Identifying the Set of Always-Active Constraints in a System of Linear Inequalities by a Single Linear Program

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Abstract: Given a system of m linear inequalities in n unknowns of the form $Ax \leq b$, a constraint index i is called always-active if $Ax \leq b$ has a solution and every solution satisfies $A_i x = b_i$. Our interest lies in identifying the set of always-active constraints of $Ax \leq b$ by solving one linear program generated from the data (A,b).

Key Words: Linear Inequalities, Active Constraints, Linear Program.

Abbreviated Title: Always-Active Constraints.

Given a system of m linear inequalities in n unknowns, of the form

$$Ax \le b, \qquad (*)$$

a constraint index i is called <u>always-active</u> if $Ax \le b$ has a solution and every solution satisfies $A_i x = b_i$, i.e. the i^{th} constraint is active in every solution. In large optimization problems where the constraint set is given by the system (*), identifying the always-active constraints before processing the problem enables the user to explicitly reduce both the dimension of the feasible region and the number of constraints, and has the potential therefore to simplify the original problem. Our interest lies in identifying the set of always-active constraints by solving just one linear program generated from the data (A,b).

A first attempt (also noted by several of our colleagues) is to solve

$$\begin{aligned} \underset{x,y}{\text{maximize}}\ & e^T y \\ & Ax+y \le b \qquad \text{(P1)} \\ & 0 \le y \le \varepsilon e \end{aligned}$$

where e is the vector of ones of appropriate dimension and ε is positive and "sufficiently small." A suitable value of ε seems however hard to determine. If the data (A,b) is integer (or rational), then Orlin has pointed out [3] that a sufficiently small ε can be predetermined (using arguments similar to those used in the ellipsoid method).

Our first solution is closely related to (P1), and employs homogenization as similar to those reducing questions about polyhedra to questions about convex cones. Consider the linear program:

$$\begin{aligned} \underset{x,y,\alpha}{\text{maximize}}\ & e^T y \\ & Ax+y-b\alpha \le 0 \qquad \text{(P2)} \\ & 0 \le y \le e \\ & \alpha \ge 1 \end{aligned}$$

We have the following straightforward result:

Proposition 1. If the system (*) is feasible, then (P2) is feasible and finite, and for any optimal solution (x^*,y^*,α^*) to (P2), the set of always-active constraint indices is the set $\{i|y_i^* = 0\}$. Furthermore, $\frac{x^*}{\alpha^*}$ is an element of the relative interior of $\{x\in R^n|Ax \le b\}$. If the system (*) is not feasible, then (P2) is infeasible. [X]

Another linear program that identifies all of the always-active constraints is obtained through consideration of the following problem.

$$\begin{aligned} \underset{x,t}{\text{maximize}}\ & t \qquad \text{(P3).} \\ \text{subject to:}\ & Ax+et \le b \end{aligned}$$

Obviously, (P3) is always feasible, $t^* < 0$ if and only if there is no solution to (*), and $t^* > 0$ if and only if there are no always-active constraints. When $t^*=0$, it can easily be shown that if x^* and λ^* are optimal values of (P3) and its dual (D3), namely

$$\begin{aligned} \underset{\lambda}{\text{minimize}}\ & \lambda^T b \\ \text{subject to:}\ & A^T\lambda = 0 \qquad \text{(D3),} \\ & e^T\lambda = 1 \\ & \lambda \ge 0 \end{aligned}$$

then if $\lambda_i^* > 0$, the i^{th} constraint is always-active, and if $A_i x^* < b_i$, than the i^{th}-constraint is not always-active. If we could ensure that we have a strictly complementary pair of optimal solutions to (P3) and (D3) (i.e. $\lambda_i^* > 0$ or $b_i - A_i x^* > 0$ for each i), then we could identify for each constraint whether or not it is always-active. This is accomplished by solving the following linear program:

$$\begin{array}{lrll}
 & \underset{x,\lambda,t,\theta}{\text{maximize}} \ \theta & & \\
 & \text{subject to:} & & \\
(1) & Ax + et & \leq b & \\
(2) & A^T\lambda & = 0 & \\
(3) & e^T\lambda & = 1 & \qquad (P4) \\
(4) & -b^T\lambda + t & = 0 & \\
(5) & \lambda - Ax - et - e\theta & \geq -b & \\
(6) & \lambda & \geq 0 &
\end{array}$$

Constraints (1) represent primal feasibility, constraints (2), (3) and (6) represent dual feasibility, contraint (4) represents strong duality, and a positive value of θ in constraints (5) represents strict complementarity of λ and (b-Ax) when t=0. Tucker's strict complementarity theorem [4] ensures that a positive value of θ will exist if (*) has a solution. We have:

Proposition 2. If (P4) is infeasible, then (*) has no always-active constraints. If (P4) is feasible, then it is finite, and for any optimal solution $(x^*, \lambda^*, t^*, \theta^*)$, we have:

(i) If $t^* = 0$, the set of always-active constraint indices is the set $\{i^* | \lambda_i > 0\}$ and x^* lies in the relative interior of $\{x \in R^n | Ax \leq b\}$.

(ii) If $t^* > 0$, there are no always-active constraints.

(iii) If $t^* < 0$, the system (*) has no solution. [X]

Instead of the system (*), we could have worked with the system

$$Ax=b, \quad x \geq 0, \qquad (**)$$

whereby the always-active constraints would correspond to null variables of (**), i.e. variables x_j such that $x_j=0$ in every solution to (**), see Luenberger [2]. Determining null variables is also of interest in the recent linear programming algorithm of Karmarkar [1].

References

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