Online Appendix For 'Simple Policies for Dynamic Pricing with Imperfect Forecasts'

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Results in this appendix are numbered consistently with those in the main paper. Results that do not appear in the paper (auxiliary Lemmas or additional theorems omitted from the exposition in the main paper) are numbered using the convention 'SectionLetter.Number'.

A. Proofs for Section 4

We begin with establishing properties of the unit revenue function, $g(\cdot)$.

Lemma 6.

1. *$g(\cdot)$ is a non-negative, continuous, non-decreasing, and concave function on $\mathbb{R}_+$, with $g(0) = 0$.*

2. *$yg(1/y)$ is non-decreasing and concave on $\mathbb{R}_{++}$.*

3. *$g(y)/y$ is non-increasing on $\mathbb{R}_+$.*

4. *If $u, v > 0$, then $\frac{g(u)}{g(v)} \geq \min(\frac{u}{v}, 1)$, $\frac{1}{u}\int_0^u g(v)dv \leq g(u/2)$.*

Proof.

1. That $g(\cdot)$ is non-negative, continuous and non-decreasing with $g(0) = 0$ follows by definition. We show $g(\cdot)$ is a concave function. In the remainder of the proof, we use the fact that $(p\overline{F}(p))'|_{p=p^*} = \overline{F}(p^*) - p^*f(p^*) = 0$. We know that on $y \leq 1/\overline{F}(p^*)$, $g'(y) = p^*\overline{F}(p^*)$. Now on $y \geq 1/\overline{F}(p^*)$, $g(y)$ is non-decreasing in y and we have $g'(y) = \overline{F}^2(g(y))/f(g(y))$, which in turn must be non-increasing following the second part of Assumption 1 that $\overline{F}(p)/f(p)$ is non-increasing. Finally,
$$\overline{F}^2(g(1/\overline{F}(p^*)))/f(g(1/\overline{F}(p^*))) = \overline{F}^2(p^*)/f(p^*) = p^*\overline{F}(p^*).$$
so that $g(\cdot)$ is continuously differentiable on $\mathbb{R}_+$ with a non-increasing derivative. Thus, $g(\cdot)$ is concave on $\mathbb{R}_+$.

2. Note that
$$yg(1/y) = \begin{cases} p^*\overline{F}(p^*) & \text{if } y \geq \overline{F}(p^*); \\ y\overline{F}^{-1}(y) & \text{otherwise.} \end{cases}$$

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It follows that $g(y)' = 0$ on $y \geq \overline{F}(p^*)$. On the domain $(0, \overline{F}(p^*)]$, define the function $p(y) = \overline{F}^{-1}(y)$; $p(y)$ is decreasing in y. On $(0, \overline{F}(p^*)]$, we have $(yg(1/y))' = p(y) - \overline{F}(p(y))/f(p(y))$, which is non-increasing in y following the second part of Assumption 1 that $\overline{F}(p)/f(p)$ is non-increasing, and the fact that $p(y)$ is decreasing in y. Moreover, on $(0, \overline{F}(p^*)]$,

$$(yg(1/y))' \geq (yg(1/y))'|_{y=\overline{F}(p^*)} = p^* - \overline{F}(p^*)/f(p^*) = 0.$$

It follows that $yg(1/y)$ is non-decreasing and concave on $\mathbb{R}_{++}$.

3. That $g(y)/y$ in non-increasing on $\mathbb{R}_+$ follows directly from property (2) above.
4. Since $g(\cdot)$ is a non-decreasing and concave function on $\mathbb{R}_+$, this property holds due to Lemma 7.

■

Lemma 1.

$$\begin{aligned}
J^*(x^0, \lambda^0, 0) &\leq \mathsf{E}\left[J^*_{\{\Lambda_t\}}(x_0, 0)\right] \\
&\leq J^*_{\mathrm{CE}}(x_0, \lambda_0, 0) \\
&\leq x_0 g\left(\frac{\int_0^T (\lambda_t + \sigma_t/\sqrt{2\pi})dt}{x_0}\right) \qquad (4)\\
&\leq x_0 g\left(\frac{\int_0^T \lambda_t dt}{x_0}\right) + x_0 g\left(\frac{\int_0^T \sigma_t dt}{x_0\sqrt{2\pi}}\right). \qquad (5)
\end{aligned}$$

Proof. The first inequality is evident by definition. Now, by definition of the unit revenue function $g(\cdot)$ and Section 5.2 of Gallego and van Ryzin [1994], we have that

$$J^*_{\mathrm{CE}}(x_0, \lambda_0, 0) = x_0 g\left(\frac{\int_0^T \mathsf{E}[\Lambda_t]dt}{x_0}\right)$$

By the concavity of $g(\cdot)$ established in Lemma 6 and Jensen's inequality, we immediately have:

$$\mathsf{E}\left[J^*_{\{\Lambda_t\}}(x_0, 0)\right] = \mathsf{E}\left[x_0 g\left(\frac{\int_0^T \Lambda_t dt}{x_0}\right)\right] \leq x_0 g\left(\frac{\int_0^T \mathsf{E}[\Lambda_t]dt}{x_0}\right) = J^*_{\mathrm{CE}}(x_0, \lambda_0, 0)$$

which is the second inequality. The fact that $J^*_{\{\Lambda_t\}}(x_0, 0) = x_0 g\left(\frac{\int_0^T \Lambda_t dt}{x_0}\right)$ follows from the definition of $g(\cdot)$ and Section 5.2 in Gallego and van Ryzin [1994].

Now for a Normal random variable X with mean μ and variance σ^2, we know that $\mathsf{E}[X^+] \leq \mu + \sigma/\sqrt{2\pi}$. Thus, $\mathsf{E}[\Lambda_t] = \mathsf{E}\left[\overline{\Lambda}_t^+\right] \leq \lambda_t + \sigma_t/\sqrt{2\pi}$. Since, by Lemma 6, $g(\cdot)$ is non-decreasing, it then follows that

$$J^*_{\mathrm{CE}}(x_0, \lambda_0, 0) = x_0 g\left(\frac{\int_0^T \mathsf{E}[\Lambda_t]dt}{x_0}\right) \leq x_0 g\left(\frac{\int_0^T (\lambda_t + \sigma_t/\sqrt{2\pi})dt}{x_0}\right).$$

The sub-additivity of $g(\cdot)$ from the fourth part of Lemma 6 then yields the final inequality. ■

Lemma 3.

$$J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0) \geq 0.342\alpha x_0 g\left(\frac{\int_0^T \sigma_t dt}{x_0\sqrt{2\pi}}\right) + \frac{1-\alpha}{2} x_0 g\left(\frac{\int_0^T \lambda_t dt}{x_0}\right).$$

Proof. We have:

$$\begin{aligned}
J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0) &= \mathsf{E}\left[\int_0^T \pi_{\mathrm{RFP}}(X^t,\Lambda^t,t)\overline{F}(\pi_{\mathrm{RFP}}(X^t,\Lambda^t,t))\Lambda_t dt\right] \\
&= \mathsf{E}\left[\int_0^T \frac{X_t h(t,\alpha)}{\Lambda_t(T-t)} g\left(\frac{\Lambda_t(T-t)}{X_t h(t,\alpha)}\right)\Lambda_t dt\right] \\
&\geq x_0 \mathsf{E}\left[\int_0^T \left(\frac{\alpha}{T} + (1-\alpha)\frac{\lambda_t}{\int_0^T \lambda_s ds}\right) g\left(\Lambda_t \Big/ x_0\left(\frac{\alpha}{T} + (1-\alpha)\frac{\lambda_t}{\int_0^T \lambda_s ds}\right)\right) dt\right] \\
&\geq \alpha\frac{x_0}{T}\mathsf{E}\left[\int_0^T g\left(\frac{\Lambda_t T}{x_0}\right) dt\right] + (1-\alpha)\frac{x_0}{\int_0^T \lambda_s ds}\mathsf{E}\left[\int_0^T \lambda_t g\left(\frac{\Lambda_t \int_0^T \lambda_s ds}{x_0 \lambda_t}\right) dt\right]. \qquad (6)
\end{aligned}$$

where the second equality holds by the definition of $g(\cdot)$, the first inequality follows from the lower bound on X_t established in Lemma 2 on the inventory balancing property and the property that $zg(1/z)$ is a non-decreasing function. The final inequality holds because $zg(1/z)$ is a concave function.

