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ONLY AVAILABLE IN ELECTRONIC FORM

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A Model justification

In recent years, there has been a lot of research in an attempt to provide a deeper understanding of optimal control models from a theoretical as well as an application point of view. In particular, an attractive feature of these models is that they provide good scheduling, production and inventory policies in a variety of settings. Furthermore, they approximate well the underlying stochasticity of problems in a deterministic way. Fluid models arise in applications as diverse as routing and communication systems as well as queueing, supply chain and transportation systems. A continuous time approach has the advantage of not introducing any approximation to the real setting: it provides the exact solution of the system. When taking a discrete time approach, one has to decide what a reasonable time step should be, and to allow price and production changes only at those times. In reality, in some settings a supplier may need more flexibility. In order to avoid being too restrictive, the time step needs be very small, and if the time horizon is large the size of the problem may become exceedingly large, in terms of number of variables and number of constraints. Therefore, the problem size usually implies significant delays in obtaining good solutions. Examples of supply chain industries where continuous-time optimal control models of the type we discuss in this paper are relevant, include industries with a high volume of throughput and data on costs and demand that change a lot. The hardware as well as the semiconductor industries are such examples. Moreover, we believe that a similar approach can be applied to problems in areas other than dynamic pricing and inventory control, where the evolution of the system evolves dynamically and justify a continuous time approach. We believe that the techniques presented in this paper may be helpful to those areas as well.

We assume that multiple products share a single common production capacity. This assumption is a standard one in the literature that considers multiclass systems. For example, Bertsimas and Paschalidis [5] consider a multiclass make-to-stock system and assume that a single facility produces several products, with the production process over time taken as an arbitrary stationary stochastic process. Also in a make-to-stock manufacturing setting with multiple products, Kachani and Perakis [17] study this problem assuming that the total production capacity rate across all products is bounded. Gilbert [13] addresses the problem of jointly determining prices and production schedules for a set of items that are produced on the same production equipment and with a limited capacity. Maglaras and Meissner [20] consider a single firm that owns a fixed capacity of a resource that is consumed in the production of multiple products. Finally, Biller et al. [6] extend the single product model of dynamic pricing to cover supply chains with multiple products, each of which is assembled from a set of parts and shares common production capacity. In order to keep the model simple in this paper, we make a similar assumption of a single production capacity constraint. We leave as a direction of future research the case of multiple capacity constraints.

In this paper we assume that the demand for a product depends only on the price for this product and not on the prices of other products. This assumption is standard in multi-product pricing problems when the products are considered distinct so that they target distinct classes of customers. The automotive industry is one example of an industry where such an assumption is valid (see [6]). Bertsimas and de Boer [4] study a joint pricing and resource allocation problem in which a finite supply of a resource can be used to produce multiple products and the demand for each product depends on its price. They apply this problem to airline revenue management. Paschalidis and Liu [24] consider a communication network with fixed routing that can accommodate multiple service classes and in which the arrival rate of a given class (or demand for that class) depends on the price per call of that class only. In their multi-product case, Biller et al. [6] assume that there are no diversions among products, i.e. that a change in the price for one product does not affect the demand for another product. They motivate this assumption by focusing on items that appeal to various consumer market segments, such as for example, luxury cars, SUVs, small pickups, etc. in the automotive industry. For similar reasons, in this paper, we make a similar assumption on the demand being independent on the prices of other products. A more general model would allow the demand to depend on prices of all products with various price sensitivities. However, such a model would significantly increase the complexity of the problem. As a result, at this stage, we feel that this problem would go beyond the scope of this paper but could be an interesting topic for follow-up research.

A stream of the literature on inventory systems assumes that demand can be satisfied even when no inventory is available, i.e. that demand can be backlogged. In other words, in that setting, inventory levels may be negative. In this paper, we assume that no backorders are allowed. This might occur for example, when there is a contract between a supplier and a retailer that does not allow delays in the delivery of the products, or when there is very high fixed backlog cost.

Pekelman [25] studies a problem of optimal pricing and production for a single product with no backorders. Axsäter and Juntti [2] study echelon stock reorder policies with no backorders. While a stream of research considers that demand that cannot be met is lost and incurs a penalty for loss of goodwill, in practice it is difficult to estimate the penalty in terms of numerical loss. Moreover, not satisfying the demand may have long term negative effect on the firm, and it might be preferable to increase the price so that demand lowers and no demand is unmet.

We assume in this paper that production and inventory costs are quadratic. This type of cost have been used often in the literature on inventory control. Goh [15] assumes that the holding cost is a nonlinear function of the amount of the on-hand inventory. He motivates the model by discussing its application to products whose inventory value is very high and many precautionary steps are to be taken to ensure its safety and quality. He cites in particular luxury items like expensive jewelry and designer watches, for which as the on-hand stock inventory grows, some firms employ higher dimensions of security such as hidden cameras and infrared sensors. Similarly, Giri and Chaudhuri [14] consider a model with nonlinear holding cost depending on the stock level with the form $hI^n,\ n > 1$, where I is the inventory level. They justify this assumption by taking the example of electronic components, radioactive substances, or volatile liquids which are costly and require more sophisticated arrangements for their security and safety. Furthermore, we point out that the use of quadratic inventory costs partially addresses the fact that when inventory levels grow too high, it may be necessary for the firm to purchase additional storage space which would justify the convexity of the cost. Holt et al. [16] introduce a linear-quadratic inventory model in which the production and the holding cost are respectively the sum of a linear and a quadratic term in the production rate or the inventory rate. Our model is a particular case where the coefficient of the linear term is zero. They justify this approximation for production costs from a connection with workforce costs. They observe that the cost of hiring and training people rises with the number hired, and the cost of laying off workers, including terminal pay, reorganization, etc., rises with the number laid off. Moreover, for fixed workforce, increasing production may incur overtime costs. Pindyck [26] models production costs for commodities such as copper, lumber and heating oil as quadratic costs. Finally, Sethi et al. [28] assume general convex production and inventory costs. Our model is also a particular case of this model.

We focus on an open-loop Nash equilibrium: the competitors decide at time $t = 0$ their strategy for the entire time horizon. In an open-loop equilibrium, a firm makes an irreversible commitment to a future course of action. This situation may arise in practice if a contract with another firm or with a labor union forces the firm to commit to prices or workforce at the beginning of the time horizon. In such an equilibrium, the policy depends on time and the initial state vector only, the players do not use any other information, on the state variable in particular. In contrast, a feedback/ Markovian Nash equilibrium induces strategies that are based on time and on the current state vector. Then the competitors can observe at all times their current inventory level to choose an optimal policy over the rest of the time horizon. In a closed-loop strategy, the firms may review their course of action as time evolves. A closed-loop Nash equilibrium yields optimal policies that depend on time and all state vectors from time zero up to the current time. See [12] for more details on the difference between closed-loop and open-loop equilibria. Note that a closed-loop solution may be approximated by deriving an open-loop solution and using rolling-horizon techniques.

The assumption of a linear demand function of prices is common in the revenue management and pricing literature. For example, Zabel [30], Pekelman [25], Whitin [29], Mills [21], Lai [19], Chen and Min [8], Cohen [9], Kunreuther and Schrage [18], Palaka, Erlebacher, and Kropp [23], Farahat and Perakis [11], and Carr et al. [7] consider a similar demand model.

The motivation behind our model of uncertainty is, on the one hand, to not require a particular probability distribution on the data, which in practice is very difficult to determine, and on the other hand, to find an alternative to worst case reasoning in which one assumes that the value of the data that is the least favorable occurs at all times. This type of approach has been criticized for being unnecessarily overly conservative and therefore producing a solution that performs poorly in order to protect against the worst, even though unlikely, realization of the data. Indeed, even though data may vary, it is very unlikely that they equal their worst case scenario value at all times. Introducing a budget of uncertainty on the data is an efficient way to measure the trade-off between conservativeness and performance by providing a bound on the allowed spread of the data around the nominal value over time. The budget of uncertainty is data in the model, the modeler can decide whether she wants to obtain a more conservative solution (by choosing a large budget of uncertainty) while sacrificing optimality, or a solution that performs well and is less immune to data uncertainty (by choosing a smaller budget of uncertainty).

When dealing with uncertainty on input parameters, a natural question is to decide what the objective value should be. A common approach when assuming demand follows a probability distribution is to maximize the expected value of the objective. Depending on the context and on risk preferences, it is possible to optimize the best or worst case objective, or to model a utility function based on expected value and standard deviation of the objective. In robust optimization, since no such distribution is assumed, one option is to aim at optimizing the realized objective, which is reformulated as maximizing the worst case objective within the uncertainty set of the parameters. In this paper, we consider a robust approach that maximizes the nominal objective function. In other words, we maximize the "expected" objective function value, not in the probabilistic sense, but in terms of considering the values at the center of the range of realized values. However, we still consider demand uncertainty in the feasibility constraints. Another motivation for this approach comes from the following qualitative observation. The worst case objective corresponds to low demand, while the worst case for the no backorder constraint corresponds to high demand. Therefore the worst case cannot occur simultaneously for both the objective and the constraints, and it would be overly conservative to protect against both occurrences simultaneously. As a result, we choose to focus on ensuring the feasibility of the problem and solve for the worst case of the constraints, but maximize the nominal objective. In the numerical study in Section 7.2, we evaluated the realized objective value in two cases of price sensitivities, 6 cases of budget of uncertainty and for uncertain parameters generated from either a normal distribution or a uniform distribution. We simulated 1000 realizations and averaged the objective value. In all cases, we found that the average realized objective was within 1% of the nominal objective.

B Notations

Superscript k ($k = A,\ B$) denotes supplier k. Superscript $-k$ denotes supplier k's competitor (i.e. if $k = A$, then $-k = B$ and if $k = B$, then $-k = A$).

Inputs

$[0,T]$	time horizon;
N	number of products;
$K^k(t)$	shared production capacity rate at time t (non negative) for supplier k;
$I_i^{k^0}$	initial non negative inventory level for product i for supplier k;
$h_i^k(t)$	coefficient of quadratic holding cost for product i at time t for supplier k;
$\gamma_i^k(t)$	coefficient of quadratic production cost for product i at time t for supplier k;
$\alpha_i^k(t),\ \beta_i^{k,k}(t),\ \beta_i^{k,-k}(t)$	nominal coefficients (fixed term and price sensitivities) for product i at time t in the linear relationship between price and demand for supplier k

Outputs

$p_i^k(t)$	price of one unit of product i at time t for supplier k (control variable);
$u_i^k(t)$	production flow rate of product i at time t for supplier k (control variable);
$I_i^k(t)$	inventory level (number of units) of product i at time t for supplier k (state variable).

C Vector space and associated norm, feasible set

C.1 Vector space and associated norm

Let E_1 be the vector space such that any element of E_1 has $3N$ components (price, production and inventory vectors) that are real bounded functions defined over $[0,T]$. The integral of the square of their absolute value is well-defined. Let $E = E_1 \times E_1$ be the Hilbert space (we use the L^2 norm on E so we have a reflexive Banach space):

$$||(x^1, x^2)|| = \sqrt{\int_0^T \sum_{i=1}^{3N} \sum_{k=1,2} (x_i^k(t))^2 dt} \quad \forall x^1,\ x^2\ \in E_1,$$

associated with the inner product

$$< (x^1, x^2),\ (\bar{x}^1, \bar{x}^2) >= \int_0^T \sum_{i=1}^{3N} \sum_{k=1,2} x_i^k(t) \bar{x}_i^k(t) dt.$$

Note that this space has an infinite dimension.

We will denote by $x^k \in E_1$ the vector representing a pricing and production strategy along with the state variables of player k in the following way:

$$x^k = (p^k,\ u^k,\ I^k) \quad \text{where } \ p^k = (p_1^k(.), \ldots, p_N^k(.)),\ \ u^k = (u_1^k(.), \ldots, u_N^k(.)),\ \ I^k = (I_1^k(.), \ldots, I_N^k(.)).$$

The vector $(x^A, x^B) \in E$ represents the collective strategy and state vector.

As a result the norm for a collective strategy and state vector is given by:

$$||x|| = \sqrt{\int_0^T \sum_{i=1}^N \sum_{k=A,B} \Big[(p_i^k(t))^2 + (u_i^k(t))^2 + (I_i^k(t))^2\Big] dt}, \quad x \in E$$

associated with the inner product

$$< x,\ \bar{x} >= \int_0^T \sum_{i=1}^N \sum_{k=A,B} \Big(p_i^k(t)\bar{p}_i^k(t) + u_i^k(t)\bar{u}_i^k(t) + I_i^k(t)\bar{I}_i^k(t)\Big) dt, \quad x, \bar{x} \in E.$$

C.2 Feasible set

Let's denote $X^k \subset E_1$ the set of strategy and state vectors for player k satisfying the constraints that are independent of the competitor's strategy:

$$X^k = \big\{x = (p, u, I) \in E_1 : \ \ u_i(t),\ p_i(t),\ I_i(t) \geq 0\ \forall i,\ t,\ \ \sum_{i=1}^N u_i(t) \leq K^k(t)\ \forall t,\ \ I_i(0) = I_i^{k^0}\ \forall i\big\}.$$

Let $X \subset E$ such that $X = X^A \times X^B$.

The following lemma follows directly from the definition of X.

Lemma 1. *X is a non empty, convex, closed subset of E.*

For a fixed strategy and state vector of the competitor, let's denote $Q^k(\bar{x}^{-k}) \subset X^k$ the subset of all feasible strategy and state vectors for player k, given the strategy and state vector $\bar{x}^{-k}$ of her competitor:

$$Q^k(\bar{x}^{-k}) = \{ \quad \begin{aligned} & x = (p_1(.), \ldots, p_N(.),\ u_1(.), \ldots, u_N(.),\ I_1(.), \ldots, I_N(.)) \in X^k : \\ & p_i(t) \leq \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t)}{\beta_i^{k,k}(t)} \forall i,\ t, \\ & \dot{I}_i(t) = u_i(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i(t) - \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t)\ \forall i,\ t \quad \}. \end{aligned}$$

Lemma 2. *For all $\bar{x}^{-k} \in X^{-k}$, $Q^k(\bar{x}^{-k})$ is a non empty, closed, convex subset of X^k.*

Proof. Convexity follows from the fact that the constraints defining the set are linear. To see that the set is closed, note that if we take a convergent sequence of vectors of X^k (even not uniformly convergent), since the inventory levels are bounded and the time horizon is finite, we can interchange the limit and the integral, and as a result the limit belongs to the set as well.
It is easy to verify that the vector $x = (p, u, I)$ such that

$$p_i(t) = \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t)}{\beta_i^{k,k}(t)}, \quad u_i(t) = \frac{K^k(t)}{N}, \quad I_i(t) = I_i^{k^0} + \int_0^t \frac{K^k(s)}{N}\, ds \quad \forall i$$

is an element of $Q^k(\bar{x}^{-k})$. □

We denote $Y \subset X$ the set of collectively feasible strategy and state vectors for both players:

$$Y = \{x \in X : x^k \in Q^k(x^{-k}),\ k = A, B\}.$$

Lemma 3. *Y is a convex, closed, non empty subset of X.*

Proof. Convexity and closedness follow from the fact that sets X and $Q^k(\bar{x}^{-k})$ are convex and closed.
It is easy to verify that vector $x = (p, u, I)$ such that

$$p_i^k(t) = \frac{\alpha_i^k(t)\beta_i^{-k,-k}(t) + \alpha_i^{-k}(t)\beta_i^{k,-k}(t)}{\beta_i^{B,B}(t)\beta_i^{A,A}(t) - \beta_i^{B,A}(t)\beta_i^{A,B}(t)}, \quad u_i^k(t) = \frac{K^k(t)}{N}, \quad I_i^k(t) = I_i^{k^0} + \int_0^t \frac{K^k(s)}{N}\, ds$$

is an element of Y. □

We denote $Q : X \mapsto 2^X$ the mapping such that $Q(x) = Q^A(x^B) \times Q^B(x^A)$ represents the set of feasible collective strategy and state vectors for both players when the competitor keeps her strategy fixed at x^{-k}.

