac{1}{2} v_t^2]dt$, respectively, and we continue to assume that $(\rho, \eta, \theta) \epsilon R^{+3}$, and $\gamma > \eta$.

The bound on the controls makes the problem very difficult. Depending on U and the realizations of *(1)* and *(2)* the state can be in one of four regions: Neither player is constrained, one of the players is constrained, or both players are constrained. The value functions must take into account that the state may enter and leave each of these regions several times and they would need value matching and smooth pasting on each such occasion. We can, however, get a limiting result by taking advantage of the fact that the incentives to choose a high jump intensity go to zero as $t \rightarrow T$. So the state will, no matter what happens, spend some time in the region in which neither player is constrained and the probability that it spends the entire $[0, T]$ interval in that region goes to 1 as $U \rightarrow \infty$.

The problem no longer satisfies the conditions of Theorem 5.1 in Øksendal and Sulem (2019) because the jump intensities $u(x_{t-}, y_{t-}, t-)$ and $v(y_{t-}, x_{t-}, t-)$ depend on the very states they govern. Since the coefficients in *(20)* and *(21)* therefore do not satisfy Lipschitz conditions, we cannot apply classical existence and uniqueness results. Fortunately, we can rely on a more general result first obtained by Jacod and Protter (1982, Corollary 31) and Protter (1983, Corollary 3.11).[11] They show that *(20)* and *(21)* have solutions that are unique in law if there exists finite-valued increasing processes p and q such that $\int_0^t \boldsymbol{u}(x_s, y_s, s)ds \leq p_t$ and $\int_0^t \boldsymbol{v}(y_s, x_s, s)ds \leq q_t$ for all $t \geq 0$.[12] These conditions are clearly satisfied by $p_t = q_t = tU$.

[11] The idea in the proof is to inductively define an increasing sequence of stopping times (jump times) $\tau_1, \tau_2, \ldots \tau_n, ..$ and piece together the entire solution from the intervals $[\tau_1, \tau_2-)$ in which the classical theory applies.
[12] Uniqueness in law means that all solutions produce the same distribution over future realizations given the same starting point and time. Classical conditions give pathwise uniqueness, a stronger property under which all solutions follow the same paths everywhere.

Given the above, we will take U to infinity, disregard the constraint, and apply Theorem 7.1 in Hernandez-Hernandez et al. (2019) to justify the HJB equation. This gives us the following limiting result:

Proposition 2: *Suppose that all jumps are of size* φ*, that the arrival intensities* (u_t, v_t) *are bounded by* $U > 0$*, and that these are controlled by players X and Y, respectively. Further, define* $A(t), B(t), C(t), D(t), E(t)$ *and* $F(t)$ *as the solutions to:*

$A'(t) = - \gamma - [2A(t)^2 + F(t)^2]\varphi^2 + \rho A(t), A(T) = 0,$

$B'(t) = - [A(t)\varphi + B(t)]\varphi^2[2A(t) + F(t)] - \varphi^2[D(t)\varphi + E(t)] + \rho B(t), B(T) = 0,$

$C'(t) = - \varphi^2[A(t)\varphi + B(t)]^2/2 - \varphi^2[A(t)\varphi + B(t)][D(t)\varphi + E(t)] + \rho C(t), C(T) = 0$

$D'(t) = \eta - \varphi^2[4A(t)D(t) + F(t)/2] + \rho D(t), D(T) = 0$

$E'(t) = - \varphi^2 F(t)[A(t)\varphi + 2B(t)] - 2\varphi^2[2A(t)D(t)\varphi + A(t)E(t) + B(t)D(t)] + \rho E(t), E(T) = 0$

$F'(t) = \theta - 2\varphi^2 F(t)[2A(t) + D(t)] + \rho fFt), F(T) = 0$

In this case, as $U \to \infty$*:*

(2. 1) The probability of $Max_t\{u_{t-}, v_{t-}\} < U$ *converges to 1.*

(2. 2) The value functions $W^*(x_t, y_t, t)$ *and* $W^*(y_t, x_t, t)$ *converge to* $e^{-\rho t}[A(t)x_t^2 + B(t)x_t + C(t) + D(t)y_t^2 + E(t)y_t + F(t)x_{t-}y_{t-}]$ *and* $e^{-\rho t}[A(t)y_t^2 + B(t)y_t + C(t) + D(t)x_t^2 + E(t)x_t + F(t)x_{t-}y_{t-}]$, respectively.

