Operations Research
http://dx.doi.org/10.1287/opre.1110.1017ec

e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Generalized Quantity Competition for Multiple Products and Loss of Efficiency" by Jonathan Kluberg and Georgia Perakis, *Operations Research*, http://dx.doi.org/10.1287/opre.1110.1017.

Appendix

A Proof of Lemma 1

Lemma. *In a market with differentiated substitute products, a single product per firm and separate capacity constraints for each product, colluding firms always sell less quantity of each product than if they compete freely:* $\mathbf{d}^{MP} \leq \mathbf{d}^{OP}$.

Proof. To prove this lemma we first formulate the oligopoly problem (OP) under capacity constraints. It can be written as:

$$\begin{aligned} \max_{d_i} \quad & d_i \cdot \left\{ \bar{p}_i - \left(\ \mathbf{B}_i \ \right) \cdot \begin{pmatrix} d_i \\ \mathbf{d}_{-i}^{OP} \end{pmatrix} \right\} \\ \text{s.t.} \quad & 0 \leq d_i \leq C_i \leq \bar{d}_i \end{aligned}$$

where $\mathbf{B}_i$ denotes the row of matrix $\mathbf{B}$ corresponding to firm i.

Using notation $\mathbf{\Gamma} = \text{diag}(\mathbf{B})$, the corresponding (OP) KKT conditions are:

$$\bar{\mathbf{p}} - \mathbf{B}\mathbf{d}^{OP} - \mathbf{\Gamma}\mathbf{d}^{OP} - \boldsymbol{\lambda}^{OP} + \boldsymbol{\mu}^{OP} = 0 \qquad \begin{cases} \lambda_i^{OP}(C_i - d_i^{OP}) = 0 \\ \lambda_i^{OP} \geq 0 \\ d_i^{OP} \leq C_i \leq \bar{d}_i \end{cases} \qquad \begin{cases} \mu_i^{OP} d_i^{OP} = 0 \\ \mu_i^{OP} \geq 0 \\ d_i^{OP} \geq 0 \end{cases}$$

Similarly, we write down the monopoly problem (MP) under capacity constraints.

$$\begin{aligned} \max_{\mathbf{d}} \quad & \mathbf{d} \cdot \{\bar{\mathbf{p}} - \mathbf{B} \cdot \mathbf{d}\} \\ \text{s.t.} \quad & 0 \leq \mathbf{d} \leq \mathbf{C} \leq \bar{\mathbf{d}} \end{aligned}$$

The corresponding (MP) KKT conditions are:

$$\bar{\mathbf{p}} - 2\mathbf{B}\mathbf{d}^{MP} - \boldsymbol{\lambda}^{MP} + \boldsymbol{\mu}^{MP} = 0 \qquad \begin{cases} (\boldsymbol{\lambda}^{MP})^T(\mathbf{C} - \mathbf{d}^{MP}) = 0 \\ \boldsymbol{\lambda}^{MP} \geq 0 \\ \mathbf{d}^{MP} \leq \mathbf{C} \leq \bar{\mathbf{d}} \end{cases} \qquad \begin{cases} (\boldsymbol{\mu}^{MP})^T \mathbf{d}^{MP} = 0 \\ \boldsymbol{\mu}^{MP} \geq 0 \\ \mathbf{d}^{MP} \geq 0 \end{cases}$$

Step 1: We will prove that $\boldsymbol{\mu}^{OP} = 0$

Let us consider the problem that ignores the constraint $\mathbf{d}^{OP} \geq 0$. This suggests we ignore $\boldsymbol{\mu}^{OP}$ and the KKT conditions of problem (OP) become:

$$\bar{\mathbf{p}} - (\mathbf{B} + \mathbf{\Gamma})\mathbf{d}^{OP} - \boldsymbol{\lambda}^{OP} = 0 \qquad \text{or} \qquad \mathbf{d}^{OP} = (\mathbf{B} + \mathbf{\Gamma})^{-1}(\bar{\mathbf{p}} - \boldsymbol{\lambda}^{OP})$$

with $\mathbf{B}+\mathbf{\Gamma}$ being an inverse M-Matrix (see [1]).

There are two cases to distinguish.