The following lemma follows from Lemma 2.

Lemma 4. *$Q(x)$ is a non empty convex closed subset of X.*

The following proposition is immediate.

Proposition 1. *The following equivalence holds:*

$$x \in Q(x) \Leftrightarrow x \in Y.$$

D Existence of a Nash Equilibrium for the Deterministic Problem

D.1 Quasi-variational inequality formulation

We observe that the payoff function of player k may be formulated as
$J^k(x) = -a^k(x^k, x^k) - 2b^k(x^{-k}, x^k) + 2L^k(x^k)$ where

- $a^k : E_1 \times E_1 \mapsto \mathbb{R}$ is the continuous bilinear form, symmetric and non-negative along the diagonal such that $a^k(x, \bar{x}) = \int_0^T \sum_{i=1}^N \big(\beta_i^{k,k}(t)p_i(t)\bar{p}_i(t) + \gamma_i^k(t)u_i(t)\bar{u}_i(t) + h_i^k(t)I_i(t)\bar{I}_i(t)\big)dt$
- $b^k : E_1 \times E_1 \mapsto \mathbb{R}$ is the continuous bilinear form such that $b^k(x, \bar{x}) = -\frac{1}{2}\int_0^T \sum_{i=1}^N \beta_i^{k,-k}(t)\bar{p}_i(t)p_i(t)dt$
- $L^k : E_1 \mapsto \mathbb{R}$ is the continuous linear functional such that $L^k(x) = \frac{1}{2}\int_0^T \sum_{i=1}^N \big(\alpha_i^k(t)p_i(t)\big)dt$.

Let $a : E \times E \mapsto \mathbb{R}$ and $L : E \mapsto \mathbb{R}$ defined by

$$\begin{aligned}
a(x, \bar{x}) &= a^A(x^A, \bar{x}^A) + a^B(x^B, \bar{x}^B) + b^B(x^A, \bar{x}^B) + b^A(\bar{x}^A, x^B) && (1)\\
&= \int_0^T \sum_{i=1}^N \sum_{k=A,B} \Big(\beta_i^{k,k}(t)p_i^k(t)\bar{p}_i^k(t) + \gamma_i^k(t)u_i^k(t)\bar{u}_i^k(t) + h_i^k(t)I_i^k(t)\bar{I}_i^k(t) - \frac{1}{2}\beta_i^{k,-k}(t)\bar{p}_i^k(t)p_i^{-k}(t)\Big)dt.\\
L(x) &= L^A(x^A) + L^B(x^B). && (2)
\end{aligned}$$

We reformulate the Nash equilibrium problem as a quasi-variational inequality problem. The following proposition follows from (1) and (2).

Proposition 2. *There exists $f \in E$ and a linear operator A on E such that*

$$\begin{aligned}
a(x,\bar{x}) &= \quad < Ax, \bar{x} > \quad \forall x, \bar{x} \in E, \qquad L(x) = \quad < f, x > \quad \forall x \in E \\
\textit{with} \qquad f &= \frac{1}{2}\Big(\alpha_1^A(.), \dots, \alpha_N^A(.), 0, \dots, 0, 0, \dots, 0, \alpha_1^B(.), \dots, \alpha_N^B(.), 0, \dots, 0, 0, \dots, 0\Big) \\
Ax &= \Big(\beta_1^{A,A}(.)p_1^A(.) - \frac{1}{2}\beta_1^{A,B}(.)p_1^B(.), \dots, \beta_N^{A,A}(.)p_N^A(.) - \frac{1}{2}\beta_N^{A,B}(.)p_N^B(.), \\
&\quad \gamma_1^A(.)u_1^A(.), \dots, \gamma_N^A(.)u_N^A(.), \quad h_1^A(.)I_1^A(.), \dots, h_N^A(.)I_N^A(.), \\
&\quad \beta_1^{B,B}(.)p_1^B(.) - \frac{1}{2}\beta_1^{B,A}(.)p_1^A(.), \dots, \beta_N^{B,B}(.)p_N^B(.) - \frac{1}{2}\beta_N^{B,A}(.)p_N^A(.), \\
&\quad \gamma_1^B(.)u_1^B(.), \dots, \gamma_N^B(.)u_N^B(.), \quad h_1^B(.)I_1^B(.), \dots, h_N^B(.)I_N^B(.)\ \Big).
\end{aligned}$$

Theorem 1. *[3] $x \in Y$ is a Nash Equilibrium if and only if*

$$a(x, x - \bar{x}) \leq L(x - \bar{x}) \quad \forall \bar{x} \in Q(x). \tag{3}$$

Corollary 1. *$x \in Y$ is a solution of (3) if and only if*

$$< Ax - f, x - \bar{x} > \quad \leq \ 0 \quad \forall \bar{x} \in Q(x). \tag{4}$$

We observe that the problem is thus reformulated as a *quasi*-variational inequality (QVI), since the set in which the inequality must be satisfied depends on the QVI solution x.

D.2 Properties

Lemma 5. *Under Assumption 1, operator a (and thus A) is coercive.*

The proof can be found in the next section of the Appendix.

Definition 1. *$Q : X \mapsto 2^X$ is lower semi continuous on D_0 if and only if*

for a generalized sequence x_n converging to x in D_0, for every $\bar{x} \in Q(x)$, there exists an integer n_0 and a sequence $\bar{x}_n \in X$ converging to $\bar{x}$, such that $\bar{x}_n \in Q(x_n)$, for all $n \geq n_0$.

Definition 2. *$Q : X \mapsto 2^X$ is upper semi continuous on D_0 if and only if*

for every generalized sequence $(x_n, \bar{x}_n)$ converging to $(x, \bar{x})$ in $D_0 \times D_0$ and satisfying $\bar{x}_n \in Q(x_n)$, then in the limit $\bar{x} \in Q(x)$.

Definition 3. *$Q : X \mapsto 2^X$ is continuous on $D_0 \subset X$ if and only if it is lower semi continuous and upper semi continuous on D_0.*

Let S be the selection map corresponding to the quasi-variational inequality (4): $S : X \mapsto E$ associates with any fixed vector $u \in X$ the unique solution $v \in E$ of the following variational inequality:

$$v \in Q(u), \quad < Av - f,\ v - w > \leq 0 \ \ \forall w \in Q(u).$$

Definition 4. *[22] A set D_0 is stable under selection map S if set $S(u)$ is contained in set D_0 whenever u belongs to D_0.*

Theorem 2. *[22] If*

- *$a(.,.)$ is a coercive continuous bilinear form on the Hilbert space E*
- *f is a continuous linear functional on E*
- *Q is a map that associates with each vector u of the convex closed subset X of E a non empty convex closed subset $Q(u)$ of E*
- *There exists a Hilbert space E_0, which has a continuous injection $\hookrightarrow$ into E, and a non empty convex closed subset D_0 of E_0, with $D_0 \hookrightarrow X$, such that D_0 is stable under S, Q is continuous on D_0 and $S(D_0)$ is bounded in E_0*

then (4) admits a solution.

We are going to show that the assumptions from this theorem hold with $E_0 = E$, $D_0 = X$ and the injection $\hookrightarrow$ being the identity function. Since the space E consists of bounded functions, it is immediate that $S(X)$ is bounded in E.

Proposition 3. *X is stable under S.*

Proof. Let $x \in X$ and let $\bar{x} \equiv S(x)$. Then $\bar{x} \in Q(x) \subset X$. As a result, $S(X) \subset X$. □

Proposition 4. *The mapping Q is upper semi continuous on X.*

Proof. Consider a sequence $(x_n, \bar{x}_n)$ converging to $(x, \bar{x}) = (p, u, I; \bar{p}, \bar{u}, \bar{I})$ in $X \times X$ such that $\bar{x}_n \in Q(x_n)$, i.e. $\bar{x}_n \in X$ and $\forall\ n, i, t$

$$\bar{p}^k_{n,i}(t) \leq \frac{\alpha^k_i(t) + \beta^{-k}_i(t) p^{-k}_{n,i}(t)}{\beta^k_i(t)} \text{ and } \dot{\bar{I}}^k_{n,i}(t) = \bar{u}^k_{n,i}(t) - \alpha^k_i(t) + \beta^{k,k}_i(t)\bar{p}^k_{n,i}(t) - \beta^{k,-k}_i(t) p^{-k}_{n,i}(t)$$

where $x_n = (x^A_n, x^B_n) = (p^A_n, u^A_n, I^A_n, p^B_n, u^B_n, I^B_n)$, $\bar{x}_n = (\bar{x}^A_n, \bar{x}^B_n) = (\bar{p}^A_n, \bar{u}^A_n, \bar{I}^A_n, \bar{p}^B_n, \bar{u}^B_n, \bar{I}^B_n)$, $x = (x^A, x^B) = (p^A, u^A, I^A, p^B, u^B, I^B)$, and $\bar{x} = (\bar{x}^A, \bar{x}^B) = (\bar{p}^A, \bar{u}^A, \bar{I}^A, \bar{p}^B, \bar{u}^B, \bar{I}^B)$.

Since $\lim p^{-k}_{n,i}(t) = p^{-k}_i(t)$, $\lim \bar{p}^k_{n,i}(t) = \bar{p}^k_i(t)$, $\lim \bar{u}^k_{n,i}(t) = \bar{u}^k_i(t)$, and $\lim \bar{I}^k_{n,i}(t) = \bar{I}^k_i(t)$, we obtain that $\bar{I}$ is differentiable and

$$\dot{\bar{I}}^k_i(t) = \bar{u}^k_i(t) - \alpha^k_i(t) + \beta^{k,k}_i(t)\bar{p}^k_i(t) - \beta^{k,-k}_i(t) p^{-k}_i(t)\ \forall i,\ t \quad \text{and} \quad \bar{p}^k_i(t) \leq \frac{\alpha^k_i(t) + \beta^{k,-k}_i(t) p^{-k}_i(t)}{\beta^{k,k}_i(t)}.$$

As a result, $\bar{x} \in Q(x)$. □

Proposition 5. *The mapping Q is lower semi continuous on X.*

Proof. Consider a sequence $x_n = (x^A_n, x^B_n) = (p^A_n, u^A_n, I^A_n, p^B_n, u^B_n, I^B_n) \in X$ converging to $x = (x^A, x^B) = (p^A, u^A, I^A, p^B, u^B, I^B)$. Let $\bar{x} = (\bar{x}^A, \bar{x}^B) = (\bar{p}^A, \bar{u}^A, \bar{I}^A, \bar{p}^B, \bar{u}^B, \bar{I}^B) \in Q(x)$. Since X is closed, $x \in X$. Let $\bar{x}_n = (\bar{x}^A_n, \bar{x}^B_n) = (\bar{p}^A_n, \bar{u}^A_n, \bar{I}^A_n, \bar{p}^B_n, \bar{u}^B_n, \bar{I}^B_n)$ such that

$$\begin{aligned}
\bar{u}^k_{n,i}(t) &= \bar{u}^k_i(t)\ \ \forall\ n, i, t \\
\bar{p}^k_{n,i}(t) &= \begin{cases} \bar{p}^k_i(t) & \text{if } \bar{p}^k_i(t) = 0 \text{ and } p^{-k}_{n,i}(t) - p^{-k}_i(t) < 0 \\ \bar{p}^k_i(t) + \frac{\beta^{k,-k}_i(t)}{\beta^{k,k}_i(t)}(p^{-k}_{n,i}(t) - p^{-k}_i(t))\ \ \forall\ n, i, t & \text{if } \bar{p}^k_i(t) > 0 \text{ or } p^{-k}_{n,i}(t) - p^{-k}_i(t) > 0 \end{cases} \\
\bar{I}^k_{n,i}(t) &= I^{k^0}_i + \int_0^T (\bar{u}^k_{n,i}(s) - \alpha^k_i(s) + \beta^{k,k}_i(s)\bar{p}^k_{n,i}(s) - \beta^{k,-k}_i(s) p^{-k}_{n,i}(s))ds\ \ \forall\ n, i, t.
\end{aligned}$$

We want to show that $\bar{x}_n$ constructed above satisfies $\bar{x}_n \to \bar{x}$ and $\bar{x}_n \in Q(x_n)$. We clearly have $\bar{I}_n(0) = I^0$, $\bar{u}_n(.) \geq 0$ and $\sum_i \bar{u}^k_{n,i}(t) \leq K^k(t)$. We notice that

$$\begin{aligned}
\bar{I}^k_{n,i}(t) - \bar{I}^k_i(t) &= \int_0^t (\beta^{k,k}_i(s)(\bar{p}^k_{n,i}(s) - \bar{p}^k_i(s)) - \beta^{k,-k}_i(s)(p^{-k}_{n,i}(s) - p^{-k}_i(s)))ds \\
&= -\int_{D^k_i \cap [0,t]} \beta^{k,-k}_i(s)(p^{-k}_{n,i}(s) - p^{-k}_i(s)))ds \geq 0
\end{aligned}$$

where $D_i^k = \{t \in [0,T] : \bar{p}_i^k(t) = 0 \text{ and } p_{n,i}^{-k}(t) - p_i^{-k}(t) < 0\}$. Therefore, $\bar{I}_n \geq \bar{I} \geq 0$. Also, when $\bar{p}_{n,i}^k(t)$ is equal to the expression $\bar{p}_i^k(t) + \frac{\beta_i^{k,-k}(t)}{\beta_i^{k,k}(t)}(p_{n,i}^{-k}(t) - p_i^{-k}(t))$, since $p_{n,i}^{-k}(t) - p_i^{-k}(t) \to 0$, we notice that for n sufficiently large $\bar{p}_{n,i}^k(t) \geq 0$ (either it is equal to a positive term to which we add a term that tends to zero, or it is zero plus a positive term that tends to zero). Clearly, when $\bar{p}_{n,i}^k(t)$ is given by the first expression, this still holds. As a result, $\bar{x}_n \in X$. Moreover, $\bar{u}_n \to \bar{u}$ and since $p_n^{-k} - p^{-k} \to 0$ we also have $\bar{I}_n \to \bar{I}$, $\bar{p}_n \to \bar{p}$, so that $\bar{x}_n \to \bar{x}$. Finally, we notice that when $\bar{p}_{n,i}^k(t)$ is equal to the expression $\bar{p}_i^k(t) + \frac{\beta_i^{k,-k}(t)}{\beta_i^{k,k}(t)}(p_{n,i}^{-k}(t) - p_i^{-k}(t))$, then

$$\bar{p}_{n,i}^k(t) - \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_{n,i}^{-k}(t)}{\beta_i^{k,k}(t)} = \bar{p}_i^k(t) - \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)}{\beta_i^{k,k}(t)} \leq 0.$$

Clearly, when $\bar{p}_{n,i}^k(t)$ is equal to $\bar{p}_i^k(t)$, then $\bar{p}_{n,i}^k(t) = \bar{p}_i^k(t) = 0$ so the inequality $\bar{p}_{n,i}^k(t) \leq \frac{\alpha_i^k(t)+\beta_i^{k,-k}(t)p_{n,i}^{-k}(t)}{\beta_i^{k,k}(t)}$ is also satisfied. As a result $\bar{x}_n \in Q(x_n)$. $\square$

Corollary 2. *Q is continuous on X.*

The following result then follows from Theorem 2.