(2. 3) The Markov equilibrium strategies converge to:

$u_{t-}^* = \varphi[2A(t)x_{t-} + \varphi A(t) + B(t) + F(t)y_{t-}]$ *and* $v_{t-}^* = \varphi[2A(t)y_{t-} + \varphi A(t) + B(t) + F(t)x_{t-}]$.

Proof: Consider a time $t < T$ and a pair of finite-valued functions from $[0, t]$ to $[0, U]$. Imagine that the latter pair of functions are arrival intensities and hold them and t constant. Then, for any pair of reals k_x, k_y, the probability of a pair of sequences of realized arrival times such that $x_t > k_x$ or $y_t > k_y$ is decreasing in U. This establishes *(2. 1)* and that the extent to which the value functions in the constrained game depart from those in the unconstrained game goes to zero as U goes to infinity.

Next, note that the HJB equation for player X in the unconstrained game is:

$$(22)\ Max_{u(x,\,y,\,t)}\{e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_{t-} -- \tfrac{1}{2}u_t^2] + \partial W(x_t, y_t, t)/\partial t + u_{t-}[W(x_{t-} + \varphi, y_{t-}, t-) -$$
$$W(x_{t-}, y_{t-}, t-)] + v_{t-}[W(x_{t-}, y_{t-} + \varphi, t-) - W(x_{t-}, y_{t-}, t-)] = 0.$$

The Markov equilibrium controls u_{t-}^* and v_{t-}^* are therefore given by the first order conditions:

$$(23)\ u_{t-}^*(x_{t-}, y_{t-}, t-) = e^{\rho t}[W(x_{t-} + \varphi, y_{t-}, t-) - W(x_{t-}, y_{t-}, t-)]$$

$$(24)\ v_{t-}^*(y_{t-}, x_{t-}, t-) = e^{\rho t}[W(y_{t-} + \varphi, x_{t-}, t-) - W(y_{t-}, x_{t-}, t-)]$$

We can substitute (23) and (24) into (22) and rewrite the HJB equation as:

$$(25)\ e^{-\rho t}(\gamma x_t^2 - \eta y_t^2 -- \theta x_t y_{t-}) + \partial W(x_{t-}, y_{t-}, t_-)/\partial t + e^{\rho t}[W(x_{t-} + \varphi, y_{t-}, t-) - W(x_{t-}, y_{t-}, t-)]^2/2 +$$
$$e^{\rho t}[W(y_{t-} + \varphi, x_{t-}, t-) - W(y_{t-}, x_{t-}, t-)][W(x_{t-}, y_{t-} + \varphi, t-) - W(x_{t-}, y_{t-}, t-)] = 0.$$

We again guess a solution of the form:

$$(26)\ W^g(x_t, y_t, t) \equiv e^{-\rho t}[A(t)x_t^2 + B(t)x_t + C(t) + D(t)y_t^2 + E(t)y_t + F(t)x_{t-}y_{t-}].$$

Substituting (26) into (25) gives:

$$(27)\ \{A'(t) - \rho A(t) + \gamma + [2A(t)^2 + F(t)^2]\varphi^2\}x_t^2 + \{B'(t) - \rho B(t) + [A(t)\varphi + B(t)]\varphi^2[2A(t) + F(t)] -$$
$$\varphi^2[D(t)\varphi + E(t)]\}x_t + \{C'(t) - \rho C(t) + \varphi^2[A(t)\varphi + B(t)]^2/2 - \varphi^2[A(t)\varphi + B(t)][D(t)\varphi + E(t)]\} +$$

$\{D'(t) - \eta - \rho D(t) + \varphi^2[4A(t)D(t) + F(t)/2]\}y_t^2 + \{E'(t) - \rho E(t) + \varphi^2 F(t)[A(t)\varphi + 2B(t)] +$

$2\varphi^2[2A(t)D(t)\varphi + A(t)E(t) + B(t)D(t)]\}y_t + \{F'(t) - \rho F(t) - \theta + 2\varphi^2 F(t)[2A(t) + D(t)]\}x_t y_t = 0$

Our guess $W^g(x_t, y_t, t)$ therefore solves *(27)* if the coefficients in $W^g(x_t, y_t, t)$, solve the following differential equations:

(28) $A'(t) = -\gamma - [2A(t)^2 + F(t)^2]\varphi^2 + \rho A(t), A(T) = 0,$

(29) $B'(t) = -[A(t)\varphi + B(t)]\varphi^2[2A(t) + F(t)] - \varphi^2[D(t)\varphi + E(t)] + \rho B(t), B(T) = 0,$

(30) $C'(t) = -\varphi^2[A(t)\varphi + B(t)]^2/2 - \varphi^2[A(t)\varphi + B(t)][D(t)\varphi + E(t)] + \rho C(t), C(T) = 0$