Next, we prove the lower bounds of two terms in 6 respectively. For the first term, we have:

$$\begin{aligned}
\frac{\mathsf{E}\left[\int_0^T g\left(\frac{\Lambda_t T}{x_0}\right)dt\right]}{Tg\left(\frac{\int_0^T \sigma_t dt}{x_0\sqrt{2\pi}}\right)} &= \frac{\int_0^T\int_{-\infty}^{\infty} g\left(\frac{T(\lambda_t+y)^+}{x_0}\right)\frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}}dydt}{Tg\left(\frac{\int_0^T \sigma_t dt}{x_0\sqrt{2\pi}}\right)} \\
&\geq \frac{\int_0^T\int_{-\infty}^{\infty} g\left(\frac{Ty^+}{x_0}\right)\frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}}dydt}{Tg\left(\frac{\int_0^T \sigma_t dt}{x_0\sqrt{2\pi}}\right)} \\
&\geq \frac{1}{T}\int_0^T\int_{-\infty}^{\infty}\min\left\{1, \frac{y^+}{\int_0^T \sigma_t dt/T\sqrt{2\pi}}\right\}\frac{\exp\left(-y^2/2\sigma_t^2\right)}{\sqrt{2\pi\sigma_t^2}}dydt \\
&= \frac{1}{T}\int_0^T\left[1-\Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \int_0^{\overline{\sigma}_{T,1}/\sqrt{2\pi}}\frac{y}{\overline{\sigma}_{T,1}\sigma_t}\exp\left(-y^2/2\sigma_t^2\right)dy\right]dt \\
&\geq 0.342.
\end{aligned}$$

The first inequality holds due to Property 1 in Lemma 6, and the positivity of λ_t. The second inequality holds due to Property 4 in Lemma 6. The final inequality was derived as a property of the class of market-size processes we consider in Property 3 in Lemma 8.

For the second term in 6, we have

$$\begin{aligned}
&\frac{1}{\int_0^T \lambda_s ds}\mathsf{E}\left[\int_0^T \lambda_t g\left(\frac{\Lambda_t\int_0^T \lambda_s ds}{x_0\lambda_t}\right)dt\right] \\
&= \frac{1}{\int_0^T \lambda_s ds}\int_0^T \lambda_t\int_{-\infty}^{\infty} g\left(\frac{(\lambda_t+y)^+\int_0^T \lambda_s ds}{x_0\lambda_t}\right)\frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}}dydt
\end{aligned}$$

$$\geq \frac{1}{\int_0^T \lambda_s ds} \int_0^T \lambda_t \int_0^\infty g\left(\frac{(\lambda_t + y)\int_0^T \lambda_s ds}{x_0 \lambda_t}\right) \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dydt$$

$$\geq \frac{1}{\int_0^T \lambda_s ds} \int_0^T \lambda_t \int_0^\infty g\left(\frac{\lambda_t \int_0^T \lambda_s ds}{x_0 \lambda_t}\right) \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dydt$$

$$= \frac{1}{\int_0^T \lambda_s ds} \int_0^T \lambda_t g\left(\frac{\int_0^T \lambda_s ds}{x_0}\right) \int_0^\infty \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dt$$

$$= \frac{1}{2} g\left(\frac{\int_0^T \lambda_t dt}{x_0}\right).$$

The first and second inequalities follow respectively from the fact that $g(\cdot)$ is non-negative and non-decreasing. ■

Theorem 1. *Assume Λ_t is a generalized moving average process. Then,*

1. *For the RFP policy with $\alpha = 1$, we have*

$$\frac{J^{\pi_{\text{RFP}}}(x^0, \lambda^0, 0)}{J^*(x^0, \lambda^0, 0)} \geq \max\left\{0.342, \frac{1}{1+B} - \frac{B}{1+B}\left(\exp(-1/4\pi B^2) + 0.853\right)\right\},$$

where $B \triangleq \sigma_T/\sqrt{2\pi}\lambda^2$, and we assume $\lambda_t = \lambda$ for all t.

2. *For the RFP policy with $\alpha = 0.594$, and arbitrary forecasts $\{\lambda_t\}$, we have:*

$$\frac{J^{\pi_{\text{RFP}}}(x^0, \lambda^0, 0)}{J^*(x^0, \lambda^0, 0)} \geq 0.203.$$

Proof. We provide a proof of the first part of the theorem; the second part is proved in Section 4. By Lemma 1 we have that $J^*(x^0, \lambda^0, 0) \leq J^*_{\text{CE}}(x^0, \lambda^0, 0)$. Consequently, if $\lambda_t = \lambda$ for all t, then for an arbitrary α,

$$\frac{J^{\pi_{\text{RFP}}}(x^0, \lambda^0, 0)}{J^*(x^0, \lambda^0, 0)} \geq \frac{J^{\pi_{\text{RFP}}}(x^0, \lambda^0, 0)}{J^*_{\text{CE}}(x^0, \lambda^0, 0)}.$$

Now, we have:

$$\begin{aligned}
\frac{J^{\pi_{\text{RFP}}}(x^0, \lambda^0, 0)}{J^*_{\text{CE}}(x^0, \lambda^0, 0)} &\geq \frac{\mathsf{E}\left[\int_0^T \frac{X_t}{\Lambda_t(T-t)} g\left(\frac{\Lambda_t(T-t)}{X_t}\right)\Lambda_t dt\right]}{x_0 g\left(\frac{\lambda T + \int_0^T \sigma_t dt/\sqrt{2\pi}}{x_0}\right)} \\
&\geq \frac{\mathsf{E}\left[\int_0^T g\left(\frac{\Lambda_t T}{x_0}\right) dt\right]}{T g\left(\frac{\lambda T + \int_0^T \sigma_t dt/\sqrt{2\pi}}{x_0}\right)} \\
&= \frac{\int_0^T \int_{-\infty}^\infty g\left(\frac{T(\lambda+y)^+}{x_0}\right) \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dydt}{T g\left(\frac{\lambda T + \int_0^T \sigma_t dt/\sqrt{2\pi}}{x_0}\right)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} \min\left\{1, \frac{(\lambda+y)^+}{\lambda + \int_0^T \sigma_t dt/T\sqrt{2\pi}}\right\} \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dydt \\
&= \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \int_{-\lambda}^{\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{\lambda+y}{\lambda+\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dy\right] dt \\
&\geq \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \int_{0}^{\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{y}{\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dy\right] dt \\
&\geq 0.342.
\end{aligned}
$$

The first equality holds by definition of $g(\cdot)$ and Lemma 1. The second inequality follows by applying the inventory balancing Lemma (Lemma 2) to obtain a lower bound on X_t along with the property that $zg(1/z)$ is a non-decreasing function, which is established in Lemma 6. The third inequality follows by Property 4 in Lemma 6. The final inequality follows by Property 3 in Lemma 8.