Theorem 3. *Under Assumption 1, there exists a Nash equilibrium to the deterministic problem under competition.*

E Proofs

E.1 Proof of Proposition 1

Proof. It is clear from the definition of a Nash Equilibrium that the inequality is a necessary condition. For the reverse, suppose $x \in Y$ satisfies the inequality above for all $\bar{x} \in Q(x)$ and $\exists k,\ \bar{x}^k \in Q^k(x^{-k})$ such that $J^k(x^k, x^{-k}) < J^k(\bar{x}^k, x^{-k})$. Let y such that $y^k = \bar{x}^k,\ y^{-k} = x^{-k}$. Then $y \in Q(x)$ and

$$J^A(y^A, x^B) + J^B(x^A, y^B) = J^k(\bar{x}^k, x^{-k}) + J^{-k}(x^k, x^{-k}) > J^k(x^k, x^{-k}) + J^{-k}(x^k, x^{-k}) = J^A(x) + J^B(x)$$

which is a contradiction. $\square$

E.2 Proof of Theorem 1

Proof. We obtain that in the robust counterpart, the price constraint (8) is written as $p_i^k(t) \leq \frac{\alpha_i^k(t)-\hat{\alpha}_i^k(t)+(\beta_i^{k,-k}(t)-\hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t)+\hat{\beta}_i^{k,k}(t)}$ $\forall i,\ t$. We observe that we may write the realized inventory level at time t as follows:

$$\tilde{I}_i^k(t) \;\;=\;\; I_i^k(t) - \int_0^t (z_i^k(s)\hat{\alpha}_i^k(s) - y_i^{k,k}(s)\hat{\beta}_i^{k,k}(s)p_i^k(s) + y_i^{k,-k}(s)\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s))ds$$

where $I_i^k(t) = I_i^{k^0} + \int_0^t (u_i^k(s) - \alpha_i^k(s) + \beta_i^{k,k}(s)p_i^k(s) - \beta_i^{k,-k}(s)p_i^{-k}(s))ds$ is the nominal inventory level. The no backorder constraint at time t is a constraint that is instantaneous on the inventory level. It indirectly involves the control decisions on prices and production rates from time 0 to time t, and as such, the budget of uncertainty has an impact on it. As a result, constraint (9) is equivalent to

$$I_i^k(t) \;\geq\; \Omega_i^k(t) \;\; \forall i, t \tag{5}$$

where $\Omega_i^k(t)$ can be viewed as a minimum inventory security level that can be computed via the following deterministic continuous linear program

$$\begin{aligned}
\Omega_i^k(t) = \max_{z_i^k(.),y_i^{k,k}(.),y_i^{k,-k}(.)} \quad & \int_0^t (z_i^k(s)\hat{\alpha}_i^k(s) - y_i^{k,k}(s)\hat{\beta}_i^{k,k}(s)p_i^k(s) + y_i^{k,-k}(s)\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s))ds \\
\text{s.t.} \quad & z_i^k(s),\ y_i^{k,k}(s),\ y_i^{k,-k}(s) \;\in [-1,1] \;\; \forall s \in [0,t] \\
& \int_0^t |z_i^k(s)|ds \leq \Gamma_i^k(t), \quad \int_0^t |y_i^{k,k}(s)|ds \leq \Theta_i^{k,k}(t), \quad \int_0^t |y_i^{k,-k}(s)|ds \leq \Theta_i^{k,-k}(t).
\end{aligned}$$

Notice that $\Omega_i^k(t)$ depends on the pricing strategies of both suppliers on $[0,t]$ via the objective function. To make this dependence clear, we will denote it by $\Omega_i^k(t, p^k(.), p^{-k}(.))$. After a change of variables, this problem separates into three subproblems that are continuous linear programs as follows:

$$\Omega_i^k(t, p^k(.), p^{-k}(.)) = \Omega_i^{k^1}(t) + \Omega_i^{k^2}(t, p^k(.)) + \Omega_i^{k^3}(t, p^{-k}(.)) \tag{6}$$

with

$$\begin{aligned} \Omega_i^{k^1}(t) = \max_{z_i^k(.)} \quad & \int_0^t z_i^k(s)\hat{\alpha}_i^k(s)ds \\ \text{s.t.} \quad & 0 \le z_i^k(t) \le 1 \ \ \forall s \in [0,t], \quad \int_0^t z_i^k(s)ds \le \Gamma_i^k(t) \end{aligned}$$

and $\Omega_i^{k^2}(t, p^k(.))$, $\Omega_i^{k^3}(t, p^{-k}(.))$ are obtained similarly after substituting respectively $\hat{\beta}_i^{k,k}(.)p_i^k(.)$ and $\hat{\beta}_i^{k,-k}(.)p_i^{-k}(.)$ for $\hat{\alpha}_i^k(.)$, and $\Theta_i^{k,k}(.)$ and $\Theta_i^{k,-k}(.)$ for $\Gamma_i^k(.)$. Notice that $\Omega_i^{k^2}(t, p^k(.))$ depends on $p_i^k(s),\ 0 \le s \le t$ and $\Omega_i^{k^3}(t, p^{-k}(.))$ depends on $p_i^{-k}(s),\ 0 \le s \le t$. Under regularity assumptions (see [1]) we have strong duality and the respective dual subproblems are given by the continuous linear programs:

$$\begin{aligned} \Omega_i^{k^1}(t) = \min_{\omega_i^k(t), r_i^k(.,t)} \quad & \omega_i^k(t)\Gamma_i^k(t) + \int_0^t r_i^k(s,t)ds \\ \text{s.t.} \quad & \omega_i^k(t) + r_i^k(s,t) \ge \hat{\alpha}_i^k(s) \ \ \forall s \in [0,t] \\ & \omega_i^k(t) \ge 0, \quad r_i^k(s,t) \ge 0 \ \ \forall s \in [0,t] \end{aligned} \tag{7}$$

$$\begin{aligned} \Omega_i^{k^2}(t, p^k(.)) = \min_{\theta_i^{k,k}(t), q_i^{k,k}(.,t)} \quad & \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \int_0^t q_i^{k,k}(s,t)ds \\ \text{s.t.} \quad & \theta_i^{k,k}(t) + q_i^{k,k}(s,t) \ge \hat{\beta}_i^{k,k}(s)p_i^k(s) \ \ \forall s \in [0,t] \\ & \theta_i^{k,k}(t) \ge 0, \quad q_i^{k,k}(s,t) \ge 0 \ \ \forall s \in [0,t] \end{aligned} \tag{8}$$

$$\begin{aligned} \Omega_i^{k^3}(t, p^{-k}(.)) = \min_{\theta_i^{k,-k}(t), q_i^{k,-k}(.,t)} \quad & \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t) + \int_0^t q_i^{k,-k}(s,t)ds \\ \text{s.t.} \quad & \theta_i^{k,-k}(t) + q_i^{k,-k}(s,t) \ge \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \ \ \forall s \in [0,t] \\ & \theta_i^{k,-k}(t) \ge 0, \quad q_i^{k,-k}(s,t) \ge 0 \ \ \forall s \in [0,t]. \end{aligned} \tag{9}$$

By noticing that the inventory level constraint holds when the threshold is the lowest possible, we obtain the result. $\square$

E.3 Proof of Lemma 5

Proof. Let $\lambda > 0$ a constant.

$$\begin{aligned} a(x,x) - \lambda ||x||^2 \;\; &= \;\; \int_0^T \sum_{i=1}^N \Big[(\beta_i^{A,A}(t) - \lambda)(p_i^A(t))^2 + (\beta_i^{B,B}(t) - \lambda)(p_i^B(t))^2 \\ &\quad - \frac{1}{2}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t)) p_i^A(t) p_i^B(t) \\ &\quad + (\gamma_i^A(t) - \lambda)(u_i^A(t))^2 + (\gamma_i^B(t) - \lambda)(u_i^B(t))^2 \\ &\quad + (h_i^A(t) - \lambda)(I_i^A(t))^2 + (h_i^B(t) - \lambda)(I_i^B(t))^2 \Big] dt. \end{aligned}$$

Let

$$\lambda_1 = \min_k \min_i \inf_{t \in [0,T]} h_i^k(t), \qquad \lambda_2 = \min_k \min_i \inf_{t \in [0,T]} \gamma_i^k(t).$$

A sufficient condition for the expression above to be positive is that $\lambda < \lambda_1, \quad \lambda < \lambda_2$ and the symmetric matrix (defined at fixed i, t)

$$Q = \begin{bmatrix} \beta^{A,A} - \lambda & -\frac{(\beta^{A,B} + \beta^{B,A})}{4} \\ -\frac{(\beta^{A,B} + \beta^{B,A})}{4} & \beta^{B,B} - \lambda \end{bmatrix}$$

is positive semi-definite for all i, t (we omit the product index and time variable for the sake of clarity).

We notice that

$$\begin{aligned} Q \succeq 0 \;\; &\Leftrightarrow \;\; \big(\mathrm{Tr}(Q) \geq 0 \text{ and } \mathrm{Det}(Q) \geq 0\big) \\ &\Leftrightarrow \;\; \Big(\beta^{A,A} + \beta^{B,B} - 2\lambda \geq 0 \text{ and } (\beta^{A,A} - \lambda)(\beta^{B,B} - \lambda) - \frac{1}{16}(\beta^{A,B} + \beta^{B,A})^2 \geq 0\Big) \\ &\Leftrightarrow \;\; \begin{cases} \lambda \leq \frac{\beta^{A,A} + \beta^{B,B}}{2} \\ \lambda^2 - \lambda(\beta^{A,A} + \beta^{B,B}) + \beta^{A,A}\beta^{B,B} - \frac{1}{16}(\beta^{A,B} + \beta^{B,A})^2 \geq 0. \end{cases} \end{aligned}$$

The determinant of the polynomial above is

$$\begin{aligned}\Delta &= (\beta^{A,A}+\beta^{B,B})^2 - 4\big(\beta^{A,A}\beta^{B,B} - \frac{1}{16}(\beta^{A,B}+\beta^{B,A})^2\big)\\ &= (\beta^{A,A}-\beta^{B,B})^2 + \frac{1}{4}(\beta^{A,B}+\beta^{B,A})^2 > 0\end{aligned}$$

so the polynomial has two real roots $\frac{\beta^{A,A}+\beta^{B,B}\pm\sqrt{\Delta}}{2}$ and only one satisfies $\lambda \leq \frac{\beta^{A,A}+\beta^{B,B}}{2}$. Since we are interested in positive parameters λ, we obtain that

$$\begin{aligned}(\lambda>0 \text{ and } Q\succeq 0) &\Leftrightarrow \begin{cases}\frac{\beta^{A,A}+\beta^{B,B}-\sqrt{\Delta}}{2}>0\\ 0<\lambda<\frac{\beta^{A,A}+\beta^{B,B}-\sqrt{\Delta}}{2}\end{cases}\\ &\Leftrightarrow \begin{cases}(\beta^{A,A}-\beta^{B,B})^2+\frac{1}{4}(\beta^{A,B}+\beta^{B,A})^2<(\beta^{A,A}+\beta^{B,B})^2\\ 0<\lambda<\frac{\beta^{A,A}+\beta^{B,B}}{2}-\frac{\sqrt{4\Delta}}{4}\end{cases}\\ &\Leftrightarrow \begin{cases}\frac{1}{4}(\beta^{A,B}+\beta^{B,A})^2<4\beta^{A,A}\beta^{B,B}\\ 0<\lambda<\frac{\beta^{A,A}+\beta^{B,B}}{2}-\frac{\sqrt{4\Delta}}{4}\end{cases}\end{aligned}$$

which is satisfied under Assumption 1 and provided that $0<\lambda<\lambda_3$ where

$$\lambda_3 = \min_i \inf_{t\in[0,T]} \frac{\beta_i^{A,A}(t)+\beta_i^{B,B}(t)}{2} - \frac{1}{4}\sqrt{4(\beta_i^{A,A}(t)-\beta_i^{B,B}(t))^2+(\beta_i^{A,B}(t)+\beta_i^{B,A}(t))^2} > 0.$$

As a result, by taking $0<\lambda<\min\{\lambda_1,\lambda_2,\lambda_3\}$, we obtain that

$$a(x,x)-\lambda||x||^2>0 \qquad \forall x\in E. \qquad \square$$

E.4 Proof of Proposition 2

Proof. We consider the dual subproblem (P_t) and (P_{t+dt}) respectively at times t and $t+dt$, whose optimal objective values equal respectively $\Omega_i^{k^1}(t)$ and $\Omega_i^{k^1}(t+dt)$:

$$\begin{aligned} \min_{\omega_i^k(t), r_i^k(.,t)} \quad & \omega_i^k(t)\Gamma_i^k(t) + \int_0^t r_i^k(s,t)ds \\ \text{s.t.} \quad & \omega_i^k(t) + r_i^k(s,t) \geq \hat{\alpha}_i^k(s) \quad \forall s \in [0,t] \\ & \omega_i^k(t) \geq 0 \\ & r_i^k(s,t) \geq 0 \quad \forall s \in [0,t], \end{aligned}$$

$$\begin{aligned} \min_{\omega_i^k(t+dt), r_i^k(.,t+dt)} \quad & \omega_i^k(t+dt)\Gamma_i^k(t+dt) + \int_0^{t+dt} r_i^k(s,t+dt)ds \\ \text{s.t.} \quad & \omega_i^k(t+dt) + r_i^k(s,t+dt) \geq \hat{\alpha}_i^k(s) \quad \forall s \in [0,t+dt] \\ & \omega_i^k(t+dt) \geq 0 \\ & r_i^k(s,t+dt) \geq 0 \quad \forall s \in [0,t+dt]. \end{aligned}$$

We denote by $(\omega_i^{k*}(t), r_i^{k*}(.,t))$, $(\omega_i^{k*}(t+dt), r_i^{k*}(.,t+dt))$ the respective optimal solutions. It is clear that $(\omega_i^{k*}(t+dt), r_i^{k*}(.,t+dt))$ is feasible for (P_t), therefore, we have

$$\omega_i^{k*}(t)\Gamma_i^k(t) + \int_0^t r_i^{k*}(s,t)ds \leq \omega_i^{k*}(t+dt)\Gamma_i^k(t) \int_0^t r_i^{k*}(s,t+dt)ds.$$

As a result, we observe that

$$\begin{aligned} \Omega_i^{k^1}(t+dt) &= \omega_i^{k*}(t+dt)\Gamma_i^k(t+dt) + \int_0^{t+dt} r_i^{k*}(s,t+dt)ds \\ &= \omega_i^{k*}(t+dt)\Gamma_i^k(t) + \omega_i^{k*}(t+dt)\dot{\Gamma}_i^k(t)dt + \int_0^t r_i^{k*}(s,t+dt)ds + \int_t^{t+dt} r_i^{k*}(s,t+dt)ds \\ &\geq \omega_i^{k*}(t)\Gamma_i(t) + \int_0^t r_i^{k*}(s,t)ds + \omega_i^{k*}(t+dt)\dot{\Gamma}_i^k(t)dt + \int_t^{t+dt} r_i^{k*}(s,t+dt)ds \\ &= \Omega_i^{k^1}(t) + \omega_i^{k*}(t+dt)\dot{\Gamma}_i^k(t)dt + \int_t^{t+dt} r_i^{k*}(s,t+dt)ds. \end{aligned}$$

Since $\dot{\Gamma}_i^k(t) \geq 0$ by assumption, and for feasibility of (P_{t+dt}), $\omega_i^{k*}(t+dt) \geq 0$, $r_i^{k*}(s,t+dt) \geq 0 \;\; \forall s \in [t, t+dt]$, we obtain that

$$\Omega_i^{k^1}(t+dt) \geq \Omega_i^{k^1}(t).$$

□

E.5 Proof of Proposition 3

Proof. We prove this result by showing that $\Omega_i^{k^1}, \Omega_i^{k^2}$, and $\Omega_i^{k^3}$ are respectively non decreasing with $\Gamma_i^k(t)$, $\Theta_i^{k,k}(t)$ and $\Theta_i^{k,-k}(t)$. We describe the proof for $\Omega_i^{k^1}$, and it is similar for $\Omega_i^{k^2}$ and $\Omega_i^{k^3}$. Let $\Gamma_i'^k(t)$ such that $\Gamma_i^k(t) < \Gamma_i'^k(t)$, and let $(\omega(t), r(., t))$ and $(\omega'(t), r'(., t))$ the respective optimal solutions of the dual subproblems, which we denote (D) and (D'), and $\Omega_i^{k^1}(t)$ and $\Omega_i'^{k^1}(t)$ the respective optimal objective values. Notice that $(\omega'(t), r'(., t))$ is feasible for (D), since (D) and (D') have the same feasible sets. Therefore

$$\Omega_i^{k^1}(t) \quad = \quad \omega(t)\Gamma_i^k(t) + \int_0^t r(s,t)ds \leq \omega'(t)\Gamma_i^k(t) + \int_0^t r'(s,t)ds,$$

since $(\omega'(t), r'(., t))$ is feasible suboptimal

$$\leq \quad \omega'(t)\Gamma_i'^k(t) + \int_0^t r'(s,t)ds = \Omega_i'^{k^1}(t), \quad \text{since } \omega'(t) \geq 0, \quad \Gamma_i^k(t) < \Gamma_i'^k(t) \ . \qquad \square$$