- Either $\lambda_j^{OP} > 0$, in which case: $d_j^{OP} = C_j > 0$
- Or $\lambda_j^{OP} = 0$,

$$\begin{aligned} d_j^{OP} &= (\mathbf{B}+\mathbf{\Gamma})_j^{-1}(\bar{\mathbf{p}} - \boldsymbol{\lambda}^{OP}) \\ &= (- \cdots - \underbrace{+}_{jj} - \cdots -) \begin{pmatrix} \bar{p}_1 - \lambda_1^{OP} \\ \bar{p}_j \\ \bar{p}_n - \lambda_n^{OP} \end{pmatrix} \geq (- \cdots - + - \cdots -)\ \bar{\mathbf{p}} \\ d_j^{OP} &\geq (\mathbf{B}+\mathbf{\Gamma})_j^{-1}\ \mathbf{B}\ \bar{\mathbf{d}} \quad = (I+\mathbf{M\Gamma})_j^{-1}\ \bar{\mathbf{d}} \quad > 0 \end{aligned}$$

Since $\mathbf{M}$ is an M-matrix, so is $I+\mathbf{M\Gamma}$ (see [1]). Hence $(I+\mathbf{M\Gamma})^{-1}$ has non-negative elements, and the last inequality follows from $\bar{\mathbf{d}} > 0$.

Hence, it is always the case that $\mathbf{d}^{OP} \geq 0$ even without including this constraint (i.e. the constraint that $\mathbf{d}^{OP} \geq 0$). As a result, $\boldsymbol{\mu}^{OP} = 0$.

Step 2: Similarly, we now show that $\boldsymbol{\mu}^{MP} = 0$

Following a similar thought process as before, we first consider the problem that ignores $\boldsymbol{\mu}^{MP}$ (that is, ignores the constraint $\mathbf{d}^{MP} \geq 0$). Then the KKT conditions of problem (MP) become:

$$\bar{\mathbf{p}} - 2\mathbf{B}\ \mathbf{d}^{MP} - \boldsymbol{\lambda}^{MP} = 0 \qquad \text{or} \qquad \mathbf{d}^{MP} = 1/2\ \mathbf{M}(\bar{\mathbf{p}} - \boldsymbol{\lambda}^{MP})$$

- Either $\lambda_j^{MP} > 0$, in which case: $d_j^{MP} = C_j > 0$
- Or $\lambda_j^{MP} = 0$,

$$\begin{aligned} d_j^{MP} &= 1/2\ \mathbf{M}_j\ (\bar{\mathbf{p}} - \lambda^{MP}) \\ &= (- \cdots - \underbrace{+}_{jj} - \cdots -) \begin{pmatrix} \bar{p}_1 - \lambda_1^{MP} \\ \bar{p}_j \\ \bar{p}_n - \lambda_n^{MP} \end{pmatrix} \geq 1/2\ \mathbf{M}_j\ \bar{\mathbf{p}} \\ d_j^{MP} &\geq 1/2\ \bar{d}_j \quad > 0 \end{aligned} \tag{1}$$

Step 3: *Characterization of* $\mathbf{d}^{OP}$

Let $K_1 = \{$Set of active constraints for the oligopoly problem$\} = \{i = 1, \ldots, n, \quad \lambda_i^{OP} > 0\}$. We denote by K_1^c the complement set of K_1 and by $\mathbf{H}_{AB}$ and $\mathbf{u}_A$ the restrictions of matrix $\mathbf{H}$ and vector $\mathbf{u}$ to rows indexed by A and columns indexed by B. Since K_1 is the set of active capacity constraints for problem (OP), $\mathbf{d}^{OP} = \begin{pmatrix} d^{OP}_{K_1} \\ d^{OP}_{K_1^c} \end{pmatrix} = \begin{pmatrix} c_{K_1} \\ d^{OP}_{K_1^c} \end{pmatrix}$.

Since $\boldsymbol{\mu}^{OP} = 0$, the oligopoly KKT conditions become:

$$\bar{\mathbf{p}} - (\mathbf{B} + \boldsymbol{\Gamma})\mathbf{d}^{OP} - \boldsymbol{\lambda}^{OP} = 0$$

Restricting attention to the set K_1^c of inactive constraints ($\lambda^{OP}_{K_1^c} = 0$) and noting that $\boldsymbol{\Gamma}$ disappears in off-diagonal block matrices:

$$\bar{\mathbf{p}}_{K_1^c} - \ \mathbf{B}_{K_1^c K_1}\ c_{K_1} - (\mathbf{B} + \boldsymbol{\Gamma})_{K_1^c K_1^c}\ d^{OP}_{K_1^c} = 0$$