In addition, we have:

$$
\begin{aligned}
\frac{J^{\pi_{RFP}}(x^0,\lambda^0,0)}{J^*(x^0,\lambda^0,0)} &\geq \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \int_{-\lambda}^{\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{\lambda+y}{\lambda+\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dy\right] dt \\
&= \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \frac{\lambda}{\lambda+\overline{\sigma}_{T,1}/\sqrt{2\pi}}\left(\Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) - \Phi\left(-\frac{\lambda}{\sigma_t}\right)\right)\right. \\
&\quad \left. + \frac{\sigma_t}{\sqrt{2\pi}\lambda+\overline{\sigma}_{T,1}}\left(\exp(-\lambda^2/2\sigma_t^2) - \exp(-\overline{\sigma}_{T,1}^2/4\pi\sigma_t^2)\right)\right] dt \\
&\geq \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \frac{1}{1+B}\left(\Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) - \Phi\left(-\frac{1}{\sqrt{2\pi}B}\right)\right)\right. \\
&\quad \left. - \frac{\sigma_t}{\sqrt{2\pi}\lambda+\overline{\sigma}_{T,1}} \exp(-\overline{\sigma}_{T,1}^2/4\pi\sigma_t^2)\right] dt \\
&= \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \frac{1}{1+B}\left(\Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) - \Phi\left(-\frac{1}{\sqrt{2\pi}B}\right)\right)\right. \\
&\quad \left. - \frac{\overline{\sigma}_{T,1}}{\sqrt{2\pi}\lambda+\overline{\sigma}_{T,1}} \frac{\sigma_t}{\overline{\sigma}_{T,1}} \exp(-\overline{\sigma}_{T,1}^2/4\pi\sigma_t^2)\right] dt \\
&\geq \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \frac{1}{1+B}\left(\Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) - \Phi\left(-\frac{1}{\sqrt{2\pi}B}\right)\right)\right. \\
&\quad \left. - \frac{B}{1+B} \frac{\sigma_t}{\overline{\sigma}_{T,1}} \exp(-\overline{\sigma}_{T,1}^2/4\pi\sigma_t^2)\right] dt \\
&= \frac{1}{T}\int_0^T \left[\frac{1}{1+B}\Phi\left(\frac{1}{\sqrt{2\pi}B}\right) + \frac{B}{1+B}\left(1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) - \frac{\sigma_t}{\overline{\sigma}_{T,1}}\exp(-\overline{\sigma}_{T,1}^2/4\pi\sigma_t^2)\right)\right] dt \\
&\geq \frac{1}{1+B}\Phi\left(\frac{1}{\sqrt{2\pi}B}\right) \\
&\quad + \frac{B}{1+B}\frac{1}{T}\int_0^T \left[1 - \Phi\left(\sqrt{\frac{\max\left\{\frac{T}{2t}, 1\right\}}{2\pi}}\right) - \frac{1}{1-t/3T}\exp(-(1-t/3T)^2/4\pi)\right] dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1+B}\Phi\left(\frac{1}{\sqrt{2\pi}B}\right) \\
&\quad + \frac{B}{1+B}\int_0^1 \left[1-\Phi\left(\sqrt{\frac{\max\left\{\frac{1}{2v},1\right\}}{2\pi}}\right) - \frac{1}{1-v/3}\exp(-(1-v/3)^2/4\pi)\right] dv \\
&= \frac{1}{1+B}\Phi\left(\frac{1}{\sqrt{2\pi}B}\right) - 0.853\frac{B}{1+B} \\
&\geq \frac{1}{1+B} - \frac{B}{1+B}\left(\exp(-1/4\pi B^2) + 0.853\right).
\end{aligned}
$$

Here Φ is the C.D.F of a standard normal random variable. The first inequality follows from the third inequality in the proof of Theorem 1, the second and third inequalities hold because $\sigma_t \vee \overline{\sigma}_{T,1} \leq \sigma_T \leq \sqrt{2\pi}\lambda B$, the fourth inequality follows Property 2 in Lemma 8, and the last inequality is derived from the fact that $1-\Phi(x) \leq \exp(-x^2/2)/x\sqrt{2\pi}$ for $x > 0$.

Combined with the lower bound derived in Theorem 1, we have

$$
\frac{J^{\pi_{RFP}}(x^0,\lambda^0,0)}{J^*(x^0,\lambda^0,0)} \geq \max\left\{0.342, \frac{1}{1+B} - \frac{B}{1+B}\left(\exp(-1/4\pi B^2)+0.853\right)\right\}.
$$

■

A.1. Performance Guarantees for Alternate Market Size Processes

While we focused on providing performance guarantees for market size processes satisfying Assumption 2, our analysis is easily extended to a number of distinct classes of market size processes. The analysis schema is essentially identical to what we have seen thus far, except for the final steps of the analysis where one must specialize to properties of the marginals of the market size process in question. To illustrate this, we present analogues to Theorem 1 for two market size processes outside of those specified by Assumption 2. The first class of processes we consider are 'reflected' generalized moving average processes, where as opposed to considering $\Lambda_t = (\overline{\Lambda}_t)^+$ we consider $\Lambda_t = |\overline{\Lambda}_t|$ where $\overline{\Lambda}_t$ is constructed as before. Here we have:

Theorem 3. *Consider the* RFP *policy with $\alpha = 0$. Let $\overline{\Lambda}_t$ satisfy the requirements of Assumption 2. Moreover, assume that $\lambda_t = \lambda$ for all t. Then, if $\Lambda_t = |\overline{\Lambda}_t|$, we must have:*

$$
\frac{J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0)}{J^*(x^0,\lambda^0,0)} \geq 0.243.
$$

Proof. Now, we have:

$$
\begin{aligned}
\frac{J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0)}{J^*_{\mathrm{CE}}(x^0,\lambda^0,0)} &\geq \frac{\mathsf{E}\left[\int_0^T \frac{X_t}{\Lambda_t(T-t)} g\left(\frac{\Lambda_t(T-t)}{\tilde{X}_t}\right)\Lambda_t dt\right]}{x_0 g\left(\frac{\lambda T + \int_0^T \sqrt{2}\sigma_t dt/\sqrt{\pi}}{x_0}\right)} \\
&\geq \frac{\mathsf{E}\left[\int_0^T g\left(\frac{\Lambda_t T}{x_0}\right) dt\right]}{Tg\left(\frac{\lambda T + \int_0^T \sqrt{2}\sigma_t dt/\sqrt{\pi}}{x_0}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\int_0^T \int_{-\infty}^{\infty} g\left(\frac{T|\lambda+y|}{x_0}\right) \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dydt}{Tg\left(\frac{\lambda T + \int_0^T \sqrt{2}\sigma_t dt/\sqrt{\pi}}{x_0}\right)} \\
&\geq \frac{\int_0^T \int_{-\infty}^{\infty} g\left(\frac{T(\lambda+y)^+}{x_0}\right) \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dydt}{Tg\left(\frac{\lambda T + \int_0^T \sqrt{2}\sigma_t dt/\sqrt{\pi}}{x_0}\right)} \\
&\geq \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} \min\left\{1, \frac{(\lambda+y)^+}{\lambda + \int_0^T \sqrt{2}\sigma_t dt/T\sqrt{\pi}}\right\} \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dydt \\
&= \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}\sqrt{2}}{\sigma_t\sqrt{\pi}}\right) + \int_{-\lambda}^{\overline{\sigma}_{T,1}\sqrt{2/\pi}} \frac{\lambda+y}{\lambda + \overline{\sigma}_{T,1}\sqrt{2/\pi}} \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dy\right] dt \\
&\geq \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}\sqrt{2}}{\sigma_t\sqrt{\pi}}\right) + \int_{0}^{\overline{\sigma}_{T,1}\sqrt{2/\pi}} \frac{y}{\overline{\sigma}_{T,1}\sqrt{2/\pi}} \frac{\exp(-y^2/2\sigma_t^2)}{\sqrt{2\pi\sigma_t^2}} dy\right] dt \\
&\geq 0.243.
\end{aligned}
$$