E.6 Proof of Proposition 4

Proof. Let us for example, show the convexity of $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ in $p_i^k(.)$. $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ is the sum of the objective values of the three corresponding dual subproblems. Two of these subproblems are independent of $p_i^k(.)$ (the ones dealing with uncertainty on respectively $\hat{\alpha}_i^k$ and $\hat{\beta}_i^{k,-k}$). Therefore, it is necessary and sufficient to show that $\Omega_i^{k^2}(t, p_i^k(.))$ given by (8) is convex in $p_i^k(.)$. Let $p_i^{k^1}(.)$, $p_i^{k^2}(.)$, $\lambda \in (0,1)$ and $p_i^{k^3}(.) = \lambda p_i^{k^1}(.) + (1-\lambda)p_i^{k^2}(.)$. We will use superscripts 1, 2, 3 similarly to denote the optimal solutions of the corresponding subproblems with input $p_i^{k^1}(.)$, $p_i^{k^2}(.)$, $p_i^{k^3}(.)$. Clearly,

$$\lambda\theta_i^{k,k^1}(t) + \lambda q_i^{k,k^1}(s,t) \geq \hat{\beta}_i^{k,k}(s)\lambda p_i^{k^1}(s) \quad \forall s \in [0,t]$$

$$(1-\lambda)\theta_i^{k,k^2}(t) + (1-\lambda)q_i^{k,k^2}(s,t) \geq \hat{\beta}_i^{k,k}(s)(1-\lambda)p_i^{k^2}(s) \quad \forall s \in [0,t].$$

Adding these inequalities shows that $\lambda\theta_i^{k,k^1}(t) + (1-\lambda)\theta_i^{k,k^2}(t)$ along with $\lambda q_i^{k,k^1}(., t) + (1-\lambda)q_i^{k,k^2}(., t)$, is feasible for the subproblem with input $p_i^{k^3}(.)$ (non negativity is clearly satisfied).

Since it may not yield the optimal objective value, we have

$$\begin{aligned}
\Omega_i^{k^2}(t, p_i^{k^3}(.)) &\leq (\lambda\theta_i^{k,k^1}(t) + (1-\lambda)\theta_i^{k,k^2}(t))\Theta_i^{k,k}(t) + \int_0^t (\lambda q_i^{k,k^1}(s,t) + (1-\lambda)q_i^{k,k^2}(s,t))ds \\
&= \lambda\Omega_i^{k^1}(t, p_i^{k^1}(.)) + (1-\lambda)\Omega_i^{k^1}(t, p_i^{k^2}(.)). \qquad \square
\end{aligned}$$

E.7 Proof of Theorem 2

We start by reformulating the robust counterpart problem.

The following lemma follows from the derivation of the robust counterpart by integrating the dual subproblems into the main optimization problem.

Lemma 6. *The robust counterpart for the best response problem faced by supplier k (at $p_i^{-k}(.)$ fixed) can be written:*

$$\begin{aligned}
\max \quad & \int_0^T \sum_{i=1}^N \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(u_i^k(t))^2 - h_i^k(t)(I_i^k(t))^2 \Big) dt \qquad (10) \\
s.t. \quad & \dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t) \quad \forall i \quad \forall t \in [0,T] \\
& I_i^k(0) = I_i^{k^0} \quad \forall i \\
& \sum_{i=1}^N u_i^k(t) \leq K^k(t) \quad \forall t \in [0,T] \\
& p_i^k(t) \leq \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)} \quad \forall i \quad \forall t \in [0,T] \\
& p_i^k(t), \ u_i^k(t) \geq 0 \quad \forall i \quad \forall t \in [0,T] \\
& I_i^k(t) \geq \omega_i^k(t)\Gamma_i^k(t) + \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t) \\
& \qquad\qquad + \int_0^t (r_i^k(s,t) + q_i^{k,k}(s,t) + q_i^{k,-k}(s,t))ds \quad \forall i \quad \forall t \in [0,T] \\
& \omega_i^k(t) + r_i^k(s,t) \geq \hat{\alpha}_i^k(s) \quad \forall i \quad \forall s \in [0,t] \quad \forall t \in [0,T] \\
& \theta_i^{k,k}(t) + q_i^{k,k}(s,t) \geq \hat{\beta}_i^{k,k}(s)p_i^k(s) \quad \forall i \quad \forall s \in [0,t] \quad \forall t \in [0,T] \\
& \theta_i^{k,-k}(t) + q_i^{k,-k}(s,t) \geq \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \quad \forall i \quad \forall s \in [0,t] \quad \forall t \in [0,T] \\
& \omega_i^k(t), \ \theta_i^{k,k}(t), \ \theta_i^{k,k}(t) \geq 0 \quad \forall i \quad \forall t \in [0,T] \\
& r_i^k(s,t), \ q_i^{k,k}(s,t), \ q_i^{k,-k}(s,t) \geq 0 \quad \forall i \quad \forall s \in [0,t] \quad \forall t \in [0,T].
\end{aligned}$$

Adding the constraint that prices must be lower than their maximum price leaves the problem unchanged. After introducing new variables in order to reformulate the constraint involving an integral expression, we obtain the following:

Lemma 7. *The robust counterpart problem can be reformulated equivalently as a deterministic fluid model with linear constraints:*

$$
\begin{aligned}
\max \quad & \int_0^T \sum_{i=1}^N \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(u_i^k(t))^2 - h_i^k(t)(I_i^k(t))^2\Big)dt \qquad (11)\\
s.t. \quad & \dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t) \quad \forall i \quad \forall t \in [0,T]\\
& I_i^k(0) = I_i^{k^0} \quad \forall i\\
& \sum_{i=1}^N u_i^k(t) \le K^k(t) \quad \forall t \in [0,T]\\
& p_i^k(t) \le \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)} \quad \forall i \quad \forall t \in [0,T]\\
& p_i^k(t) \le p_{i_{max}}^k(t) \quad \forall i \quad \forall t \in [0,T]\\
& p_i^k(t),\ u_i^k(t) \ge 0 \quad \forall i \quad \forall t \in [0,T]\\
& I_i^k(t) \ge \omega_i^k(t)\Gamma_i^k(t) + \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t) + R_i^k(t,t)\\
& \qquad\qquad + S_i^{k,k}(t,t) + S_i^{k,-k}(t,t) \quad \forall i \quad \forall t \in [0,T]\\
& \omega_i^k(t) + r_i^k(s,t) \ge \hat{\alpha}_i^k(s) \quad \forall i \quad \forall s \in [0,t] \quad \forall t \in [0,T]\\
& \theta_i^{k,k}(t) + q_i^{k,k}(s,t) \ge \hat{\beta}_i^{k,k}(s)p_i^k(s) \quad \forall i \quad \forall s \in [0,t] \quad \forall t \in [0,T]\\
& \theta_i^{k,-k}(t) + q_i^{k,-k}(s,t) \ge \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \quad \forall i \quad \forall s \in [0,t] \quad \forall t \in [0,T]\\
& \omega_i^k(t),\ \theta_i^{k,k}(t),\ \theta_i^{k,k}(t) \ge 0 \quad \forall i \quad \forall t \in [0,T]\\
& R_i^k(0,t) = S_i^{k,k}(0,t) = S_i^{k,-k}(0,t) \ = 0 \quad \forall i \quad \forall t \in [0,T]\\
& \frac{\partial R_i^k}{\partial s}(s,t) = r_i^k(s,t) \quad \forall i \quad \forall s \in [0,t], \quad \forall t \in [0,T]\\
& \frac{\partial S_i^{k,k}}{\partial s}(s,t) = q_i^{k,k}(s,t) \quad \forall i \quad \forall s \in [0,t], \quad \forall t \in [0,T]\\
& \frac{\partial S_i^{k,-k}}{\partial s}(s,t) = q_i^{k,-k}(s,t) \quad \forall i \quad \forall s \in [0,t], \quad \forall t \in [0,T]\\
& r_i^k(s,t),\ q_i^{k,k}(s,t),\ q_i^{k,-k}(s,t) \ge 0 \quad \forall i \quad \forall s \in [0,t], \quad \forall t \in [0,T].
\end{aligned}
$$

Note that the fluid equations as well as the constraints remain linear, even though there are more variables than in the nominal problem: only the size has increased, but the complexity is of the same order.

We now prove Theorem 2.

Proof. The variables space is now the space of vectors $x = (x^A, x^B)$ with

$$x^k = (p_i^k(.),\ u_i^k(.),\ I_i^k(.),\ \omega_i^k(.),\ \theta_i^{k,k}(.),\ \theta_i^{k,-k}(.),\ r_i^k(.,.),\ q_i^{k,k}(.,.),\ q_i^{k,-k}(.,.),$$

$$R_i^k(.,.),\ S_i^{k,k}(.,.),\ S_i^{k,-k}(.,.),\ i = 1, \dots, N).$$

To ease the exposition, we will denote $y^k = (p_i^k(.),\ u_i^k(.),\ I_i^k(.),\ i = 1, \dots, N)$ and

$$\lambda_i^k = (\omega_i^k(.),\ \theta_i^{k,k}(.),\ \theta_i^{k,-k}(.),\ r_i^k(.,.),\ q_i^{k,k}(.,.),\ q_i^{k,-k}(.,.), R_i^k(.,.),\ S_i^{k,k}(.,.),\ S_i^{k,-k}(.,.),\ i = 1, \dots, N)$$

so that $x^k = (y^k, \lambda^k)$.

By defining X^k as the set of variables x^k that satisfy the constraints that are independent from p^{-k}, and X such that $X = X^A \times X^B$, it is clear that X is convex and closed. We notice that it is non empty by taking all variables equal to 0 except $r_i^k(s,t) = \hat{\alpha}_i^k(s)$, $R_i^k(\tau,t) = \int_0^\tau \hat{\alpha}_i^k(s)ds$, and $I_i^k(t) = I_i^{k^0} + R_i^k(t,t)\ \forall i, t,\ k = A, B$.

As previously, we denote $Q^k(\bar{x}^{-k}) \subset X^k$ the subset of all feasible strategy and state vectors x^k for player k including all constraints, given the strategy and state vector $\bar{x}^{-k}$ of her competitor. Again, it is clear that for all $\bar{x}^{-k} \in X^{-k}$, $Q^k(\bar{x}^{-k})$ is a closed and convex subset of X^k. We will prove that it is non empty by showing that the solution (feasible under Assumption 2) such that $\forall i,\ t$:

$$\omega_i^k(t) = \theta_i^{k,k}(t) = \theta_i^{k,-k}(t) = 0$$

$$p_i^k(t) = \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))\bar{p}_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)}$$

$$r_i^k(s,t) = \hat{\alpha}_i^k(s),\ q_i^{k,k}(s,t) = \hat{\beta}_i^{k,k}(s)p_i^k(s),\ q_i^{k,-k}(s,t) = \hat{\beta}_i^{k,-k}(s)\bar{p}_i^{-k}(s) \quad \forall s \in [0,t]$$

$$R_i^k(\tau,t) = \int_0^\tau r_i^k(s,t)ds, \quad S_i^{k,k}(\tau,t) = \int_0^\tau q_i^{k,k}(s,t)ds, \quad S_i^{k,-k}(\tau,t) = \int_0^\tau q_i^{k,-k}(s,t)ds \quad \forall \tau \in [0,t]$$

$$u_i^k(t) = 2\hat{\alpha}_i^k(t) + 2\hat{\beta}_i^{k,k}(t)p_i^k(t) + 2\hat{\beta}_i^{k,-k}(t)\bar{p}_i^{-k}(t)$$

$$I_i^k(t) = I_i^{k^0} + \int_0^t (u_i^k(s) - \alpha_i^k(s) + \beta_i^{k,k}(s)p_i^k(s) - \beta_i^{k,-k}(s)\bar{p}_i^{-k}(s))ds$$

belongs to the set $Q^k(\bar{x}^{-k})$ for $\bar{x}^{-k} \in X^{-k}$.

Since $\bar{x}^{-k} \in X^{-k}$, $\bar{p}_i^{-k}(t) \leq p_{i_{max}}^{-k}(t)$ and therefore $p_i^k(t) \leq p_{i_{max}}^k(t)$.
Since both prices are below their maximum threshold, it is clear that under Assumption 2, $\sum_i u_i^k(t) \leq K^k(t)$.
Finally, it is easy to derive that $\dot{I}_i^k(t) = \hat{\alpha}_i^k(t) + \hat{\beta}_i^{k,k}(t)p_i^k(t) + \hat{\beta}_i^{k,-k}(t)\bar{p}_i^{-k}(t)$, and since $I_i^{k^0} \geq 0 = R_i^k(0,0) + S_i^{k,k}(0,0) + S_i^{k,-k}(0,0)$, using inequality (11), the security level for $I_i^k(t)$ is satisfied.
We denote $Y \subset X$ the set of feasible collective strategy and state vectors:

$$Y = \{x \in X : x^k \in Q^k(x^{-k}),\ k = A, B\}.$$

Then clearly Y is a convex closed subset of X. To show that it is non empty, we take the same solution as above except that both prices are set to their maximum threshold. Using the same reasoning, this point is an element of set Y.
The objective function is unchanged, so all the properties we proved regarding it earlier still hold for this problem.
The proof of upper semi continuity of Q can be adapted from the proof of Proposition 5 in a straightforward way.

Now let's prove that Q is lower semi continuous. Consider $x_n \in X$ such that $x_n \to x(\in X)$ and $\bar{x} \in Q(x)$. We want to construct $\bar{x}_n$ such that $\bar{x}_n \to \bar{x}$ and for n large enough, $\bar{x}_n \in Q(x_n)$. We observe that the difficulty comes from the inventory security level guarantee; it is straightforward to satisfy the constraints that are involving directly the control variables.
Let's denote

$$m_i^k(t,\lambda) \equiv \omega_i^k(t)\Gamma_i^k(t) + \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t) + R_i^k(t,t) + S_i^{k,k}(t,t) + S_i^{k,-k}(t,t)$$

the minimum security level for the inventory of product i at time t for supplier k. The constraint guaranteeing no backorder is written $I_i^k(t) \geq m_i^k(t,\lambda)$. Note that $m_i^k(t,.)$ is a continuous function.
First, we notice that if $\bar{I}_i^k(t) > m_i^k(t,\bar{\lambda})\ \forall i,t$, then for any $\bar{x}_n$ such that $\bar{x}_n \to \bar{x}$, we will have $\bar{I}_{n,i}^k(t) > m_i^k(t,\bar{\lambda}_n)$ for n large enough since $\bar{I}_n(t) \to \bar{I}(t)$ and $m_i^k(t,\bar{\lambda}_n) \to m_i^k(t,\bar{\lambda})$. It is therefore

easy to construct $\bar{x}_n$ such that $\bar{x}_n \to \bar{x}$ and $\bar{x}_n \in Q(x_n)$ in that case, so let's assume we have a time t and a product i such that $\bar{I}_i^k(t) = m_i^k(t, \bar{\lambda})$ for supplier k.