Using the relation $\bar{\mathbf{p}}_{K_1^c} = \mathbf{B}_{K_1^c}\ \bar{\mathbf{d}}$, we get:

$$\begin{aligned} & (\mathbf{B} + \boldsymbol{\Gamma})_{K_1^c K_1^c}\ d^{OP}_{K_1^c} = \mathbf{B}_{K_1^c K_1}\ \bar{\mathbf{d}}_{K_1} + \mathbf{B}_{K_1^c K_1^c}\ \bar{\mathbf{d}}_{K_1^c} - \ \mathbf{B}_{K_1^c K_1}\ c_{K_1} \\ \Rightarrow \quad & (\mathbf{B} + \boldsymbol{\Gamma})_{K_1^c K_1^c}\ d^{OP}_{K_1^c} = \mathbf{B}_{K_1^c K_1}(\bar{\mathbf{d}}_{K_1} - c_{K_1}) + \mathbf{B}_{K_1^c K_1^c}\ \bar{\mathbf{d}}_{K_1^c} \end{aligned} \tag{2}$$

Clearly, on K_1 we have: $d^{OP}_{K_1} = c_{K_1} \geq d^{MP}_{K_1}$. Hence, to prove the lemma above, we only need to show: $d^{OP}_{K_1^c} \geq d^{MP}_{K_1^c}$.

<u>Step 4</u>: *Characterization of* $\mathbf{d}^{MP}$
Let $K_2 = \{\text{Set of active constraints for the monopoly problem}\} = \{i = 1, \ldots, n, \quad \lambda_i^{MP} > 0\}$. We denote by K_2^c the complement set of K_2. Since K_2 is the set of active capacity constraints for problem (MP), $\mathbf{d}^{MP} = \begin{pmatrix} c_{K_2} \\ d^{MP}_{K_2^c} \end{pmatrix}$.

Since $\boldsymbol{\mu}^{MP} = 0$, the monopoly KKT conditions become:

$$\bar{\mathbf{p}} - 2\ \mathbf{B}\ \mathbf{d}^{MP} - \boldsymbol{\lambda}^{MP} = 0$$

Restricting attention to the set K_2^c of inactive constraints ($\lambda^{MP}_{K_2^c} = 0$):

$$\bar{\mathbf{p}}_{K_2^c} - \ 2\ \mathbf{B}_{K_2^c}\ d^{MP} = 0 \tag{3}$$

Without loss of generality, we now assume $K_2 \subseteq K_1$ (and hence $K_2^c \supseteq K_1^c$). If there were constraints in $K_2 \setminus K_1$, we simply remove them. We show that without these constraints $d^{MP}_{K_1^c} \leq d^{OP}_{K_1^c}$ which proves that capacity constraints cannot be active on $d^{MP}_{K_1^c}$ as they are not active on $d^{OP}_{K_1^c}$.

Restricting further (3) to K_1^c ($\subseteq K_2^c$) and splitting variables according to $K_1 \mid K_1^c$, we get:

$$\bar{\mathbf{p}}_{K_1^c} - \ 2\ \mathbf{B}_{K_1^c K_1} \begin{pmatrix} c_{K_2} \\ d^{MP}_{K_1 \setminus K_2} \end{pmatrix} - 2\ \mathbf{B}_{K_1^c K_1^c}\ d^{MP}_{K_1^c} = 0$$

Using the relation $\bar{\mathbf{p}}_{K_1^c} = \mathbf{B}_{K_1^c}\ \bar{\mathbf{d}}$, we get:

$$\begin{aligned} & 2\ \mathbf{B}_{K_1^c K_1^c}\ d^{MP}_{K_1^c} = \mathbf{B}_{K_1^c K_1}\ \bar{\mathbf{d}}_{K_1} + \mathbf{B}_{K_1^c K_1^c}\ \bar{\mathbf{d}}_{K_1^c} - \ 2\ \mathbf{B}_{K_1^c K_1} \begin{pmatrix} c_{K_2} \\ d^{MP}_{K_1 \setminus K_2} \end{pmatrix} \\ \Rightarrow \quad & 2\ \mathbf{B}_{K_1^c K_1^c}\ d^{MP}_{K_1^c} = \mathbf{B}_{K_1^c K_1} \left(\bar{\mathbf{d}}_{K_1} - \begin{matrix} 2\ c_{K_2} \\ 2\ d^{MP}_{K_1 \setminus K_2} \end{matrix} \right) + \mathbf{B}_{K_1^c K_1^c}\ \bar{\mathbf{d}}_{K_1^c} \qquad (4) \end{aligned}$$