The first equality holds by definition of $g(\cdot)$ and Lemma 1. In addition, we use the fact that for a Normal random variable X with mean μ and variance σ^2, we know that $\mathsf{E}[|X|] \leq \mu + \sqrt{2}\sigma/\sqrt{\pi}$ ao that $\mathsf{E}[\Lambda_t] = \mathsf{E}[|\overline{\Lambda}_t|] \leq \lambda + \sqrt{2}\sigma_t/\sqrt{\pi}$. The second inequality follows by applying the inventory balancing Lemma (Lemma 2) to obtain a lower bound on X_t along with the property that $zg(1/z)$ is a non-decreasing function, which is established in Lemma 6. The fourth inequality follows by Property 4 in Lemma 6. Finally, by Lemma 1 we have that $J^*(x^0, \lambda^0, 0) \leq J^*_{\mathrm{CE}}(x^0, \lambda^0, 0)$ so that

$$
\frac{J^{\pi_{\mathrm{RFP}}}(x^0, \lambda^0, 0)}{J^*(x^0, \lambda^0, 0)} \geq \frac{J^{\pi_{\mathrm{RFP}}}(x^0, \lambda^0, 0)}{J^*_{\mathrm{CE}}(x^0, \lambda^0, 0)},
$$

and the guarantee follows. ■

As a second example of an alternate market size process, we consider a market-size process specified by the Cox-Ingersoll-Ross (CIR) process

$$
d\Lambda_t = \theta(\lambda - \Lambda_t)dt + \sigma\sqrt{\Lambda_t}dZ_t,
$$

where $\theta, \lambda, \sigma > 0$. As is customary for the use of this process in applications we consider the regime where $2\theta\lambda > \sigma^2$ wherein the process above becomes an example of a strictly positive and ergodic affine process. In this model, θ controls the speed of market-size adjustment, λ and σ corresponds to mean and volatility of the process respectively. The stationary distribution for this process is Gamma distributed with shape parameter $2\theta\lambda/\sigma^2$ and scale parameter $\sigma^2/2\theta$. We assume Λ_0 is distributed according to this stationary distribution and define $\lambda = \Lambda_0$.

Theorem 4. *Consider the RFP policy with $\alpha = 0$. Then if Λ_t is driven by the CIR process above, we have:*

$$
\frac{J^{\pi_{\mathrm{RFP}}}(x^0, \lambda^0, 0)}{J^*(x^0, \lambda^0, 0)} \geq 0.632.
$$

Proof. Now, we have:

$$\begin{aligned}
\frac{J^{\pi_{\rm RFP}}(x^0,\lambda^0,0)}{J^*_{\rm CE}(x^0,\lambda^0,0)} &\geq \frac{\mathsf{E}\left[\int_0^T \frac{X_t}{\Lambda_t(T-t)} g\left(\frac{\Lambda_t(T-t)}{X_t}\right)\Lambda_t dt\right]}{x_0 g\left(\frac{\lambda T}{x_0}\right)} \\
&\geq \frac{\mathsf{E}\left[\int_0^T g\left(\frac{\Lambda_t T}{x_0}\right)dt\right]}{T g\left(\frac{\lambda T}{x_0}\right)} \\
&\geq \frac{1}{T}\int_0^T \mathsf{E}[\min\{\frac{\Lambda_t}{\lambda},1\}]dt \\
&= \mathsf{E}\left[\min\{\frac{\Lambda_0}{\lambda},1\}\right] \\
&= 1-\frac{\Gamma(a+1,a)}{\Gamma(a+1)}+\frac{\Gamma(a,a)}{\Gamma(a)} \\
&\geq 0.632.
\end{aligned}$$

The second inequality follows by applying the inventory balancing Lemma (Lemma 2) to obtain a lower bound on X_t along with the property that $zg(1/z)$ is a non-decreasing function, which is established in Lemma 6. The third inequality follows by Property 4 in Lemma 6. $\Gamma(\cdot,\cdot)$ is an incomplete Gamma function and is given by $\Gamma(x,y)=\int_y^\infty s^{x-1}e^{-s}ds$, and $a \triangleq 2\theta\lambda/\sigma^2 \geq 1$. By Lemma 1 we have that $J^*(x^0,\lambda^0,0) \leq J^*_{\rm CE}(x^0,\lambda^0,0)$ so that

$$\frac{J^{\pi_{\rm RFP}}(x^0,\lambda^0,0)}{J^*(x^0,\lambda^0,0)} \geq \frac{J^{\pi_{\rm RFP}}(x^0,\lambda^0,0)}{J^*_{\rm CE}(x^0,\lambda^0,0)}.$$

and the guarantee follows. ■

B. Proofs for Section 5

Lemma 4. *(Sample Path Modulus of Continuity) Assume that $\overline{\Lambda}_t$ is a generalized moving average process with $\phi \in \mathcal{C}_2$ and $\lambda_t = \lambda$. Then, for $\Delta > 0$, and any $t \in [0,T)$, we have:*

$$\limsup_{\Delta\to 0}\sup_{0\leq t\leq T-\tau, 0\leq\tau\leq\Delta}\frac{|\Lambda_{t+\tau}-\Lambda_t|}{\sigma\sqrt{2\Delta\log(1/\Delta)}} \leq 1 \text{ a.s.}$$

Proof. For any $0 \leq t \leq T-\tau$ and $0 \leq \tau \leq \Delta$, we have

$$\begin{aligned}
|\Lambda_{t+\tau}-\Lambda_t| &= \left|\left(\lambda+\int_0^{t+\tau}\phi(t+\tau-s)dZ_s\right)^+ - \left(\lambda+\int_0^t\phi(t-s)dZ_s\right)^+\right| \\
&\leq \left|\int_t^{t+\tau}\phi(t+\tau-s)dZ_s\right| + \left|\int_0^t(\phi(t-s)-\phi(t+\tau-s))dZ_s\right| \\
&= \left|\sigma Z_{t+\tau}-\phi(\tau)Z_t+\int_t^{t+\tau}\phi^{'}(t+\tau-s)Z_s ds\right| \\
&\quad +\left|(\phi(0)-\phi(\tau))Z_t+\int_0^t(\phi^{'}(t-s)-\phi^{'}(t+\tau-s))Z_s ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma|Z_{t+\tau}-Z_t| + (\phi(0)-\phi(\tau))|Z_t| + \left|\int_t^{t+\tau}\phi^{'}(t+\tau-s)Z_s ds\right| \\
&\quad +(\phi(0)-\phi(\tau))|Z_t| + \left|\int_0^t(\phi^{'}(t-s)-\phi^{'}(t+\tau-s))Z_s ds\right| \\
&\leq \sigma \sup_{0\leq s\leq T-u, 0\leq u\leq \Delta}|Z_{s+u}-Z_s| + L_{\phi 1}\tau B + L_{\phi 1}\tau B + L_{\phi 1}\tau B + L_{\phi 2}\tau B t \\
&= \sigma \sup_{0\leq s\leq T-u, 0\leq u\leq \Delta}|Z_{s+u}-Z_s| + (3L_{\phi 1}B + L_{\phi 2}Bt)\tau
\end{aligned}
$$

where $B \triangleq \sup_{0\leq t\leq T} Z_t$. The first inequality follows property that $|(A+B)^+-(A+C)^+| \leq |B-C|$, the second equality follows from the integration by parts formulas for stochastic integrals, the third inequality follows from the assumed differentiability properties of $\phi(t)$ (the constants correspond to bounds on the appropriate differentials) and the definition of B.