To prove the result, it would be sufficient to construct feasible $\bar{x}_n$ such that in particular

$$\bar{I}_{n,i}^k(t) - \bar{I}_i^k(t) \geq m_i^k(t, \bar{\lambda}_n) - m_i^k(t, \bar{\lambda}),$$

(in addition to other feasibility constraints) i.e.

$$\begin{aligned}\int_0^t (\bar{u}_{n,i}^k(t) - \bar{u}_i^k(t)) + (\beta_i^{k,k}(s)(\bar{p}_{n,i}^k(s) - \bar{p}_i^k(s)) - \beta_i^{k,-k}(s)(p_{n,i}^{-k}(s) - p_i^{-k}(s)))ds \geq \\ (\bar{\omega}_{n,i}^k(t) - \bar{\omega}_i^k(t))\Gamma_i^k(t) + (\bar{\theta}_{n,i}^{k,k}(t) - \bar{\theta}_i^{k,k}(t))\Theta_i^{k,k}(t) + (\bar{\theta}_{n,i}^{k,-k}(t) - \bar{\theta}_i^{k,-k}(t))\Theta_i^{k,-k}(t) \\ + \int_0^t (\bar{r}_{n,i}^k(s,t) - \bar{r}_i^k(s,t) + \bar{q}_n^{k,k}(s,t) - \bar{q}^{k,k}(s,t) + \bar{q}_n^{k,-k}(s,t) - \bar{q}^{k,-k}(s,t))ds. \quad (12)\end{aligned}$$

In order to satisfy this inequality, we should attempt to choose $\bar{x}_n$ such that $\bar{u}_{n,i}^k - \bar{u}_i^k, \bar{p}_{n,i}^k - \bar{p}_i^k$ are as large as possible (while converging to zero) and $m_i^k(t, \bar{\lambda}_n) - m_i^k(t, \bar{\lambda})$ as small as possible (while converging to zero). Our goal is thus to construct $\bar{\lambda}_n$ by modifying $\bar{\lambda}$ (this modification converging to zero) while decreasing its value is possible, and satisfying all feasibility constraints.

Note that for a given x and $\bar{y}$ (in particular their price components), the vector $\bar{\lambda}$ that minimizes $m_i^k(t, \bar{\lambda})$ under the constraint $\bar{x} \in Q(x)$ is obtained if $\bar{\lambda}$ is formed by the variables that solve the dual subproblems presented in section 5.2. Let's denote $\Omega_i^{k*}(t, \bar{p}^k(.), p^{-k}(.))$ the minimum value of the security level obtained with the solution above. Let $\bar{\lambda}_i^{k*}(t, \bar{p}^k(.), p^{-k}(.))$ the corresponding components (we write explicitly the arguments for the same reason as just explained).

Let $\epsilon > 0$. We claim that given $x_n \to x$ and $\bar{y}_n \to \bar{y}$, if $\bar{\lambda}_i^k \neq \bar{\lambda}_i^{k*}(t, \bar{p}^k(.), p^{-k}(.))$ (and thus $m_i^k(t, \bar{\lambda}) > \Omega_i^{k*}(t, \bar{p}^k(.), p^{-k}(.)))$, there exists $\bar{\bar{\lambda}}_{n,i}^k \to \bar{\lambda}_i^k$ such that for n large enough, $m_i^k(t, \bar{\bar{\lambda}}_n) = m_i^k(t, \bar{\lambda}) - \epsilon m_{n,i}^k$ for some positive $m_{n,i}^k$ that converges toward zero, and such that $\bar{\bar{\lambda}}_{n,i}^k$ satisfies the feasibility constraints depending on $\bar{p}_{n,i}^k$ and $p_{n,i}^{-k}$ for n sufficiently large. To see this, notice that $\bar{\lambda}$ is not the optimal solution of the continuous LPs shown above; therefore the linearity of the problem implies that it is possible to perturb its components (in a way that converges to zero at $n \to \infty$) while decreasing the objective value. Furthermore, the linearity of the constraints satisfied by $\bar{\lambda}$ that involve prices $\bar{p}_i^k$ and p_i^{-k} implies that it is again possible to perturb the components of $\bar{\lambda}$ (in a way that converges to zero at $n \to \infty$) to make the perturbed solution feasible with $\bar{p}_{n,i}^k$ and $p_{n,i}^{-k}$, since $x_n \to x$ and $\bar{y}_n \to \bar{y}$.

Moreover, the only situation in which we cannot choose $\bar{u}^k_{n,i}$ strictly greater than $\bar{u}^k_i$ is when the capacity constraint is tight under $\bar{u}^k$ and the inventory security level guarantee is satisfied with equality for all products. Indeed, otherwise we can increase infinitesimally the production rate by shifting production from a product that has a non tight inventory level constraint (shifting production from that product will slightly decrease its inventory level, but as long as the security level constraint is not tight we can do it infinitesimally and remain feasible).
Similarly, the only situation in which we cannot perturb $\bar{p}^k_i$ by increasing it while remaining feasible is when the price is already at its maximum (for fixed p^{-k}).

Let's define on $[0, t]$ (omitting the time argument for the sake of clarity) for some $\epsilon' > 0$

$$\bar{u}^k_{n,i} = \begin{cases} \bar{u}^k_i & \text{if } \sum_i \bar{u}^k_i = K^k \text{and } \bar{I}^k_j(t) = m_j(t, \bar{\lambda}) \;\; \forall j \\ \bar{u}^k_i + m^k_{n,i}\epsilon' & \text{else} \end{cases}$$

$$\bar{p}^k_{n,i} = \begin{cases} \frac{\alpha^k_i(t) - \hat{\alpha}^k_i(t) + (\beta^{k,-k}_i(t) - \hat{\beta}^{k,-k}_i(t))p^{-k}_{n,i}(t)}{\beta^{k,k}_i(t) + \hat{\beta}^{k,k}_i(t)} & \text{if } \bar{p}^k_i = \frac{\alpha^k_i(t) - \hat{\alpha}^k_i(t) + (\beta^{k,-k}_i(t) - \hat{\beta}^{k,-k}_i(t))p^{-k}_i(t)}{\beta^{k,k}_i(t) + \hat{\beta}^{k,k}_i(t)} \\ \bar{p}^k_i + m^k_{n,i}\epsilon' & \text{else} \end{cases}$$

$$\bar{\lambda}^k_{n,i} = \; = \begin{cases} \bar{\lambda}^{k*}_i(t, \bar{p}^k_n(.), p^{-k}_n) & \text{if } \bar{\lambda}^k_i = \bar{\lambda}^{k*}_i(t, \bar{p}^k(.), p^{-k}) \\ \bar{\bar{\lambda}}^k_i & \text{else.} \end{cases}$$

We observe that if for some product i, either $\bar{u}^k_{n,i}$ or $\bar{p}^k_{n,i}$ is given by its second expression on a domain with positive measure, or if $\bar{\lambda}^k_{n,i}$ is given by its second expression, then the inequality (12) that we want to prove can be rewritten $\epsilon'' m^k_{n,i} + A^k_{n,i} \geq 0$, with ϵ'' depends on ϵ, ϵ' and the measure of that domain. If $A_n \geq 0$, taking $m^k_{n,i} = 0$ will satisfy the inequality. Otherwise, we take $m^k_{n,i} = -A^k_{n,i}/\epsilon''$.

So now let's suppose that for all products i, $\bar{u}^k_{n,i}$ and $\bar{p}^k_{n,i}$ are given at all times by their first expression and that so is $\bar{\lambda}^k_{n,i}$. Therefore we are supposing that at time t, the production capacity is tight, that for all products the inventory security level is tight, and for some product i prices are equal to their upper bounds, and the variables introduced by the dual subproblems are at their optimum. We will show that this situation is impossible by displaying a contradiction.

In this case, the fact that the inventory security level is tight can be rewritten as

$$I_i^{k^0}+\int_0^t\Big(\bar{u}_i^k(s)-\alpha_i^k(s)+\beta_i^{k,k}(s)\frac{\alpha_i^k(s)-\hat{\alpha}_i^k(s)+(\beta_i^{k,-k}(s)-\hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s)+\hat{\beta}_i^{k,k}(s)}-\beta_i^{k,-k}(s)p^{-k}(s)\Big)ds=$$

$$\Omega_i^k\Big(t,\ \frac{\alpha_i^k(s)-\hat{\alpha}_i^k(s)+(\beta_i^{k,-k}(s)-\hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s)+\hat{\beta}_i^{k,k}(s)},\ s\in[0,t],\ p_i^{-k}(.)\Big)$$

$$\le\int_0^t\big(\hat{\alpha}_i^k(s)+\hat{\beta}_i^{k,k}(s)\frac{\alpha_i^k(s)-\hat{\alpha}_i^k(s)+(\beta_i^{k,-k}(s)-\hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s)+\hat{\beta}_i^{k,k}(s)}+\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s)\big)ds$$

which (after calculations) implies

$$I_i^{k^0}+\int_0^t\Big(\bar{u}_i^k(s)-2\hat{\alpha}_i^k(s)-2\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s)-2\hat{\beta}_i^{k,k}(s)\frac{\alpha_i^k(s)-\hat{\alpha}_i^k(s)+(\beta_i^{k,-k}(s)-\hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s)+\hat{\beta}_i^{k,k}(s)}\Big)ds\le 0.$$

Note that the left hand side is lower bounded by

$$I_i^{k^0}+\int_0^t\Big(\bar{u}_i^k(s)-2\hat{\alpha}_i^k(s)-2\hat{\beta}_i^{k,-k}(s)p_{i_{max}}^{-k}(s)-2\hat{\beta}_i^{k,k}(s)p_{i_{max}}^{k}(s)\Big)ds$$

which, after adding over all products, and under Assumption 2, since the capacity is tight, is lower bounded by $\sum_{i=1}^N I_i^{k^0}>0$. This is a contradiction since the right hand side in the last equality is negative.

□

E.8 Proof of Proposition 6

Proof. Let x^* a Nash equilibrium and $z\in Q(x^*)$. By definition of a Nash equilibrium, since $(z^k,x^{*-k})\in Y,\ k=A,B,$ we have $J^k(z^k,x^{*-k})-J^k(x^{*k},x^{*-k})\le 0,\ k=A,B$. Summing these inequalities over $k=A,B$ leads to $\psi(x^*,z)\le 0\ \ \forall z\in Q(x^*)$. Since $x^*\in Q(x^*)$ and $\psi(x^*,x^*)=0$, we have $\max_{z\in Q(x^*)}\psi(x^*,z)=0$.

To show the reverse, let's assume that $x^*\in Y$ is given and $\max_{z\in Q(x^*)}\psi(x^*,z)=0$, and suppose x^* is not a Nash equilibrium. Then there exists k_0 and z^{k_0} such that $(z^{k_0},x^{-k_0})\in Y$ and $J^{k_0}(z^{k_0},x^{*-k_0})-J^{k_0}(x^{*k_0},x^{*-k_0})>0$. Let z^* such that $z^{*k_0}=z^{k_0}$ and $z^{*-k_0}=x^{-k_0}$. Then $z^*\in Q(x^*)$ and $\psi(x^*,z^*)=\sum_k J^k(z^{*k},x^{*-k})-J^k(x^{*k},x^{*-k})=J^{k_0}(z^{k_0},x^{*-k_0})-J^{k_0}(x^{*k_0},x^{*-k_0})>0$ which contradicts $\max_{z\in Q(x^*)}\psi(x^*,z)=0$. □

E.9 Proof of Theorem 4

Proof. Rosen [27] provides the proof of this result in the case where all coupled constraints are shared by the agents. In this paper, the minimum inventory level constraint, the upper bound on the price, and the inventory flow constraint for supplier k are not included in supplier $-k$'s program. Specifically, in our model, $\Omega^k(\bar{x}^{-k}) \equiv \{x^k : (x^k, \bar{x}^{-k}) \in Y\} \subset Q^k(\bar{x}^{-k})$, while in Rosen's model these two sets are equal. Nevertheless, when we focus on normalized Nash equilibria, we consider the sum of the utility functions and we maximize it over the cartesian product of each set across $k = A,\ B$. Since we have $x \in Q(x) \Leftrightarrow x \in \Omega(x) \Leftrightarrow x \in Y$, where $\Omega(x) = \Omega^A(x^B) \times \Omega^B(x^A)$, Rosen's proof is still valid. $\square$

E.10 Proof of Theorem 6

Proof. In this problem we have (we abuse notations by mentioning only the component for product i at time t in order to ease the exposition):

$$g(x) = \begin{pmatrix} \nabla_{x^A} J^A(x) \\ \nabla_{x^B} J^B(x) \end{pmatrix} = \begin{pmatrix} \alpha_i^A(t) - 2\beta_i^{A,A}(t) p_i^A(t) + \beta_i^{A,B}(t) p_i^B(t) \\ -2\gamma_i^A(t) u_i^A(t) \\ -2h_i^A(t) I_i^A(t) \\ \alpha_i^B(t) - 2\beta_i^{B,B}(t) p_i^B(t) + \beta_i^{B,A}(t) p_i^A(t) \\ -2\gamma_i^B(t) u_i^B(t) \\ -2h_i^B(t) I_i^B(t) \end{pmatrix}$$

$$G(x) = \begin{pmatrix} -2\beta_i^{A,A}(t) & & & \beta_i^{A,B}(t) & & \\ & -2\gamma_i^A(t) & & & & \\ & & -2h_i^A(t) & & & \\ \beta_i^{B,A}(t) & & & -2\beta_i^{B,B}(t) & & \\ & & & & -2\gamma_i^B(t) & \\ & & & & & -2h_i^B(t) \end{pmatrix}$$

$G(x) + G^T(x)$ is a symmetric and strictly diagonally dominant matrix under Assumption 1 for $x \in Y$, with negative elements on the diagonal. Therefore, it is negative definite. The result then follows from Theorem 7. □

E.11 Proof of Proposition 7

Proof. Let's consider formulation (12). Let $x \in Y$; we can rewrite $BR(x)$ as the solution of the quadratic program $\min_{z\in Y} z^T Mz - c(x)^T z$ where $M \in \mathbb{R}^{6NT}$ is positive definite, $c(x)$ is a vector such that $c(.)$ a continuous mapping. (M is a diagonal matrix with components $\beta_i^{k,k}(t)$, $\gamma_i^k(t)$, $h_i^k(t)$, and $c(x)$ has components $\alpha_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)$, with the prices $p_i^{-k}(t)$ taken from vector x). Based on the fact that the feasible set is convex and independent of x, Daniel[1] showed using variational inequalities that if $x' \in Y$, $\epsilon \equiv ||c(x) - c(x')||$ and κ is the smallest eigenvalue of matrix M, then we have $||BR(x') - BR(x)|| \le \epsilon(\kappa - \epsilon)^{-1}(1 + ||BR(x)||)$ for $\epsilon < \kappa$. Since $c(.)$ is a continuous mapping, it follows that BR is continuous. □

E.12 Proof of Proposition 8

Proof. Let $\theta_1 \in [0, 1]$ and $\theta_2 = 1 - \theta_1$. To ease the exposition, we omit the time argument.

$$
\begin{aligned}
\Delta \;&=\; \theta_1\psi(x,\bar{\bar{x}}) + \theta_2\psi(\bar{x},\bar{\bar{x}}) - \psi(\theta_1 x + \theta_2\bar{x},\bar{\bar{x}}) \\
&=\; \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k=A,B}\Big[\theta_1\Big((\alpha_i^k + \beta_i^{k,-k}p_i^{-k})(\bar{\bar{p}}_i^k - p_i^k) - \beta_i^{k,k}(\bar{\bar{p}}_i^{k^2} - p_i^{k^2}) - \gamma_i^k(\bar{\bar{u}}_i^{k^2} - u_i^{k^2}) - h_i^k(\bar{\bar{I}}_i^{k^2} - I_i^{k^2})\Big) \\
&\qquad + \theta_2\Big((\alpha_i^k + \beta_i^{k,-k}\bar{p}_i^{-k})(\bar{\bar{p}}_i^k - \bar{p}_i^k) - \beta_i^{k,k}(\bar{\bar{p}}_i^{k^2} - \bar{p}_i^{k^2}) - \gamma_i^k(\bar{\bar{u}}_i^{k^2} - \bar{u}_i^{k^2}) - h_i^k(\bar{\bar{I}}_i^{k^2} - \bar{I}_i^{k^2})\Big) \\
&\qquad - \big(\alpha_i^k + \beta_i^{k,-k}(\theta_1 p_i^{-k} + \theta_2\bar{p}_i^{-k})\big)(\bar{\bar{p}}_i^k - \theta_1 p_i^k - \theta_2\bar{p}_i^k) + \beta_i^{k,k}\big(\bar{\bar{p}}_i^{k^2} - (\theta_1 p_i^k + \theta_2\bar{p}_i^k))^2\big) \\
&\qquad - \gamma(\bar{\bar{u}}_i^{k^2} - (\theta_1 u_i^k + \theta_2\bar{u}_i^k)^2) - h_i^k(\bar{\bar{I}}_i^{k^2} - (\theta_1 I_i^k + \theta_2\bar{I}_i^k)^2)\Big].
\end{aligned}
$$

[1]Daniel [10] proved the result under linear equality and inequality constraints. Nevertheless, it is easy to see that the proof remains the same under a more general convex set.