Step 5: $\mathbf{d}^{OP} \geq \mathbf{d}^{MP}$

As shown in (1), for all $j \in K_2^c$, $d_j^{MP} \geq 1/2\ \bar{d}_j$. In particurlar:

$$2\ d^{MP}_{K_1 \setminus K_2} \ \geq \ \bar{\mathbf{d}}_{K_1 \setminus K_2} \ \geq \ c_{K_1 \setminus K_2} \qquad (5)$$

$$2\ d^{MP}_{K_1^c} \ \geq \ \bar{\mathbf{d}}_{K_1^c} \qquad (6)$$

On the other hand, combining (2) and (4), we have:

$$\begin{aligned} (\mathbf{B}+\mathbf{\Gamma})_{K_1^c K_1^c}\ d^{OP}_{K_1^c} - \mathbf{B}_{K_1^c K_1}(\bar{\mathbf{d}}_{K_1} - c_{K_1}) \ &= \ 2\ \mathbf{B}_{K_1^c K_1^c}\ d^{MP}_{K_1^c} - \mathbf{B}_{K_1^c K_1} \left(\bar{\mathbf{d}}_{K_1} - \begin{matrix} 2\ c_{K_2} \\ 2\ d^{MP}_{K_1 \setminus K_2} \end{matrix} \right) \\ \Rightarrow (\mathbf{B}+\mathbf{\Gamma})_{K_1^c K_1^c}\ d^{OP}_{K_1^c} \ &= \ 2\ \mathbf{B}_{K_1^c K_1^c}\ d^{MP}_{K_1^c} + \mathbf{B}_{K_1^c K_1} \underbrace{\begin{pmatrix} 2\ c_{K_2} & - & c_{K_2} \\ 2\ d^{MP}_{K_1 \setminus K_2} & & c_{K_1 \setminus K_2} \end{pmatrix}}_{\geq 0 \quad \text{using (5)}} \\ \Rightarrow (\mathbf{B}+\mathbf{\Gamma})_{K_1^c K_1^c}\ d^{OP}_{K_1^c} \ &\geq \ 2\ \mathbf{B}_{K_1^c K_1^c}\ d^{MP}_{K_1^c} \qquad (7) \end{aligned}$$

Finally, let's assume there exist $i \in K_1^c$ such that $d_i^{OP} < d_i^{MP}$. Denoting $\{s_1, \cdots, s_f\}$ the indices of K_1^c, let's expand the i-th row of (7):

$$(b_{is_1} \cdots 0 \cdots b_{is_f}) \ \underbrace{d^{OP}_{K_1^c}}_{\leq\ \bar{\mathbf{d}}_{K_1^c}} + 2\ b_{ii} \ \underbrace{d_i^{OP}}_{<\ d_i^{MP}} \ \geq \ (\ b_{is_1} \cdots 0 \cdots b_{is_f}) \ \underbrace{2\ d^{MP}_{K_1^c}}_{\substack{\geq\ \bar{\mathbf{d}}_{K_1^c} \\ \text{using (6)}}} + 2\ b_{ii}\ d_i^{MP}$$

Since all the coefficients b_{ij} are non-negative, this is a contradiction.

We just showed that $d^{MP}_{K_1^c} \leq d^{OP}_{K_1^c}$, leading to $d^{MP} \leq d^{OP}$.

□

B Proof of Step 1 for Theorem 3

Ignoring $\boldsymbol{\mu}^{SP}$, the KKT conditions of problem (SP) become:

$$\bar{\mathbf{p}} - \mathbf{B}\ \mathbf{d}^{SP} - \boldsymbol{\lambda}^{SP} = 0 \qquad \text{or} \qquad \mathbf{d}^{SP} = \mathbf{M}\ (\bar{\mathbf{p}} - \boldsymbol{\lambda}^{SP})$$

- Either $\lambda_j^{SP} > 0$, in which case: $d_j^{SP} = C_j > 0$
- Or $\lambda_j^{SP} = 0$,

$$\begin{aligned} d_j^{SP} &= \mathbf{M}_j\ (\bar{\mathbf{p}} - \boldsymbol{\lambda}^{SP}) \\ &= (- \cdots - \underbrace{+}_{jj} - \cdots -) \begin{pmatrix} \bar{p}_1 - \lambda_1^{SP} \\ \bar{p}_j \\ \bar{p}_n - \lambda_n^{SP} \end{pmatrix} \geq \mathbf{M}_j\ \bar{\mathbf{p}} \\ d_j^{SP} &\geq \bar{d}_j \quad > 0 \end{aligned}$$