Now, we have

$$
\begin{aligned}
\limsup_{\Delta\to 0} \sup_{0\leq t\leq T-\tau, 0\leq\tau\leq\Delta} \frac{|\Lambda_{t+\tau}-\Lambda_t|}{\sqrt{2\Delta\log(1/\Delta)}} &\leq \limsup_{\Delta\to 0} \sup_{0\leq t\leq T-\tau, 0\leq\tau\leq\Delta} \frac{\sigma|Z_{t+\tau}-Z_t| + (3L_{\phi 1}B + L_{\phi 2}Bt)\tau}{\sqrt{2\Delta\log(1/\Delta)}} \\
&= \limsup_{\Delta\to 0} \sup_{0\leq t\leq T-\tau, 0\leq\tau\leq\Delta} \frac{\sigma|Z_{t+\tau}-Z_t|}{\sqrt{2\Delta\log(1/\Delta)}} \\
&= \sigma,
\end{aligned}
$$

where the first inequality follows from the first part of our argument, and the second inequality is Levy's theorem on the modulus of continuity of sample paths of Brownian motion. ■

Lemma 5. *Assume that $\overline{\Lambda}_t$ is a generalized moving average process with $\phi \in \mathcal{C}_2$ and $\lambda_t = \lambda$. Then, for any $t \in [0, T)$, we have almost surely:*

$$
\limsup_{\Delta\to 0} \frac{|\tilde{\Lambda}_{t(\Delta)}-\Lambda_t|}{\sigma\sqrt{\Delta\log(1/\Delta)}} \leq 2
$$

and further,

$$
\limsup_{\Delta\to 0} \frac{|X^{\Delta}_{t(\Delta)}-X_t|}{\sigma T\sqrt{\Delta\log(1/\Delta)}} \leq 4
$$

Proof. First, we prove the convergence rate of the estimated market size. We have

$$
\begin{aligned}
|\tilde{\Lambda}_{t(\Delta)}-\Lambda_t| &= \left|\frac{1}{\Delta}\int_{t(\Delta)-\Delta}^{t(\Delta)}\Lambda_s ds - \Lambda_t\right| \\
&\leq \sup_{0\leq s\leq 2\Delta}|\Lambda_{t-s}-\Lambda_t|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\limsup_{\Delta\to 0} \frac{|\tilde{\Lambda}_{t(\Delta)}-\Lambda_t|}{\sigma\sqrt{\Delta\log(1/\Delta)}} &\leq \limsup_{\Delta\to 0} \frac{\sup_{0\leq s\leq 2\Delta}|\Lambda_{t-s}-\Lambda_t|}{\sigma\sqrt{\Delta\log(1/\Delta)}} \\
&\leq \limsup_{\Delta\to 0} \frac{\sup_{0\leq t\leq T-s, 0\leq s\leq 2\Delta}|\Lambda_{t-s}-\Lambda_t|}{\sigma\sqrt{\Delta\log(1/\Delta)}} \\
&= \limsup_{\Delta\to 0} \frac{\sup_{0\leq t\leq T-s, 0\leq s\leq 2\Delta}|\Lambda_{t-s}-\Lambda_t|}{\sigma\sqrt{4\Delta\log(1/2\Delta)}} \frac{\sqrt{4\Delta\log(1/2\Delta)}}{\sqrt{\Delta\log(1/\Delta)}}
\end{aligned}
$$

$$\leq \quad 2.$$

The last inequality follows from Lemma 4.

Next, we prove the convergence rate of the inventory process under the RFP-Δ policy. Now for $i > 0$, we have

$$X^{\Delta}_{(i+1)\Delta} = \left(X^{\Delta}_{i\Delta} - \overline{F}(\pi^{\Delta}_{\text{RFP}}(X^{\Delta, i\Delta}, i\Delta)) \int_{i\Delta}^{(i+1)\Delta} \Lambda_s ds \right)^{+}.$$

We have that for $\epsilon > 0$, there exist numbers $C(\epsilon), D(\epsilon) < \infty$, such that $1/(T-s) \leq C(\epsilon)$ and $|\frac{d}{ds}\frac{1}{T-s}| \leq D(\epsilon)$ for all $s < T - \epsilon$. Now, for any s, Δ such that $2\Delta \leq s < T - \epsilon$, we have:

$$\begin{aligned}
&\left| \min\left\{ \overline{F}(p^*), \frac{X_s}{\Lambda_s (T-s)} \right\} \Lambda_s - \min\left\{ \overline{F}(p^*), \frac{X_{s(\Delta)}}{\tilde{\Lambda}_{s(\Delta)}(T - s(\Delta))} \right\} \Lambda_s \right| \\
&\leq \left| \min\left\{ \overline{F}(p^*)\Lambda_s, \frac{X_s}{T-s} \right\} - \min\left\{ \overline{F}(p^*)\tilde{\Lambda}_{s(\Delta)}, \frac{X_{s(\Delta)}}{T - s(\Delta)} \right\} \right| \\
&\quad + \min\left\{ \overline{F}(p^*), \frac{X_{s(\Delta)}}{\tilde{\Lambda}_{s(\Delta)}(T - s(\Delta))} \right\} \left| \Lambda_s - \tilde{\Lambda}_{s(\Delta)} \right| \\
&\leq \overline{F}(p^*) \left| \Lambda_s - \tilde{\Lambda}_{s(\Delta)} \right| + \left| \frac{X_s}{T-s} - \frac{X_{s(\Delta)}}{T - s(\Delta)} \right| + \overline{F}(p^*) \left| \Lambda_s - \tilde{\Lambda}_{s(\Delta)} \right| \\
&\leq 2 \sup_{0 \leq \tau \leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}| + \left| \frac{X_s}{T-s} - \frac{X_{s(\Delta)}}{T-s} \right| + \left| \frac{X_{s(\Delta)}}{T-s} - \frac{X_{s(\Delta)}}{T - s(\Delta)} \right| \\
&\leq 2 \sup_{0 \leq s \leq T-\tau, 0 \leq \tau \leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}| + (C(\epsilon)K + x_0 D(\epsilon))\Delta,
\end{aligned} \tag{7}$$

where $K \triangleq \sup_{t \in [0,T]} \Lambda_t$. The second inequality follows from the fact that $|\min\{A, B\} - \min\{C, D\}| \leq |A - C| + |B - D|$. Now, we have, for $i \geq 1$ with $(i+1)\Delta < T - \epsilon$,