After calculations,

$$\begin{aligned}\Delta &= \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k=A,B}\Big[-\theta_1\theta_2\beta_i^{k,-k}(p_i^{-k}-\bar{p}_i^{-k})(p_i^k-\bar{p}_i^k)+\theta_1\theta_2\beta_i^{k,k}(p_i^k-\bar{p}_i^k)^2+\gamma_i^k\theta_1\theta_2(u_i^k-\bar{u}_i^k)^2 \\ &\qquad\qquad +h_i^k\theta_1\theta_2(I_i^k-\bar{I}_i^k)^2\Big] \\ &= \theta_1\theta_2 v(x,\bar{x}), \quad \text{where}\end{aligned}$$

$$v(x,\bar{x}) = \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k=A,B}\Big[-\beta_i^{k,-k}(p_i^{-k}-\bar{p}_i^{-k})(p_i^k-\bar{p}_i^k)+\beta_i^{k,k}(p_i^k-\bar{p}_i^k)^2+\gamma_i^k(u_i^k-\bar{u}_i^k)^2+h_i^k(I_i^k-\bar{I}_i^k)^2\Big].$$

We have

$$v(x,\bar{x}) = -\sum_{t=1}^{T}\sum_{i=1}^{N}(\beta_i^{A,B}+\beta_i^{B,A})(p_i^B-\bar{p}_i^B)(p_i^A-\bar{p}_i^A)+O_1(||x-\bar{x}||^2),$$

and thus

$$\frac{v(x,\bar{x})}{||x-\bar{x}||} = -\frac{\sum_{t=1}^{T}\sum_{i=1}^{N}(\beta_i^{A,B}+\beta_i^{B,A})(p_i^B-\bar{p}_i^B)(p_i^A-\bar{p}_i^A)}{\sqrt{\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k=A,B}\Big[(p_i^k-\bar{p}_i^k)^2+(u_i^k-\bar{u}_i^k)^2+(I_i^k-\bar{I}_i^k)^2\Big]}}+O_1(||x-\bar{x}||)$$

which tends to 0 as $x,\bar{x}\to w\in Y$ since in the ratio above, the numerator is of the order of ϵ^2 and the denominator is of the order of ϵ, with $\epsilon\to 0$. □

E.13 Proof of Proposition 9

Proof. This is clear considering that in the expression of $\psi(x,\bar{x})$, the dependence in $\bar{x}$ appears as negative quadratic terms in $\bar{p}$, $\bar{u}$ and $\bar{I}$. More explicitly, let $\theta_1\in[0,1]$ and $\theta_2=1-\theta_1$. To ease the exposition, we omit the time argument.

$$\begin{aligned}
\Delta &= \theta_1\psi(\bar{\bar{x}},x)+\theta_2\psi(\bar{\bar{x}},\bar{x})-\psi(\bar{\bar{x}},\theta_1 x+\theta_2\bar{x}) \\
&= \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k=A,B}\Big[\theta_1\Big((\alpha_i^k+\beta_i^{k,-k}\bar{\bar{p}}_i^{-k})(p_i^k-\bar{\bar{p}}_i^k)-\beta_i^{k,k}(p_i^{k^2}-\bar{\bar{p}}_i^{k^2})-\gamma(u_i^{k^2}-\bar{\bar{u}}_i^{k^2})-h_i^k(I_i^{k^2}-\bar{\bar{I}}_i^{k^2})\Big) \\
&\quad +\theta_2\Big((\alpha_i^k+\beta_i^{k,-k}\bar{\bar{p}}_i^{-k})(\bar{p}_i^k-\bar{\bar{p}}_i^k)-\beta_i^{k,k}(\bar{p}_i^{k^2}-\bar{\bar{p}}_i^{k^2})-\gamma_i^k(\bar{u}_i^{k^2}-\bar{\bar{u}}_i^{k^2})-h_i^k(\bar{I}_i^{k^2}-\bar{\bar{I}}_i^{k^2})\Big) \\
&\quad -\big(\alpha_i^k+\beta_i^{k,-k}\bar{\bar{p}}_i^{-k}\big)(\theta_1 p_i^k+\theta_2\bar{p}_i^k-\bar{\bar{p}}_i^k)+\beta_i^{k,k}\big((\theta_1 p_i^k+\theta_2\bar{p}_i^k)^2-\bar{\bar{p}}_i^{k^2}\big) \\
&\quad -\gamma_i^k((\theta_1 u_i^k+\theta_2\bar{u}_i^k)^2-\bar{\bar{u}}_i^{k^2})-h_i^k((\theta_1 I_i^k+\theta_2\bar{I}_i^k)^2-\bar{\bar{I}}_i^{k^2})\Big].
\end{aligned}$$

After calculations,

$$\begin{aligned}
\Delta &= \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k=A,B}\Big[-\theta_1\theta_2\Big(\beta_i^{k,k}(p_i^k-\bar{p}_i^k)^2+\gamma_i^k(u_i^k-\bar{u}_i^k)^2+h_i^k(I_i^k-\bar{I}_i^k)^2\Big)\Big] \\
&= \theta_1\theta_2\mu(x,\bar{x}), \quad \text{where}
\end{aligned}$$

$$\mu(x,\bar{x}) = -\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{k=A,B}\Big[\beta_i^{k,k}(p_i^k-\bar{p}_i^k)^2+\gamma_i^k(u_i^k-\bar{u}_i^k)^2+h_i^k(I_i^k-\bar{I}_i^k)^2\Big].$$

We have $\mu(x,\bar{x})=O_1(||x-\bar{x}||^2)$, and thus $\frac{\mu(x,\bar{x})}{||x-\bar{x}||}=O_1(||x-\bar{x}||)$ which tends to 0 as $x,\bar{x}\to w\in Y$. In particular, ψ is weakly concave with respect to the second argument. □

E.14 Proof of Proposition 10

Proof. Consider $\zeta(x)=\lambda x^2$ for some $\lambda>0$.

$$\begin{aligned}
v(x,\bar{x})-\mu(\bar{x},x)-\lambda||x-\bar{x}||^2 &= \sum_{t=1}^{T}\sum_{i=1}^{N}\Big[(2\beta_i^{A,A}(t)-\lambda)(p_i^A(t)-\bar{p}_i^A(t))^2+(2\beta_i^{B,B}(t)-\lambda)(p_i^B(t)-\bar{p}_i^B(t))^2 \\
&\quad -(\beta_i^{A,B}(t)+\beta_i^{B,A}(t))(p_i^A(t)-\bar{p}_i^A(t))(p_i^B(t)-\bar{p}_i^B(t)) \\
&\quad +(2\gamma_i^A(t)-\lambda)(u_i^A(t)-\bar{u}_i^A(t))^2+(2\gamma_i^B(t)-\lambda)(u_i^B(t)-\bar{u}_i^B(t))^2 \\
&\quad +(2h_i^A(t)-\lambda)(I_i^A(t)-\bar{I}_i^A(t))^2+(2h_i^B(t)-\lambda)(I_i^B(t)-\bar{I}_i^B(t))^2\Big].
\end{aligned}$$

Let

$$\lambda_1 = 2 \min_k \min_i \min_t h_i^k(t), \qquad \lambda_2 = 2 \min_k \min_i \min_t \gamma_i^k(t).$$

A condition for the expression above to be positive is that $\lambda < \lambda_1, \quad \lambda < \lambda_2$ and the symmetric matrix

$$N = \begin{bmatrix} 2\beta_i^{A,A}(t) - \lambda & -\frac{(\beta_i^{A,B}(t)+\beta_i^{B,A}(t))}{2} \\ -\frac{(\beta_i^{A,B}(t)+\beta_i^{B,A}(t))}{2} & 2\beta_i^{B,B}(t) - \lambda \end{bmatrix}$$

is positive semi-definite for all i, t.

Indeed, this condition implies on the one hand that $2h_i^k(t) - \lambda > 0 \ \forall i, k, t$ and $2\gamma_i^k(t) - \lambda > 0 \ \forall i, k, t$, and on the other hand that for any vector $v = (v_1, v_2)$, we have

$$0 \leq v^T N v = (2\beta_i^{A,A}(t) - \lambda)v_1^2 + (2\beta_i^{B,B}(t) - \lambda)v_2^2 - (\beta_i^{A,B}(t) + \beta_i^{B,A}(t))v_1 v_2$$

In particular, using $v_1 = p_i^A(t) - \bar{p}_i^A(t)$ and $v_2 = p_i^B(t) - \bar{p}_i^B(t)$ leads to the result.

We notice that

$$\begin{aligned} N \succeq 0 \quad &\Leftrightarrow \quad \big(\mathrm{Tr}(N) \geq 0 \text{ and } \mathrm{Det}(N) \geq 0\big) \\ &\Leftrightarrow \quad \big(\beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \lambda \geq 0 \text{ and } (2\beta_i^{A,A}(t) - \lambda)(2\beta_i^{B,B}(t) - \lambda) - \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 \geq 0\big) \\ &\Leftrightarrow \quad \begin{cases} \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) \\ \lambda^2 - 2\lambda(\beta_i^{A,A}(t) + \beta_i^{B,B}(t)) + 4\beta_i^{A,A}(t)\beta_i^{B,B}(t) - \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 \geq 0. \end{cases} \end{aligned}$$

The simplified determinant of the polynomial above is (for fixed $i, \ t$)

$$\begin{aligned} \Delta \quad &= \quad (\beta_i^{A,A}(t) + \beta_i^{B,B}(t))^2 - 4\beta_i^{A,A}(t)\beta_i^{B,B}(t) + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 \\ &= \quad (\beta_i^{A,A}(t) - \beta_i^{B,B}(t))^2 + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 > 0 \end{aligned}$$

so the polynomial has two real roots $\beta_i^{A,A}(t) + \beta_i^{B,B}(t) \pm \sqrt{\Delta}$ and only one satisfies $\lambda \leq \beta_i^{A,A}(t) +$

$\beta_i^{B,B}(t)$. Since we are interested in positive parameters λ, we obtain that

$$(\lambda > 0 \text{ and } N \succeq 0) \quad \Leftrightarrow \quad \begin{cases} \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta} > 0 \\ 0 < \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} (\beta_i^{A,A}(t) - \beta_i^{B,B}(t))^2 + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 < (\beta_i^{A,A}(t) + \beta_i^{B,B}(t))^2 \\ 0 < \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 < 4\beta_i^{A,A}(t)\beta_i^{B,B}(t) \\ 0 < \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta}. \end{cases}$$

The first condition is satisfied under Assumption 1. Let

$$\lambda_3 = \min_i \inf_{t \in [0,T]} \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{(\beta_i^{A,A}(t) - \beta_i^{B,B}(t))^2 + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2} > 0.$$

As a result, by taking $0 < \lambda < \min\{\lambda_1, \lambda_2, \lambda_3\}$, we obtain that

$$v(x, \bar{x}) - \mu(\bar{x}, x) - \lambda ||x - \bar{x}||^2 > 0 \quad \forall x, \ \bar{x} \in Y.$$

$\square$

F Solution of the dual subproblems

In what follow, we omit the subscript i, the superscript k, and we fix time t and thus omit the time argument. The notations below correspond to the first subproblem concerning the uncertainty on the fixed term in the demand $\alpha^k(.)$. The method for the two other subproblems is identical after substituting respectively $\hat{\beta}^{k,k}p^k$ and $\hat{\beta}^{k,-k}p^{-k}$ for $\hat{\alpha}$, and $\Theta^{k,k}$ and $\Theta^{k,-k}$ for Γ.

Consider:

$$
\begin{aligned}
\Omega = \min_{\omega, r(.)} \quad & \omega\Gamma + \int_0^t r(s)ds \\
\text{s.t.} \quad & \omega + r(s) \geq \hat{\alpha}(s) \quad \forall s \in [0,t] \\
& \omega \geq 0 \\
& r(s) \geq 0 \quad \forall s \in [0,t]
\end{aligned}
$$

where $\Gamma \geq 0, \quad \hat{\alpha}(.) \geq 0.$

Case 1 :

If $\omega^* = 0$, then $r^*(s) = \hat{\alpha}(s), \; \forall s \in [0,t]$ and

$$\Omega_1 = \int_0^t \hat{\alpha}(s)ds.$$

Case 2 :

If $\omega^* \geq \sup_{s\in[0,t]} \hat{\alpha}(s)$, then $r^*(s) = 0, \; \forall s \in [0,t]$ and $\Omega = \Gamma\omega^*$. Therefore if this case is optimal, $\omega^* = \sup_{s\in[0,t]} \hat{\alpha}(s)$ and

$$\Omega_2 = \Gamma. \sup_{s\in[0,t]} \hat{\alpha}(s).$$

Note: a necessary condition for Case 2 to be better than Case 1 is $\Gamma < t$.

Case 3 :

If $0 < \omega^* < \sup_{s\in[0,t]} \hat{\alpha}(s)$, then

$$r(s) = \begin{cases} 0 & \text{if } \omega^* \geq \hat{\alpha}(s) \\ \hat{\alpha}(s) - \omega^* & \text{if } \omega^* < \hat{\alpha}(s) \end{cases} = (\hat{\alpha}(s) - \omega^*)^+$$

and

$$\Omega_3 = \Gamma\omega^* + \int_0^t (\hat{\alpha}(s) - \omega^*)^+ ds.$$

In other words, by denoting D_{ω^*} the domain $D_{\omega^*} = \{s \in [0,t] : \hat{\alpha}(s) > \omega^*\}$ and l_{ω^*} its measure, then

$$\Omega_3 = (\Gamma - l_{\omega^*})\omega^* + \int_{s \in D_{\omega^*}} \hat{\alpha}(s) ds.$$

We then have to determine the value $\omega^* \in (0, \sup_{s\in[0,t]} \hat{\alpha}(s))$ that minimizes the expression above for Ω_3.

Notice that if $\Gamma \geq t$, then

$$\Omega_1 - \Omega_3 = -(\Gamma - l_{\omega^*})\omega^* + \int_{s \in D^c_{\omega^*}} \hat{\alpha}(s) ds < -(\Gamma - l_{\omega^*})\omega^* + (t - l_{\omega^*})\omega^* = (t - \Gamma)\omega^* < 0$$

so Case 1 is optimal.

- **Case 3a**: if $\omega^* \leq \inf_{s\in[0,t]} \hat{\alpha}(s)$, then

  $$\Omega_{3a} = (\Gamma - t)\omega^* + \int_0^t \hat{\alpha}(s) ds.$$

  If $\Gamma \geq t$, the best value is $\omega^* = 0$ and this leads to Case 1.