(31) $D'(t) = \eta - \varphi^2[4A(t)D(t) + F(t)/2] + \rho D(t), D(T) = 0$

(32) $E'(t) = -\varphi^2 F(t)[A(t)\varphi + 2B(t)] - 2\varphi^2[2A(t)D(t)\varphi + A(t)E(t) + B(t)D(t)] + \rho E(t), E(T) = 0$

(33) $F'(t) = \theta - 2\varphi^2 F(t)[2A(t) + D(t)] + \rho F(t), F(T) = 0$

Since *(28) – (33)* have Lipschitz continuous right-hand sides, a solution exists and *(2. 2)* and *(2. 3)* follows.

QED

So just as in the game with controlled jump sizes, the limiting value functions with controlled arrival intensities are second order polynomials, and the policy functions are linear in both players' stocks. Finally, except for the quadratic $A(t)\varphi^2$, the policy functions are proportional to the jump size which therefore play a role very similar to that played by the arrival intensity in the game with controlled jump sizes. Between the jump size and the arrival intensity, the uncontrolled magnitude plays more or less the same role in both models.

It is easy to establish that the game with controlled jump intensities behaves "like" the game with controlled jump sizes in the sense that Corollaries 2.1 – 2.4 below are analogues of Corollaries 1.1 – 1.4.

Corollaries 2.1-2.4*: As $U \to \infty$, the probability that the following statements are true converges to 1:*

Corollary 2.1: *$a(t) > 0$ and $f(t) < 0$.*

Corollary 2.2: *If $x_0 > y_0$, $E_{t=0}\,(x_s/y_s)$ grows with s.*

Corollary 2.3: *$E_{t=0}\left| x_s - y_s \right|$ grows with s.*

Corollary 2.4: *The probability that $x_s - y_s$ changes sign decreases with s.*

Proof: By arguments identical to those used to prove Corollaries 1.1 – 1.4 with one difference: In this model players will get equally-sized arrivals, but whoever is ahead is more likely to get the next one.

QED

In a natural analogue of the model presented in Section II, we obtain a limiting result according to which the value - and policy functions are very similar to what we found there. By Corollaries 2 – 4, the two games share other appealing properties as well. The analysis in this Section shows that we can solve dynamic optimization problems and games with controlled jump intensities if we are willing to accept results that depend on theoretical bounds on the control variables and solutions that are unique in law only. However, it also suggests that the qualitative properties of the solution may be relatively close to those obtained in similar models with controlled jump sizes.

IV. DISCUSSION

There are some theoretical limits on the acceptability of the solution to the second formulation, but we do not see them as particularly important. It is hard to think of an application in which a solution would be disqualified because it only is unique in law. There are certainly cases in which it is natural to assume that the arrivals cannot be too close in time ("at most one per day"?), but the two formulations do not differ in that respect; only the arrival intensities are at stake. Can there be cases in which it is important to bound the arrival intensities? We cannot think of an example, but the answer may be less clear than that about the nature of uniqueness.

We would like to close with four observations: First, as discussed at the start of Section II, we claim that the result hold for general second order polynomial objective functions of the form $\int_0^T e^{-\rho t}[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t + \alpha x_t - \beta y_t + \sigma - \frac{1}{2}u_t^2 - \pi u_t]dt$. To see this, start by writing out the analogue of *(7)*. This only differs because the first term (the payoff function) is longer and because $u_{t-}^*(x_{t-}, y_{t-}, t-) = e^{\rho t-}\lambda \partial W(x_{t-}, y_{t-}, t-)/\partial x - \pi$, where the π is new. You then guess a solution of the form $W^g(x_t, y_t, t) \equiv e^{-\rho t}[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t + f(t)x_t y_t]$, which is the same as *(8)*. By going through each term in the HJB equation you can then see that it also is a second order polynomial and therefore can be solved by $W^g(x_t, y_t, t)$. Second, it is a limitation that the results only have been established for a linear-quadratic model, but such models are Taylor approximations to a much richer set and the approximation can locally be very good. It is true that relatively few models in the social sciences use our linear-quadratic functional form, but that is because it often is possible to solve models with a more appealing interpretation in the context. In the case considered here, it would be very, very hard to solve other functional forms and having an approximate model is presumably better than having no model. Third, it is not important that the games be symmetric. We chose that case because Corollaries 2 – 4 are uninteresting in other settings. Fourth, to help us think about the limitations of the results, it would be interesting to identify an application in which the two formulations yield conflicting or at least different intuitions about what is going on.

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