$$\begin{aligned}
|X^{\Delta}_{(i+1)\Delta} - X_{(i+1)\Delta}| \quad &= \quad \left| \left(X^{\Delta}_{i\Delta} - \int_{i\Delta}^{(i+1)\Delta} \min\left\{ \overline{F}(p^*), \frac{X^{\Delta}_{i\Delta}}{\tilde{\Lambda}_{i\Delta}(T - i\Delta)} \right\} \Lambda_s ds \right)^{+} \right. \\
&\qquad \left. - \left(X_{i\Delta} - \int_{i\Delta}^{(i+1)\Delta} \min\left\{ \overline{F}(p^*), \frac{X_s}{\Lambda_s(T-s)} \right\} \Lambda_s ds \right)^{+} \right| \\
&\leq \quad \left| \left(X^{\Delta}_{i\Delta} - \int_{i\Delta}^{(i+1)\Delta} \min\left\{ \overline{F}(p^*), \frac{X^{\Delta}_{i\Delta}}{\tilde{\Lambda}_{i\Delta}(T - i\Delta)} \right\} \Lambda_s ds \right)^{+} \right. \\
&\qquad \left. - \left(X_{i\Delta} - \int_{i\Delta}^{(i+1)\Delta} \min\left\{ \overline{F}(p^*), \frac{X_{i\Delta}}{\tilde{\Lambda}_{i\Delta}(T - i\Delta)} \right\} \Lambda_s ds \right)^{+} \right| \\
&\qquad + \left| \int_{i\Delta}^{(i+1)\Delta} \min\left\{ \overline{F}(p^*), \frac{X_s}{\Lambda_s(T-s)} \right\} \Lambda_s ds \right. \\
&\qquad \left. - \int_{i\Delta}^{(i+1)\Delta} \min\left\{ \overline{F}(p^*), \frac{X_{i\Delta}}{\tilde{\Lambda}_{i\Delta}(T - i\Delta)} \right\} \Lambda_s ds \right| \\
&\leq \quad |X^{\Delta}_{i\Delta} - X_{i\Delta}|
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{i\Delta}^{(i+1)\Delta} \left| \min\left\{\overline{F}(p^*), \frac{X_s}{\Lambda_s(T-s)}\right\} - \min\left\{\overline{F}(p^*), \frac{X_{s(\Delta)}}{\tilde{\Lambda}_{s(\Delta)}(T-s(\Delta))}\right\}\right| \Lambda_s ds \\
\leq \quad & |X^{\Delta}_{i\Delta} - X_{i\Delta}| + 2 \sup_{0\leq s\leq T-\tau, 0\leq\tau\leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}|\Delta + (C(\epsilon)K + x_0 D(\epsilon))\Delta^2,
\end{aligned}
$$

where the first inequality follows from the property that $|A^+ - (B+C)^+| \leq |A^+ - B^+| + |C|$, the second inequality follows from the property that $|(X - \min\{a, bX\})^+ - (Y - \min\{a, bY\})^+| \leq |X - Y|$ for $b \geq 0$, and the last inequality follows from (7). Moreover, since trivially $|X^{\Delta}_{\Delta} - X_{\Delta}| \leq \int_0^{\Delta} \Lambda_s ds \leq K\Delta$, we must have for any positive integer i with $i\Delta < T - \epsilon$,

$$
\begin{aligned}
|X^{\Delta}_{i\Delta} - X_{i\Delta}| &\leq 2T \sup_{0\leq s\leq T-\tau, 0\leq\tau\leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}| + (C(\epsilon)K + x_0 D(\epsilon))\Delta^2(i-1) + K\Delta \\
&\leq 2T \sup_{0\leq s\leq T-\tau, 0\leq\tau\leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}| + (C(\epsilon)KT + x_0 D(\epsilon)T + K)\Delta.
\end{aligned}
$$

Hence, for any $t < T - \epsilon$,

$$
\begin{aligned}
|X^{\Delta}_{t(\Delta)} - X_t| &\leq |X^{\Delta}_{t(\Delta)} - X_{t(\Delta)}| + |X_{t(\Delta)} - X_t| \\
&\leq 2T \sup_{0\leq s\leq T-\tau, 0\leq\tau\leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}| + (C(\epsilon)KT + x_0 D(\epsilon)T + K)\Delta + K\Delta
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\limsup_{\Delta\to 0} \frac{|X^{\Delta}_{t(\Delta)} - X_t|}{\sigma T\sqrt{\Delta\log(1/\Delta)}} \quad &\leq \quad \limsup_{\Delta\to 0} \frac{2T\sup_{0\leq s\leq T-\tau, 0\leq\tau\leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}| + (C(\epsilon)BT + x_0 D(\epsilon)T + B)\Delta + B\Delta}{\sigma T\sqrt{\Delta\log(1/\Delta)}} \\
&= \quad \limsup_{\Delta\to 0} \frac{2\sup_{0\leq s\leq T-\tau, 0\leq\tau\leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}|}{\sigma\sqrt{\Delta\log(1/\Delta)}} \\
&= \quad \limsup_{\Delta\to 0} \frac{2\sup_{0\leq s\leq T-\tau, 0\leq\tau\leq 2\Delta} |\Lambda_s - \Lambda_{s-\tau}|}{\sigma\sqrt{4\Delta\log(1/2\Delta)}} \frac{\sqrt{4\Delta\log(1/2\Delta)}}{\sqrt{\Delta\log(1/\Delta)}} \\
&\leq \quad 4,
\end{aligned}
$$

for all $t < T - \epsilon$. The last inequality follows from Lemma 4. Since our choice of $\epsilon > 0$ was arbitrary, the result follows. ∎

Theorem 2. *(The Price of Discretization) For generalized moving average processes and an RFP-Δ policy with $\alpha = 1$, we have:*

$$
\limsup_{\Delta\to 0} \frac{\left| J^{\pi_{\text{RFP}}}(x^0, \lambda^0, 0) - J^{\pi^{\Delta}_{\text{RFP}}}(x^0, \lambda^0, 0)\right|}{\eta(\Delta)\log(1/\eta(\Delta))} \leq 4p^*\overline{F}(p^*)\sigma\frac{\mathsf{E}K^2T^3}{x_0^2}
$$

where $\sigma \triangleq \phi(0)$, and we assume $\lambda_t = \lambda$ for all t.

Proof. Recall that by the inventory balancing property we have that:

$$
\frac{X_t}{T-t} \geq \frac{x_0}{T}.
$$

Using this fact with Lemma 5 allows us to conclude after some algebraic manipulation that for any $t < T$ that

$$
\limsup_{\Delta\to 0} \frac{1}{\sigma\eta(\Delta)} \left| \frac{\tilde{\Lambda}_{t(\Delta)}(T - t(\Delta))}{X^{\Delta}_{t(\Delta)}} - \frac{\Lambda_t(T-t)}{X_t}\right| \leq \frac{2T}{x_0} + \frac{4KT^3}{x_0^2(T-t)}
$$

Let $\kappa(\Delta) \triangleq 8T^2\sigma\eta(\Delta)/x_0$. Observe that on $t < T-\kappa(\Delta)$, we must have by the Balancing Lemma that $X_t \geq 8T\sigma\eta(\Delta)$, so that for Δ sufficiently small, Lemma 5 guarantees that $X^{\Delta}_{t(\Delta)} > 0$ as well. Consequently, we have that for Δ sufficiently small:

(8)

$$
\begin{aligned}
&|J^{\pi^{\Delta}_{\mathrm{RFP}}}(x^0,\lambda^0,0) - J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0)| \\
&= \left|\mathsf{E}\left[\int_0^T \pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta,t},t)\overline{F}\left(\pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta,t},t)\right)\Lambda_t dt\right] - \mathsf{E}\left[\int_0^T \pi_{\mathrm{RFP}}(X^t,t)\overline{F}\left(\pi_{\mathrm{RFP}}(X^t,t)\right)\Lambda_t dt\right]\right| \\
&\leq \mathsf{E}\left[\left|\int_0^T \pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta,t},t)\overline{F}\left(\pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta,t},t)\right)\Lambda_t dt - \int_0^T \pi_{\mathrm{RFP}}(X^t,t)\overline{F}\left(\pi_{\mathrm{RFP}}(X^t,t)\right)\Lambda_t dt\right|\right] \\
&\leq \mathsf{E}\left[\left|\int_0^{T-\kappa(\Delta)} \pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta}_{t(\Delta)},t)\overline{F}\left(\pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta}_{t(\Delta)},t)\right)\Lambda_t dt - \int_0^{T-\kappa(\Delta)} \pi_{\mathrm{RFP}}(X^t,t)\overline{F}\left(\pi_{\mathrm{RFP}}(X^t,t)\right)\Lambda_t dt\right|\right] \\
&\quad + \mathsf{E}K\kappa(\Delta)p^*\overline{F}(p^*).
\end{aligned}
$$

Now, by our choice of $\kappa(\Delta)$, we have that for Δ sufficiently small that

$$
\begin{aligned}
\mathcal{E}(\Delta) \;&\triangleq\; \left|\int_0^{T-\kappa(\Delta)} \pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta}_{t(\Delta)},t)\overline{F}\left(\pi^{\Delta}_{\mathrm{RFP}}(X^{\Delta}_{t(\Delta)},t)\right)\Lambda_t dt - \int_0^{T-\kappa(\Delta)} \pi_{\mathrm{RFP}}(X^t,t)\overline{F}\left(\pi_{\mathrm{RFP}}(X^t,t)\right)\Lambda_t dt\right| \\
&\leq\; Kp^*\overline{F}(p^*)\int_0^{T-\kappa(\Delta)}\left|\frac{\tilde{\Lambda}_{t(\Delta)}(T-t(\Delta))}{X^{\Delta}_{t(\Delta)}} - \frac{\Lambda_t(T-t)}{X_t}\right| dt \\
&\leq\; Kp^*\overline{F}(p^*)\left(\sigma\eta(\Delta)\int_0^{T-\kappa(\Delta)}\left(\frac{2T}{x_0} + \frac{4KT^3}{x_0^2(T-t)}\right)dt\right) \\
&\leq\; Kp^*\overline{F}(p^*)\sigma\eta(\Delta)\left(\frac{2T^2}{x_0} + \frac{4KT^3\left(\log T + \log(1/\kappa(\Delta))\right)}{x_0^2}\right)
\end{aligned}
$$

where the first inequality follows from the fact that the function $g(y)/y$ has its first derivative bounded in absolute value by $p^*\overline{F}(p^*)$, and the second inequality was established at the start of the proof. It follows that

$$
\limsup_{\Delta\to 0}\frac{\mathcal{E}(\Delta)}{\sigma\eta(\Delta)\log(1/\eta(\Delta))} \leq \frac{4K^2T^3p^*\overline{F}(p^*)}{x_0^2}
$$

Using this inequality, (8) then yields

$$
\begin{aligned}
\limsup_{\Delta\to 0}\frac{\left|J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0) - J^{\pi^{\Delta}_{\mathrm{RFP}}}(x^0,\lambda^0,0)\right|}{\eta(\Delta)\log(1/\eta(\Delta))} \;&\leq\; \limsup_{\Delta\to 0}\mathsf{E}\left[\frac{\mathcal{E}(\Delta)+\kappa(\Delta)p^*\overline{F}(p^*)K}{\eta(\Delta)\log(1/\eta(\Delta))}\right] \\
&\leq\; \frac{4\mathsf{E}K^2T^3p^*\overline{F}(p^*)\sigma}{x_0^2}
\end{aligned}
$$

where the second inequality follows from Fatou's lemma. ■

C. Miscellaneous Results and Computations

C.1. Properties of the Market-Size Process

We present in this Section, a few technical results for a class of market size processes satisfying the assumption below. It is simple to check that this class subsumes the class of generalized moving average processes we have studied in this paper.

Assumption 2.

1. *$\Lambda_t = \left(\overline{\Lambda}_t\right)^+$ where $\overline{\Lambda}_t$ is a Gaussian process with continuous sample paths.*
2. *$\mathsf{E}\left[\overline{\Lambda}_t\right] \triangleq \lambda_t$ is positive.*
3. *The variance of the random variable $\overline{\Lambda}_t$, σ_t^2, is non-decreasing as a function of t and concave.*

Indeed it is evident that our moving average processes satisfy the first two requirements; to see that the last requirement is satisfied, we observe that in the case of moving average processes, Ito isometry yields $\mathrm{Var}(\overline{\Lambda}_t) = \int_0^t \phi^2(s)ds$ which is evidently non-decreasing and concave.

Lemma 7. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing, concave function with $f(0) = 0$. Then for all $0 < y \le x$,*

$$1 \le \frac{f(x)}{f(y)} \le \frac{x}{y},$$

and

$$\frac{1}{x}\int_0^x f(t)dt \le f\left(\frac{x}{2}\right). \tag{9}$$

Proof. By definition, $f(x)/f(y) \ge 1$. Moreover, the concavity of f yields $f(x) = f(0+\frac{x}{y}y) \le \frac{x}{y}f(y)$. Thus, $f(x)/x \le f(y)/y$. Inequality (9) follows by Jensen's inequality. ∎

Now, we use Lemma 7 to characterize properties of the volatility of the market-size process, σ_t^2. Defining

$$\overline{\sigma}_{T,1} \triangleq \int_0^T \sigma_t dt/T \text{ and}$$

$$\overline{\sigma}_{T,2} \triangleq \int_0^T \sigma_t^2 dt/T.$$

we have:

Lemma 8.

1. *σ_t is non-decreasing and concave in t.*
2. *$1 - t/3T \le \overline{\sigma}_{T,1}/\sigma_t \le \sqrt{\overline{\sigma}_{T,2}/\sigma_t^2} \le \sqrt{\max\{T/2t, 1\}}$.*
3. *$\frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \int_0^{\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{y}{\overline{\sigma}_{T,1}\sigma_t}\exp(-y^2/2\sigma_t^2)dy\right] dt \ge 0.342$.*

Proof.

1. σ_t^2 is non-decreasing in t directly implies that σ_t is non-decreasing in t. Now,

$$(\sigma_t^2)'' = 2(\sigma_t')^2 + 2\sigma_t\sigma_t''$$

so that since σ_t^2 is concave, $\sigma_t'' \le 0$ and the concavity of σ_t follows.

2. To establish the first inequality, we see that:

$$\begin{aligned}
\frac{\overline{\sigma}_{T,1}}{\sigma_t} &= \frac{\int_0^T \sigma_s ds}{T\sigma_t} = \frac{1}{T}\int_0^T \sqrt{\frac{\sigma_s^2}{\sigma_t^2}} ds \\
&\geq \frac{1}{T}\int_0^T \sqrt{\min\left\{\frac{s}{t}, 1\right\}} ds \qquad (10) \\
&= 1 - \frac{t}{3T},
\end{aligned}$$

where inequality (10) follows by Lemma 7 and the concavity of σ_t^2.

That $\overline{\sigma}_{T,1}/\sigma_t \leq \sqrt{\overline{\sigma}_{T,2}/\sigma_t^2}$ is a direct consequence of Jensen's inequality.