  If $\Gamma < t$, the best value is $\omega^* = \inf_{s\in[0,t]} \hat{\alpha}(s)$ and

  $$\Omega_{3a} = -(t - \Gamma). \inf_{s\in[0,t]} \hat{\alpha}(s) + \int_0^t \hat{\alpha}(s) ds.$$

  Clearly then the objective value is lower than in Case 1. Note that depending on actual data, if $\Gamma < t$, either Case 2 or Case 3a may be optimal. Indeed,

  $$\Omega_2 - \Omega_{3a} = -\int_0^t \hat{\alpha}(s) ds + t \inf_{s\in[0,t]} \hat{\alpha}(s) + \Gamma(\sup_{s\in[0,t]} \hat{\alpha}(s) - \inf_{s\in[0,t]} \hat{\alpha}(s))$$

  which tends to a non positive value as $\Gamma \to 0^+$ and to a non negative value as $\Gamma \to t^-$.

- **Case 3b**: if $\inf_{s\in[0,t]} \hat{\alpha}(s) < \omega^* < \sup_{s\in[0,t]} \hat{\alpha}(s)$: notice that at the extreme points of this range, we are respectively in Case 3a and Case 2. However, it is possible that for some value of ω^* in this range, Ω_3 takes an even lower value than in those two other cases.

  In particular, if $\hat{\alpha}(.)$ is strictly increasing and differentiable, then $D_{\omega^*} = (\hat{\alpha}^{-1}(\omega^*), t]$,

$l_{\omega^*} = t - \hat{\alpha}^{-1}(\omega^*)$ and

$$\Omega_{3b}(\omega^*) = (\Gamma - t + \hat{\alpha}^{-1}(\omega^*))\omega^* + \int_{\hat{\alpha}^{-1}(\omega^*)}^{t} \hat{\alpha}(s)ds.$$

We have

$$\Omega'_{3b}(\omega^*) = \Gamma - t + \hat{\alpha}^{-1}(\omega^*) + \frac{\omega^*}{\hat{\alpha}'(\hat{\alpha}^{-1}(\omega^*))} - \frac{\hat{\alpha}(\hat{\alpha}^{-1}(\omega^*))}{\hat{\alpha}'(\hat{\alpha}^{-1}(\omega^*))} = \Gamma - t + \hat{\alpha}^{-1}(\omega^*)$$

so $\Omega_{3b}(\omega^*)$ reaches a minimum on the considered range if $\Gamma < t$, and then

$$\Omega_{3b} = \int_{t-\Gamma}^{t} \hat{\alpha}(s)ds$$

(note: we verify that this case yields indeed a smaller objective value than Case 3a and Case 2)

Finally, we have to compare which of Cases 1, 2, 3 provides the smallest value of Ω.

Note in particular that $\Omega \leq \Omega_1 = \int_0^t \hat{\alpha}(s)ds$ which yields inequality (11).

We observe that if $\Gamma \geq t$, then Case 1 is optimal. If $\Gamma < t$, in general, either of the cases may be optimal depending on the data.

In the particular case where $\hat{\alpha}(.)$ is strictly increasing and differentiable,

$$\Omega = \begin{cases} \int_{t-\Gamma}^{t} \hat{\alpha}(s)ds & \text{if } \Gamma < t \\ \int_0^t \hat{\alpha}(s)ds & \text{else.} \end{cases}$$

To summarize, in the general case, Ω as a function of t, Γ and $\hat{\alpha}(.)$ is as follows:

- if $\Gamma \geq t$, then $\Omega = \int_0^t \hat{\alpha}(s)ds$.
- else, $\Omega = \min\{\Omega_2, \Omega_{3a}, \Omega_{3b}\}$ where

$$\Omega_2 = \Gamma. \sup_{s\in[0,t]} \hat{\alpha}(s)$$

$$\Omega_{3a} = -(t-\Gamma). \inf_{s\in[0,t]} \hat{\alpha}(s) + \int_0^t \hat{\alpha}(s)ds$$

and

$$\Omega_{3b} = \min_{\omega^*\in(\inf_{s\in[0,t]}\hat{\alpha}(s),\sup_{s\in[0,t]}\hat{\alpha}(s))} \Big[\Gamma\omega^* + \int_0^t (\hat{\alpha}(s)-\omega^*)^+ ds\Big].$$

Notice that this can be rewritten as

$$\Omega = \min_{\omega^*\in[\inf_{s\in[0,t]}\hat{\alpha}(s),\sup_{s\in[0,t]}\hat{\alpha}(s)]} \Big[\Gamma\omega^* + \int_0^t (\hat{\alpha}(s)-\omega^*)^+ ds\Big]$$

(the minimum is taken over the closed interval).

G Additional Numerical Results

Scenario	$\beta^{A,A}(t)$	$\beta^{B,B}(t)$	$\beta^{A,B}(t)$	$\beta^{B,A}(t)$	ρ^A	ρ^B
a	$1+0.2t$	$1+0.2t$	$0.5+0.1t$	$0.5+0.1t$	0.5	0.5
b	$1+0.2t$	$1+0.2t$	$0.25+0.05t$	$0.25+0.05t$	0.25	0.25
c	$3-0.2t$	$3-0.2t$	$1.5-0.1t$	$1.5-0.1t$	0.5	0.5
d	$3-0.2t$	$3-0.2t$	$0.75-0.05t$	$0.75-0.05t$	0.25	0.25
e	$1+0.2t$	$0.6+0.2t$	$0.5+0.1t$	$0.3+0.1t$	0.5	0.5
f	$1+0.2t$	$0.6+0.2t$	$0.5+0.1t$	$0.15+0.05t$	0.5	0.25
g	$1+0.2t$	$0.6+0.2t$	$0.25+0.05t$	$0.15+0.05t$	0.25	0.25
h	$3-0.2t$	$2.2-0.2t$	$1.5-0.1t$	$1.1-0.1t$	0.5	0.5
i	$3-0.2t$	$2.2-0.2t$	$1.5-0.1t$	$0.55-0.05t$	0.5	0.25
j	$3-0.2t$	$2.2-0.2t$	$0.75-0.05t$	$0.55-0.05t$	0.25	0.25

G.1 Effect of capacity

To study the effect of the capacity level, we choose the sensitivities as given by scenario f and h (asymmetric, respectively increasing and decreasing with time. Supplier A has higher price sensitivities.) and the coefficients $\alpha^k = 15$, $I^{0^k} = 10$ also fixed as in the previous section. We

Scenario	Supplier A's obj.	Supplier B's obj.	Total obj.
a	514.83	514.83	1023.8
b	375.83	375.83	750.8
c	581.49	581.49	1157.1
d	484.80	484.80	848.7
e	578.86	611.73	1183.6
f	540.71	481.20	1018.2
g	401.45	465.57	866.1
h	1012.9	1171.2	2097.5
i	944.24	975.63	1866.5
j	586.77	934.24	1518.8

Table 1: Results: Profits under different demand sensitivity scenarios in the deterministic case

compute the equilibrium for a capacity limit taking values 6, 8, 10, 12, 14, 16 (identical for the two suppliers).

First, we notice that under scenario f, when the capacity equals 10, the production rate for supplier

	K	6	8	10	12	14	16
Scenario f	Supplier A's profit	540.86	540.22	540.71	540.71	540.71	540.71
	Supplier B's profit	505.56	492.58	481.20	481.20	481.20	481.20
	Total profits	1046.4	1032.8	1021.9	1021.9	1021.9	1021.9
Scenario h	Supplier A's profit	1031.8	1006.6	1012.9	1018.5	1018.5	1018.4
	Supplier B's profit	1270.1	1239.2	1171.2	1141.8	1135.4	1134.8
	Total profits	2283.9	2245.8	2184.1	2160.3	2153.9	2153.1

Table 2: Results: Effect of production capacity. Profits for various symmetric capacity levels

A never reaches 10. As a result, the optimal policy is identical for higher capacity levels.

In scenario f (sensitivities increasing with time), supplier A gets higher profits than supplier B, but the reverse is true in scenario h (sensitivities increasing with time).

Overall, we observe that when capacity increases the prices decrease at the equilibrium.

Interestingly, supplier B's profits tend to slightly decrease as the capacity increase. This may look surprising since a higher capacity gives more flexibility. This illustrates that the presence of competition may not yield an equilibrium that is unilaterally optimal for a given supplier.

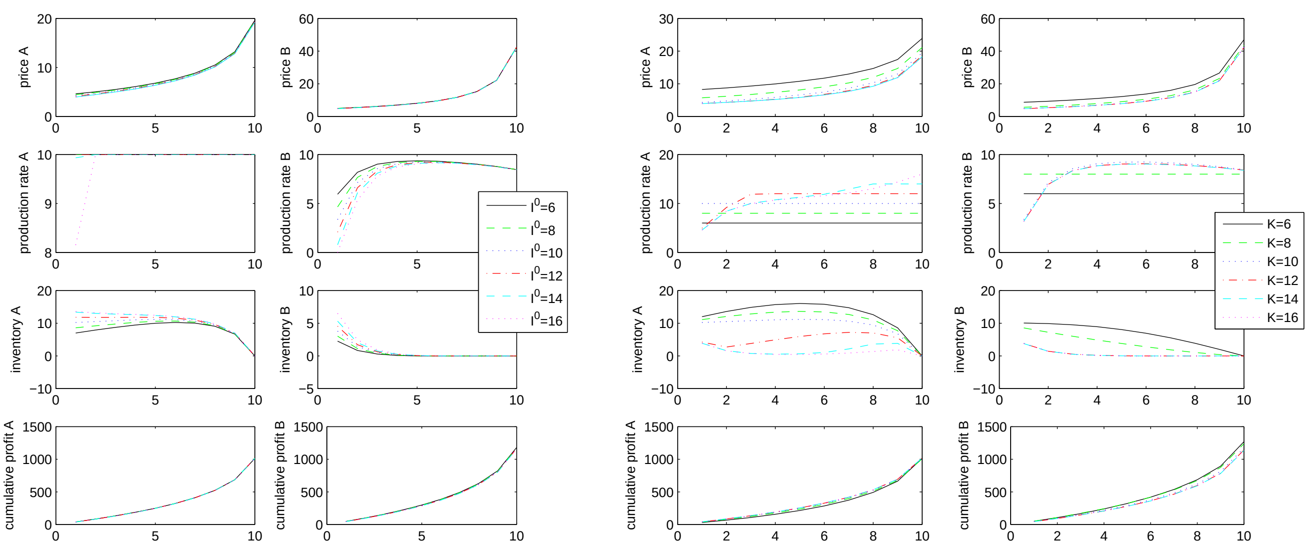
Notice also that in scenario h, when the capacity is high enough in order to enable to meet the no backorders constraint towards the end of the time horizon (when sensitivities are lower and prices can increase) without using all available capacity, at the equilibrium the inventory levels decrease from the beginning of the time horizon in order to decrease holding costs. However when the capacity is low, inventories are kept around the initial value until the sensitivities become lower, so that selling yields more significant profits.

G.2 Effect of initial inventory level

To study the effect of the initial inventory level, we choose the sensitivities as given by scenario f and h (asymmetric, increasing and decreasing with time) and the coefficients $\alpha^k = 15,\ K^k = 10$ also fixed. We compute the optimal solution for an initial inventory level identical for both suppliers, and taking values 6, 8, 10, 12, 14, 16 (identical for the two suppliers).

	I^0	6	8	10	12	14	16
Scenario f	Supplier A's profit	540.51	540.63	540.71	540.73	540.67	540.43
	Supplier B's profit	481.03	481.14	481.20	481.21	481.17	481.00
	Total profits	1021.5	1021.8	1021.9	1021.9	1021.8	1021.4
Scenario h	Supplier A's profit	1011.8	1012.7	1012.9	1012.5	1011.5	1011.5
	Supplier B's profit	1181.6	1176.4	1171.2	1166.0	1160.9	1160.2
	Total profits	2193.4	2189.1	2184.1	2178.5	2172.4	2171.8

Table 3: Results: Effect of initial inventory level. Profits for various symmetric initial inventory levels



(a) (b)

Figure 1: Results: (a) Effect of initial inventory level. Equilibrium in the case of scenario h for various initial inventory levels. (b) Effect of production capacity. Equilibrium in the case of scenario h for various symmetric capacity levels

We observe that in scenario f where the capacity constraint is not tight, when the initial inventory level changes, the production rates changes in such a way that the effects cancel out (i.e. the production rate increases by as much as the initial inventory level decreased), and the prices and cumulative profits do not vary. Therefore, the initial inventory levels seem to have a low impact. The same is true for supplier B in scenario h for the same reason. However, for supplier A in case h, the capacity constraint is tight, the only way to compensate for the increase in initial inventory level is by decreasing prices. The profits slightly increase for supplier B, and slightly decrease for supplier A. As a result, suppliers with high price sensitivities are slightly advantaged by having low initial inventories, if the capacity is a binding constraint.

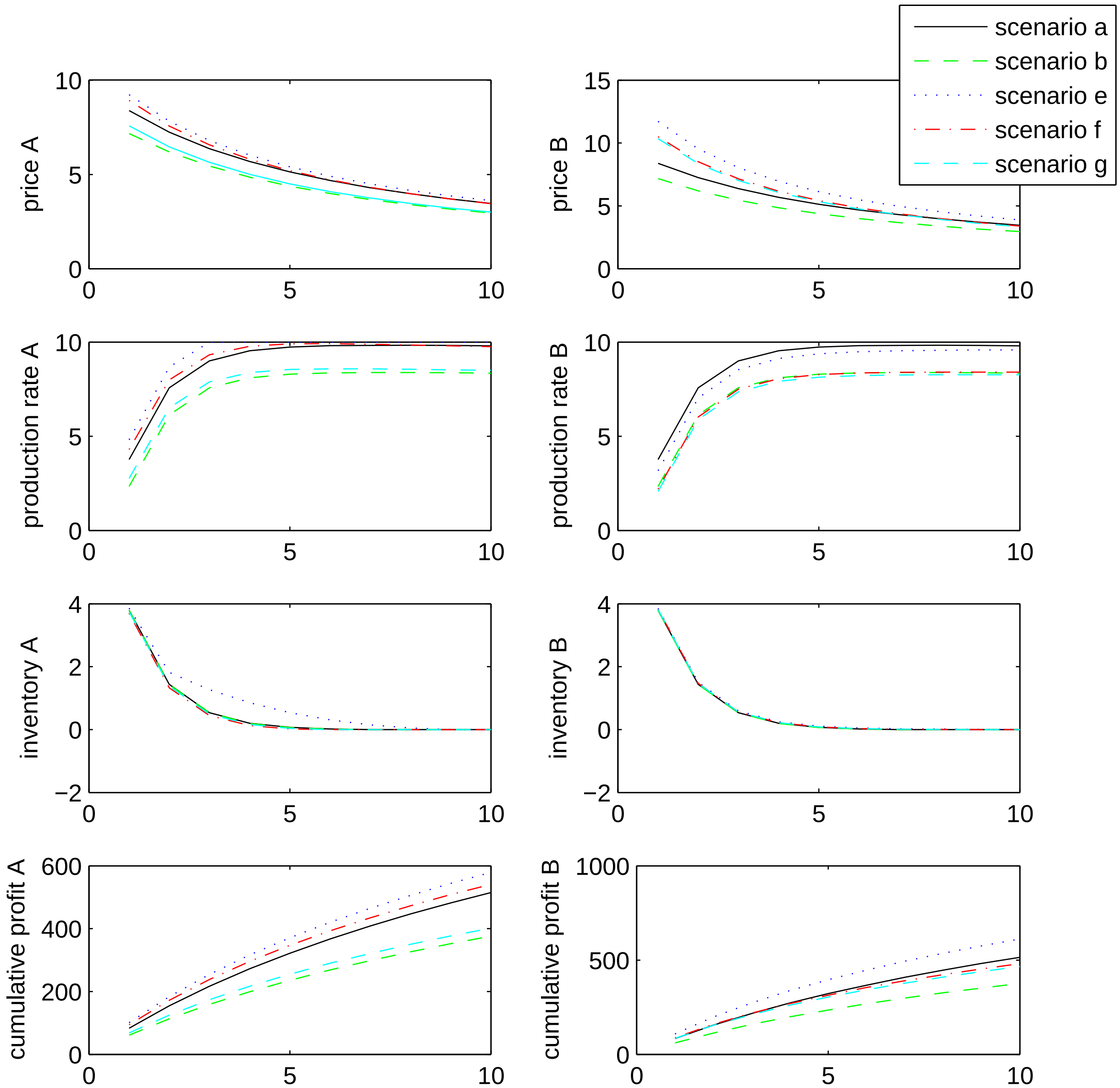


Figure 2: Results: Effect of price sensitivities. Equilibrium in the case of price sensitivities increasing with time in the deterministic problem

