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e-companion

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Electronic Companion—"Performance Analysis of Queueing Networks via Robust Optimization" by Dimitris Bertsimas, David Gamarnik, and Alexander Anatoliy Rikun, *Operations Research*, DOI 10.1287/opre.1100.0879.

Appendix. Preliminary technical results

In this section we establish some preliminary technical results. Using ϕ as defined by (8), we let $U(x) = -ax + 2b\phi(x) + c$ for some positive constants a, b, c satisfying

$$\frac{b}{a} \geq e^{2e}. \tag{34}$$

Lemma 5. *$U(x)$ is strictly concave for $x \geq e^e$.*

Proof.

$$\begin{aligned}
\frac{\partial U(x)}{\partial x} &= -a + b\sqrt{\frac{\ln\ln x}{x}} + \frac{b}{\ln x}\frac{1}{\sqrt{x \ln\ln x}} \\
\frac{\partial^2 U(x)}{\partial x^2} &= b\Big(x^{-\frac{1}{2}}\frac{1}{2}(\ln\ln x)^{-\frac{1}{2}}\frac{1}{\ln x}\frac{1}{x} + (\ln\ln x)^{\frac{1}{2}}(-\frac{1}{2}x^{-\frac{3}{2}})\Big) \\
&\quad + b\Big(-(\ln x)^{-2}\frac{1}{x}(x\ln\ln x)^{-\frac{1}{2}} + (\ln x)^{-1}(-\frac{1}{2})(x\ln\ln x)^{-\frac{3}{2}}(\frac{1}{\ln x} + \ln\ln x)\Big) \\
&= bx^{-\frac{3}{2}}(\frac{1}{2})(\ln\ln x)^{-\frac{1}{2}}\Big(\frac{1}{\ln x} - (\ln\ln x)\Big) \\
&\quad + b\Big(-(\ln x)^{-2}\frac{1}{x}(x\ln\ln x)^{-\frac{1}{2}}\Big) + b\Big((\ln x)^{-1}(-\frac{1}{2})(x\ln\ln x)^{-\frac{3}{2}}(\frac{1}{\ln x} + \ln\ln x)\Big) \\
&< 0 \;\; \text{since all three terms on RHS above are negative for } x \geq e^e
\end{aligned}$$

$\square$

Lemma 6. *Assuming (34) and $e^e > (c/b)^2$,*

$$U(x) < 0 \qquad \forall\ x > (18b^2/a^2)\ln\ln(3b/a).$$

Proof. Since $(18b^2/a^2)\ln\ln(3b/a) > e^e$, throughout the proof we restrict ourselves to the domain $x \geq e^e$. Since in addition $x > (c/b)^2$, we have $b\phi(x) \geq b\sqrt{x} > c$. In this range $-ax+2b\phi(x)+c \leq -ax+3b\phi(x) = -ax + 3b\sqrt{x\ln\ln x}$. This quantity is less than zero provided

$$\Big(\frac{x}{\ln\ln x}\Big)^{\frac{1}{2}} > \frac{3b}{a} \triangleq \alpha.$$

It is easy to check that $x/\ln\ln x$ is a strictly increasing function with $\lim_{x\to\infty}(x/\ln\ln x) = \infty$. Let x_0 be the unique solution of $x/\ln\ln x = \alpha^2$ on $x \geq e^e$. We claim that $x_0 \leq 2\alpha^2 \ln\ln\alpha$. The assertion of the lemma follows from this bound. Let $A = 2\alpha^2\ln\ln\alpha$. Then

$$\begin{aligned}
\frac{A}{\ln\ln A} &= \frac{2\alpha^2\ln\ln\alpha}{\ln(2\ln\alpha + \ln^{(3)}\alpha + \ln 2)} \\
&\geq \frac{2\alpha^2\ln\ln\alpha}{\ln(4\ln\alpha)} \quad \text{since } \ln\alpha \geq \ln^{(3)}\alpha \text{ and } \ln\alpha > \ln 2 \\
&\geq \frac{2\alpha^2\ln\ln\alpha}{2\ln(\ln\alpha)} \quad \text{since } \ln\alpha > \ln(b/a) \geq 2e > 4. \\
&= \alpha^2.
\end{aligned}$$

This implies $x_0 \leq A$ and the proof is complete. □

Proposition 2. *Under the assumption (34)*

$$\sup_{x\geq 0} U(x) \leq 7(b^2/a)\ln\ln(b/a) + c.$$

Proof. Since $a > 0$, then the supremum in $\sup_{x\geq 0} U(x)$ is achieved. Let x^* be any value achieving $\max_{x\geq 0} U(x)$. First suppose $0 \leq x^* < e^e$. It follows from the definition of ϕ in (8) that $\phi(x^*) = 1$ and thus $U(x^*) = -ax^* + 2b + c$. Using $0 \leq x^* < e^e$ and assumption (34), it is straightforward to check that $U(x^*)$ is indeed upper bounded from above by $7(b^2/a)\ln\ln(b/a) + c$. Next, we consider the case $x^* = e^e$, and using the fact that $a > 0$, we obtain $U(x^*) \leq 2b \cdot \sqrt{e^e \ln\ln(e^e)} + c$. It is again straightforward to check that the aforementioned bound is upper bounded from above by $7(b^2/a)\ln\ln(b/a) + c$.