The second part of Lemma 7 yields $\overline{\sigma}_{T,2} \leq \sigma_{T/2}^2$, so that the first part of Lemma 7 then yields:

$$\frac{\overline{\sigma}_{T,2}}{\sigma_t^2} \leq \frac{\sigma_{T/2}^2}{\sigma_t^2} \leq \max\left\{\frac{T}{2t}, 1\right\}.$$

3. We have:

$$\begin{aligned}
&\frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \int_0^{\overline{\sigma}_{T,1}/\sqrt{2\pi}} \frac{y}{\overline{\sigma}_{T,1}\sigma_t}\exp(-\frac{y^2}{2\sigma_t^2})dy\right] dt \\
&= \frac{1}{T}\int_0^T \left[1 - \Phi\left(\frac{\overline{\sigma}_{T,1}}{\sigma_t\sqrt{2\pi}}\right) + \frac{\sigma_t}{\overline{\sigma}_{T,1}}\left(1 - \exp(-(\overline{\sigma}_{T,1})^2/4\pi\sigma_t^2)\right)\right] dt \\
&\geq \frac{1}{T}\int_0^T \left[1 - \Phi\left(\sqrt{\frac{\max\left\{\frac{T}{2t}, 1\right\}}{2\pi}}\right) + \frac{1}{\sqrt{\max\left\{\frac{T}{2t}, 1\right\}}}\left(1 - \exp(-(1 - t/3T)^2/4\pi)\right)\right] dt \\
&= \int_0^1 \left[1 - \Phi\left(\sqrt{\frac{\max\left\{\frac{1}{2v}, 1\right\}}{2\pi}}\right) + \frac{1}{\sqrt{\max\left\{\frac{1}{2v}, 1\right\}}}\left(1 - \exp(-(1 - v/3)^2/4\pi)\right)\right] dv \\
&= 0.342,
\end{aligned}$$

where the first inequality follows from the previous property (i.e. Lemma 8, Property 2); the penultimate equality follows by employing the change of variables $v = t/T$, and the final equality follows from numerical evaluation of the definite integral in the penultimate line.

■

C.2. Analysis for Example 1 in Section 3

Recall, that our goal is to show that if $\sigma > 0$, then

$$\frac{J^{\pi_{\mathrm{FP}}}(x^0, \lambda^0, 0)}{J^*(x^0, \lambda^0, 0)} \leq O((\log T)^{-1}),$$

for the dynamic pricing problem described in Example 1. To show this, we will find it convenient to use properties of the RFP policy established in Section 4, as we will use performance under this policy as a lower bound to performance under an optimal policy. Now, we have

$$\begin{aligned}
\frac{J^{\pi_{\mathrm{FP}}}(x^0,\lambda^0,0)}{J^*(x^0,\lambda^0,0)} &\leq \frac{p^* x_0}{J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0)} \\
&\leq \frac{x_0}{J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0)} \\
&\leq \left[0.342 g\left(1+\frac{2T^{3/2}\sigma}{3\sqrt{2\pi}x_0} - \frac{\lambda^2\sqrt{T}}{\sigma\sqrt{2\pi}x_0}\right)\right]^{-1} \\
&= \left[0.342 \log\left(1+\frac{2T^{3/2}\sigma}{3\sqrt{2\pi}x_0} - \frac{\lambda^2\sqrt{T}}{\sigma\sqrt{2\pi}x_0}\right)\right]^{-1} \\
&= O((\log T)^{-1}).
\end{aligned}$$

The first inequality follows by the definition of J^* and also the fact that performance under the fixed price policy is trivially upper bounded by $p^* x_0$; in the case of our example, recall that $p^* = 1$. We now focus on the second inequality: Theorem 1 showed that

$$J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0) \geq 0.342 J^*_{\mathrm{CE}}(x^0,\lambda^0,0),$$

while by the definition of the unit revenue function, $g(\cdot)$, in Section 4, we know that

$$J^*_{\mathrm{CE}}(x^0,\lambda^0,0) = x_0 g\left(\frac{\int_0^T \mathsf{E}[\Lambda_t]dt}{x_0}\right).$$

Since here, $\int_0^T \mathsf{E}[\Lambda_t]dt \geq \lambda T + \frac{2T^{3/2}\sigma}{3\sqrt{2\pi}} - \frac{\lambda^2\sqrt{T}}{\sigma\sqrt{2\pi}}$ and g is non-decreasing from Lemma 6, it follows that

$$J^{\pi_{\mathrm{RFP}}}(x^0,\lambda^0,0) \geq 0.342 x_0 g\left(1+\frac{2T^{3/2}\sigma}{3\sqrt{2\pi}x_0} - \frac{\lambda^2\sqrt{T}}{\sigma\sqrt{2\pi}x_0}\right).$$

C.3. Computational Experiments Relative to a Tighter Super-Optimal Policy

In our computational experiments, we compared performance of the RFP-Δ policy against a clairvoyant upper bound that was permitted to observe the entire realization of a sample path of the market size process at time 0. While this bound was cheap to compute, we observed that in certain cases performance relative to this upper bound was worse than 10%. We conjectured that this did not reflect our pricing policies performance per se but rather simply the fact that our upper bound was loose in settings with high volatility. As such, we compute a tighter upper bound here, namely the expected revenue under an *optimal* policy with knowledge of the specification of the market size process (i.e. a probability distribution over its sample paths) and the ability to monitor the process and update prices in continuous time. This is obviously still an upper bound on the optimal value function, but nonetheless tighter than the clairvoyant bound. The results are summarized (for an OU process) in Tables 4 and 5.

Table 4: Performance Relative to a Tighter Upper Bound. Common parameters across problem instances: $\lambda = e, \beta = 1, T = 5, CV = 2.5, \Delta = 0.1$.

Initial Inventory	Load Factor	Relative Optimality		
x_0	$x_0/\lambda T$	$J^{\pi_{RFP}}/J^*$	$J^{\pi^{\Delta}_{RFP}}/J^*$	$J^{\pi^{\Delta}_{RFP}}/J^{UB}$
4	0.294	0.951	0.923	0.830
8	0.589	0.962	0.941	0.886
12	0.883	0.979	0.965	0.922
16	1.177	0.990	0.977	0.949
20	1.472	0.998	0.990	0.968

Table 5: Performance Relative to a Tighter Upper Bound. Common parameters across problem instances: $\lambda = e, \beta = 1, T = 5, CV = 5, \Delta = 0.1$.

Initial Inventory	Load Factor	Relative Optimality		
x_0	$x_0/\lambda T$	$J^{\pi_{RFP}}/J^*$	$J^{\pi^{\Delta}_{RFP}}/J^*$	$J^{\pi^{\Delta}_{RFP}}/J^{UB}$
4	0.294	0.922	0.891	0.768
8	0.589	0.938	0.915	0.828
12	0.883	0.947	0.929	0.861
16	1.177	0.951	0.936	0.887
20	1.472	0.966	0.951	0.908

In the experiments above $J^{\pi^{\Delta}_{\mathrm{RFP}}}/J^{UB}$ is the quantity reported for the bulk of our experiments – performance relevant to a clairvoyant upper bound. The quantity $J^{\pi^{\Delta}_{\mathrm{RFP}}}/J^*$ reports performance relative to the tighter upper bounds. Since even this tighter upper bound is potentially loose (since it re-optimizes continuously, and is allowed to observe the monitor the market size process), the quantity $J^{\pi_{\mathrm{RFP}}}/J^*$ report performance of the *idealized* RFP policy (that is also allowed to re-optimize continuously and monitor the market size process directly) against the tighter upper bound. We see that the results bear substantial support to the fact that a large fraction of the performance losses reported in our computational study are potentially due to the fact that we compare ourselves against an upper bound that can be fairly loose. This is not surprising given the amount of information used by the policy implicit i! n the clairvoyant upper bound.