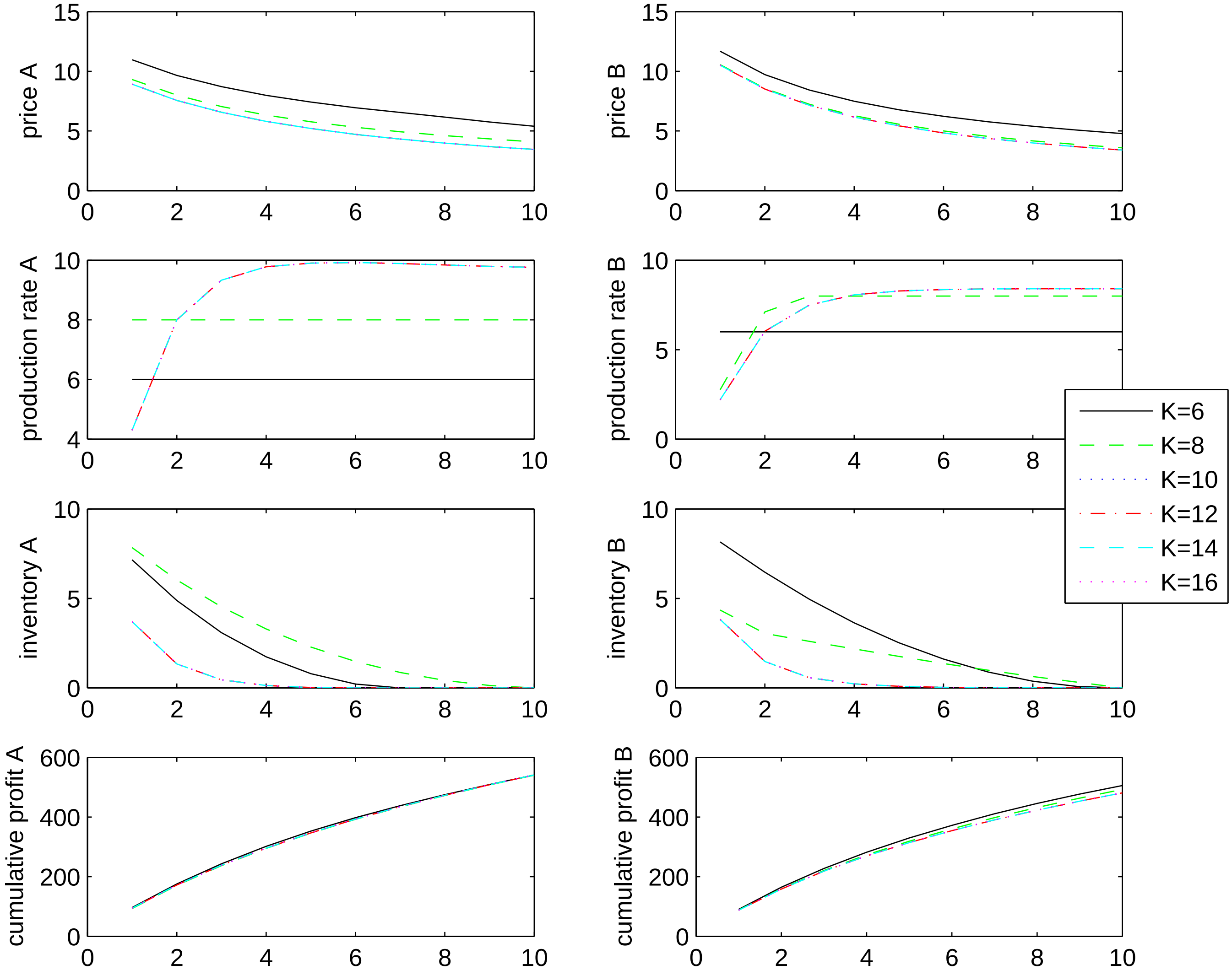


Figure 3: Results: Effect of production capacity. Equilibrium in the case of scenario f for various symmetric capacity levels in the deterministic problem

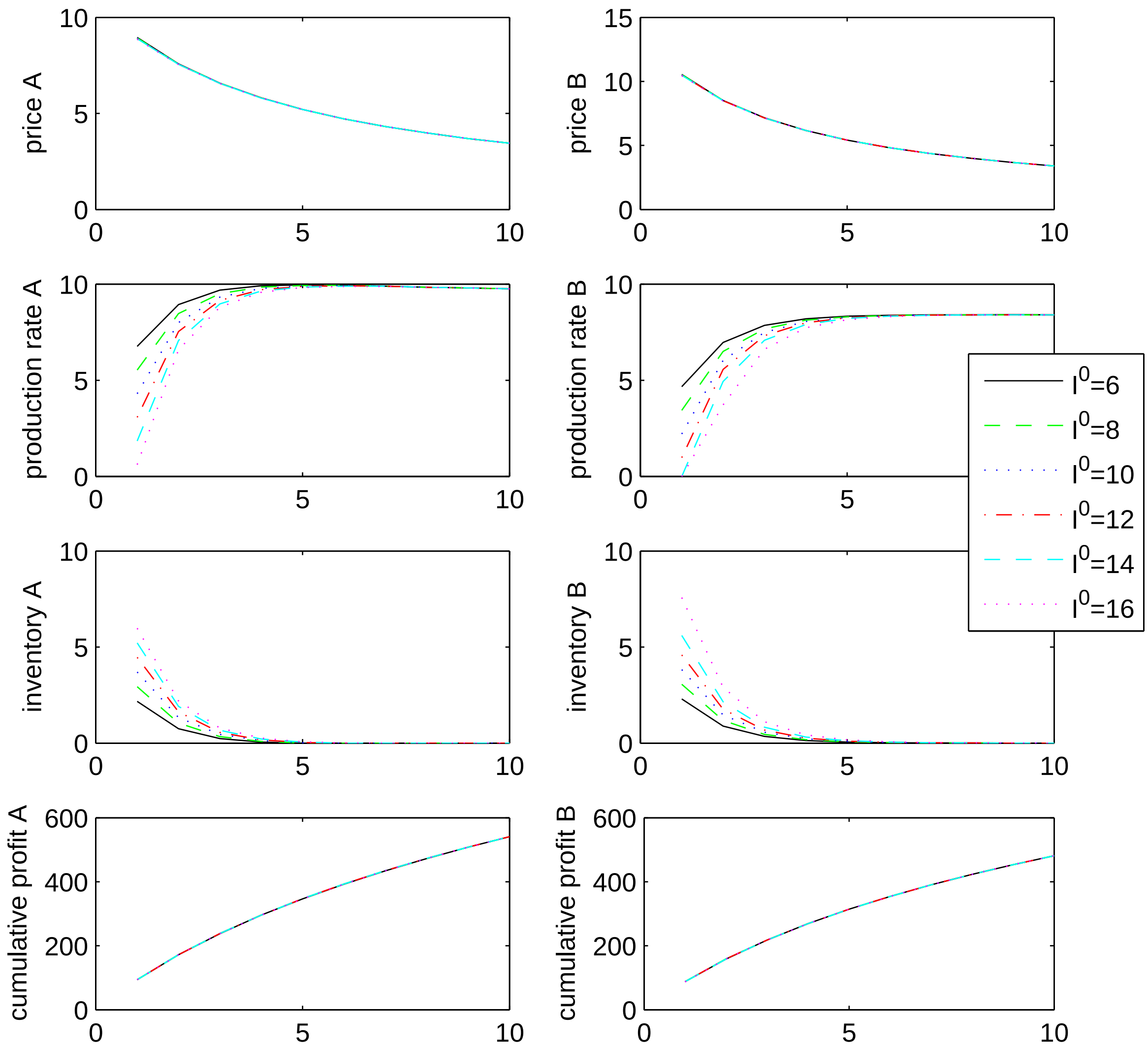


Figure 4: Results: Effect of initial inventory level. Equilibrium in the case of scenario f for various initial inventory levels

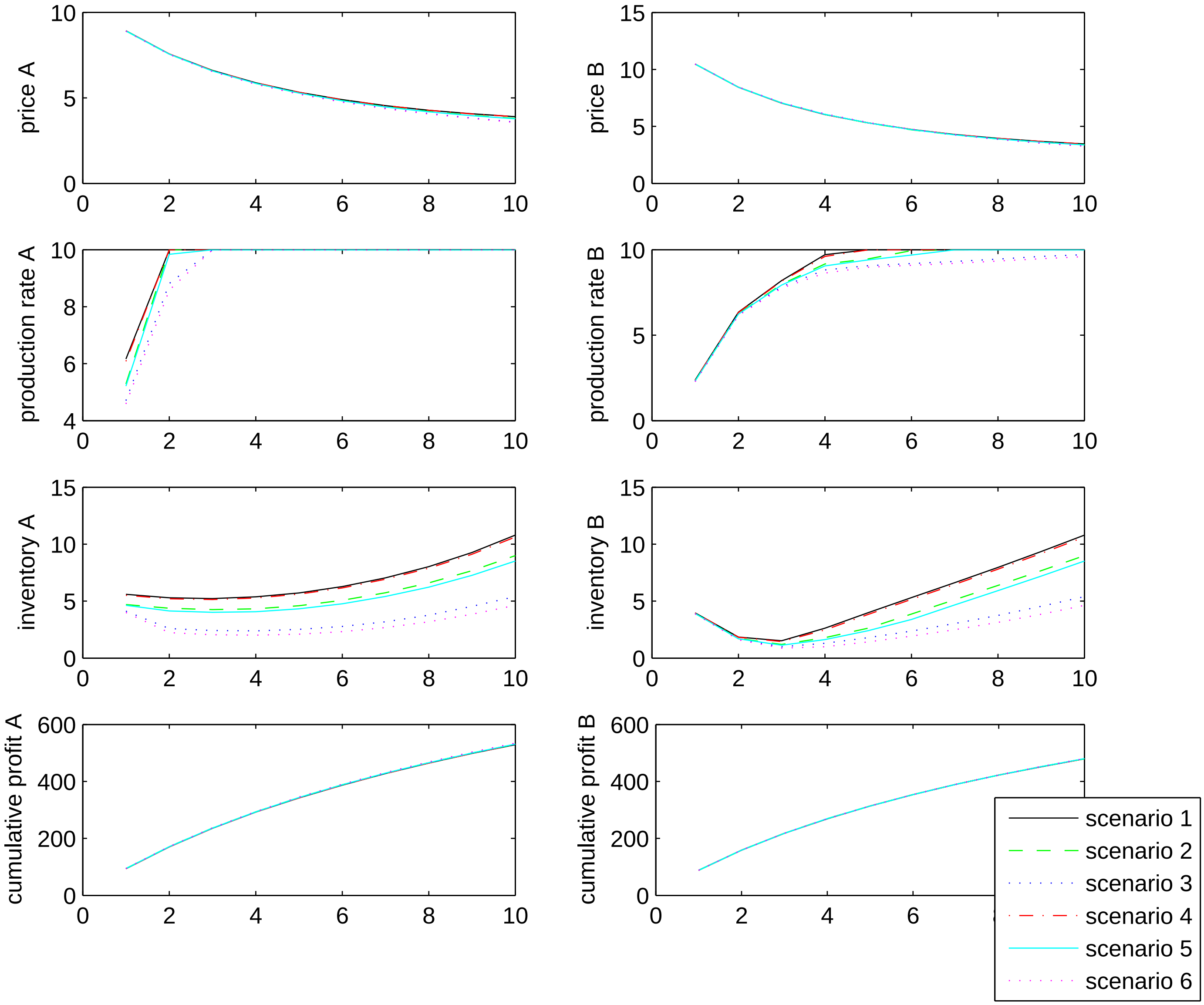


Figure 5: Results: robust formulation. Equilibrium in the case of price sensitivities scenario f for various scenarios of budget of uncertainty

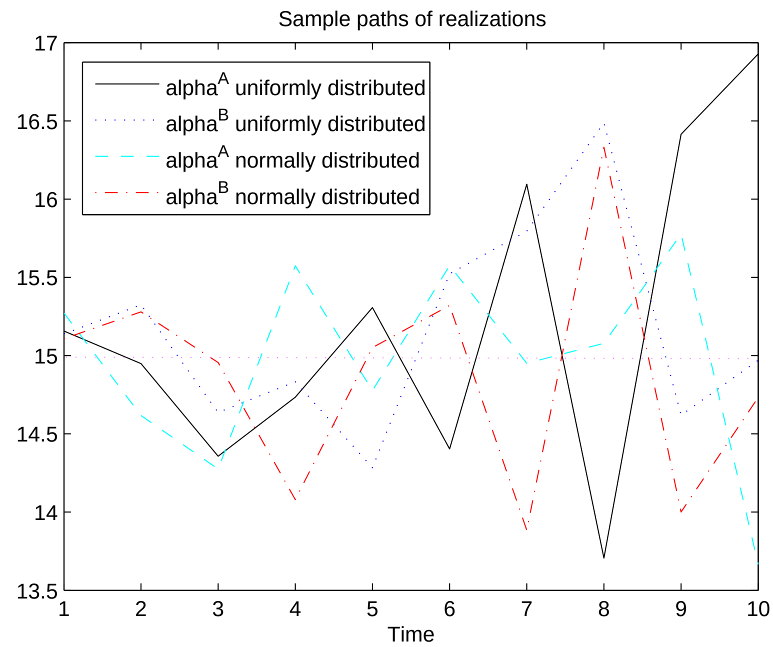


Figure 6: Example of realizations for alpha under uniform and normal distributions

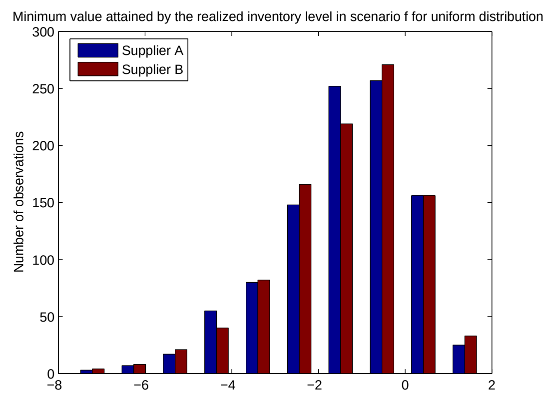


Figure 7: Histogram of minimum inventory level reached in the robust formulation for uniformly distributed realization in scenario f

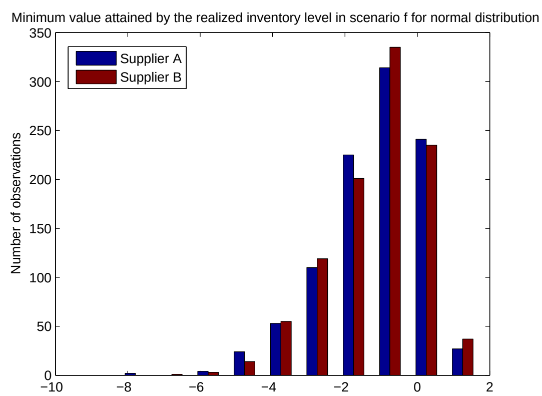


Figure 8: Histogram of minimum inventory level reached in the robust formulation for normally distributed realization in scenario f

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