We now consider the case $x^* > e^e$. By Lemma 5, x^* is the unique point satisfying $\frac{\partial U(x^*)}{\partial x^*} = 0$, if it exists. The remainder of the proof is devoted to the final case where we obtain

$$0 = \frac{\partial U(x^*)}{\partial x^*} = -a + \frac{b(\frac{1}{\ln x^*} + \ln\ln x^*)}{\sqrt{x^* \ln\ln x^*}} \tag{35}$$

Continuing further, (35) implies

$$\frac{\sqrt{x^*\ln\ln x^*}}{\ln\ln x^* + \frac{1}{\ln x^*}} = \frac{b}{a} \triangleq \alpha. \tag{36}$$

Note

$$\frac{x^*}{\ln\ln x^*} > \alpha^2$$
$$\frac{x^*}{2\ln\ln x^*} < \alpha^2 \quad \text{since } \ln\ln x^* > \frac{1}{\ln x^*} \text{ for } x \geq e^e$$

It is easy to check that $x/\ln\ln x$ is a strictly increasing function for $x \geq e^e$ and $\lim_{x\to\infty}(x/\ln\ln x) = \infty$. (34) implies that there exist unique $x_{\min}$ and $x_{\max}$ satisfying

$$\frac{x_{\min}}{\ln\ln x_{\min}} = \alpha^2 \qquad \frac{x_{\max}}{2\ln\ln x_{\max}} = \alpha^2$$

The monotonicity of $x/\ln\ln x$ implies $x_{\min} \leq x^* \leq x_{\max}$. In order to complete the proof of the proposition, we will first state and prove Lemmas 7 and 8.

Lemma 7. *$x_{\min} \geq \alpha^2 \ln\ln\alpha$ and $x_{\max} \leq 4\alpha^2\ln\ln\alpha$.*

Proof. Let $B_1 = \alpha^2\ln\ln\alpha$. Then

$$\begin{aligned}\frac{B_1}{\ln\ln B_1} &= \frac{\alpha^2\ln\ln\alpha}{\ln\ln(\alpha^2\ln\ln\alpha)}\\ &< \frac{\alpha^2\ln\ln\alpha}{\ln\ln\alpha} \quad \text{since } \ln\ln\alpha \geq 1 \text{ for } \alpha \geq e^{2e}\\ &= \alpha^2.\end{aligned}$$

Thus since $\frac{x}{\ln\ln x}$ is increasing for $x \geq e^e$, we have $x_{\min} \geq B_1$ and the first assertion is established.

Let $B_2 = 4\alpha^2\ln\ln\alpha$. Then

$$\begin{aligned}\frac{B_2}{2\ln\ln B_2} &= \frac{4\alpha^2\ln\ln\alpha}{2\ln\ln(4\alpha^2\ln\ln\alpha)}\\ &= \frac{4\alpha^2\ln\ln\alpha}{2\ln(2\ln\alpha + \ln^{(3)}\alpha + \ln 4)}\\ &\geq \frac{4\alpha^2\ln\ln\alpha}{2\ln(4\ln\alpha)} \quad \text{since } \ln\alpha \geq \ln^{(3)}\alpha \text{ and } \ln\alpha > \ln 4\\ &\geq \frac{4\alpha^2\ln\ln\alpha}{4\ln(\ln\alpha)} \quad \text{since } \ln\alpha \geq 2e > 4.\\ &= \alpha^2.\end{aligned}$$

Thus, again since $x/\ln\ln x$ is increasing for $x \geq e^e$, then the second assertion follows. □

Lemma 7 and $x_{\min} \leq x^* \leq x_{\max}$ imply

$$\alpha^2 \ln\ln\alpha \leq x^* \leq 4\alpha^2 \ln\ln\alpha. \tag{37}$$

Lemma 8. *$\sqrt{x_{\max} \ln\ln x_{\max}} \leq 4\alpha \ln\ln\alpha$.*

Proof.

$$\begin{aligned}
\sqrt{x_{\max} \ln\ln x_{\max}} &\leq \sqrt{\left(4\alpha^2 \ln\ln\alpha\right) \ln\ln\left(4\alpha^2 \ln\ln\alpha\right)} \quad \text{by Lemma 7} \\
&= \alpha\sqrt{4\ln\ln\alpha}\sqrt{\ln\left(2\ln\alpha + \ln^{(3)}\alpha + \ln 4\right)} \\
&\leq \alpha\sqrt{4\ln\ln\alpha}\sqrt{\ln\left(4\ln\alpha\right)} \quad \text{since } \ln\alpha \geq \ln^{(3)}\alpha \text{ and } \ln\alpha \geq \ln(e^{2e}) > \ln 4 \\
&\leq \alpha\sqrt{4\ln\ln\alpha}\sqrt{2\ln\ln\alpha} \quad \text{since } \ln\alpha > 4
\end{aligned}$$

and the lemma follows from the last step. □

We now complete the proof of Proposition 2. We have

$$\begin{aligned}
U(x^*) &\leq -ax^* + 2b\sqrt{x^* \ln\ln x^*} + c \\
&\leq -ax_{\min} + 2b\sqrt{x_{\max} \ln\ln x_{\max}} + c \quad \text{since } x_{\min} \leq x^* \leq x_{\max} \\
&\leq -ax_{\min} + 8b\alpha \ln\ln\alpha + c \quad \text{by Lemma 8} \\
&\leq -a\alpha^2 \ln\ln\alpha + 8b\alpha \ln\ln\alpha + c \quad \text{by Lemma 7} \\
&= 7(b^2/a) \ln\ln(b/a) + c.
\end{aligned}$$

□