C Calculations for Theorem 4

In the uniform case, matrix $\mathbf{M}$ can be written as:

$$\mathbf{M} = \begin{pmatrix} 1 & -\alpha & \dots & -\alpha \\ -\alpha & \ddots & & \vdots \\ \vdots & & \ddots & -\alpha \\ -\alpha & \dots & -\alpha & 1 \end{pmatrix} = (1+\alpha)I - \alpha H$$

$$= \Delta \begin{pmatrix} 1+\alpha-n\alpha & 0 & \dots & 0 \\ 0 & 1+\alpha & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1+\alpha \end{pmatrix} \Delta^T$$

Inverting $\mathbf{M}$, we get matrix $\mathbf{B}$:

$$\begin{aligned} \mathbf{B} &= \frac{1}{1+\alpha}(I - \frac{\alpha}{1+\alpha}H)^{-1} \\ &= \frac{1}{1+\alpha}\left[I + \frac{\alpha}{1+\alpha}(1 + \frac{\alpha}{1+\alpha}n + \cdots)H\right] \\ &= \frac{1}{1+\alpha}\left[I + \frac{\alpha}{1+\alpha-n\alpha}H\right] \end{aligned}$$

This allows us to compute:

$$\boldsymbol{\Gamma} = \text{diag}(\mathbf{B}) = \frac{1+2\ \alpha - n\alpha}{(1+\alpha)(1+\alpha-n\alpha)}I$$

On the other hand, diagonalizing $\mathbf{B}$ as we did with $\mathbf{M}$:

$$\mathbf{B} = \Delta \begin{pmatrix} \frac{1}{1+\alpha-n\alpha} & 0 & \dots & 0 \\ 0 & \frac{1}{1+\alpha} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{1+\alpha} \end{pmatrix} \Delta^T$$

We are now able to compute the diverse component of the surplus ratio.

$$I + \mathbf{M\Gamma} = \Delta \begin{pmatrix} \frac{2+3\alpha-n\alpha}{1+\alpha} & 0 & \dots & 0 \\ 0 & \frac{2+3\alpha-2n\alpha}{1+\alpha-n\alpha} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \frac{2+3\alpha-2n\alpha}{1+\alpha-n\alpha} \end{pmatrix} \Delta^T$$

$$(I + \mathbf{M\Gamma})^{-1} = \Delta \begin{pmatrix} \frac{1+\alpha}{2+3\alpha-n\alpha} & 0 & \dots & 0 \\ 0 & \frac{1+\alpha-n\alpha}{2+3\alpha-2n\alpha} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \frac{1+\alpha-n\alpha}{2+3\alpha-2n\alpha} \end{pmatrix} \Delta^T$$

Let's call $\breve{\mathbf{d}}$ the vector whose components are the eigenvectors of $\mathbf{M}$, and $[\breve{\rho}_1, \breve{\rho}_2]$ the two eigenvalues of: $(I + \mathbf{\Gamma M})^{-1}\ \mathbf{\Gamma}\ (I + \mathbf{M\Gamma})^{-1}$.

- $\breve{\rho}_1 = \frac{(1+\alpha)(1+2\alpha-n\alpha)}{(2+3\alpha-n\alpha)^2(1+\alpha-n\alpha)}$
- $\breve{\rho}_2 = \frac{(1+\alpha-n\alpha)(1+2\alpha-n\alpha)}{(2+3\alpha-2n\alpha)^2(1+\alpha)}$

The ratio of profits becomes:

$$\frac{\Pi(OP)}{\Pi(MP)} = \frac{4\ (\breve{\rho}_1 \breve{d}_1^2 + \breve{\rho}_2 \sum_{i=2}^{n} \breve{d}_i^2)}{\frac{1}{1+\alpha-n\alpha}\breve{d}_1^2 + \frac{1}{1+\alpha}\sum_{i=2}^{n} \breve{d}_i^2}$$

D Proof of Lemma 1

Lemma. *For a symmetric inverse M-matrix* $\mathbf{B}$ *and a vector* $\mathbf{d}$ *with all component positive, the following inequality holds:*

$$\|\mathbf{d}\|_{\mathbf{B}}^2 \leq (1 + r \cdot (nm - 1))\ \|\mathbf{d}\|_{\mathbf{B^{Bdiag}}}^2$$

where r *is the market power.*

Proof. Since $\mathbf{B}$ is an inverse M-matrix, Ostrowski shows in [3] that:

$$B_{ij}^{kl} \leq r_{kl} B_{ij}^{ij} \qquad \text{and} \qquad B_{ij}^{kl} = B_{kl}^{ij} \leq r_{ij} B_{kl}^{kl}$$

Introducing $r = \max_{kl}\ r_{kl}$, we have: $B_{ij}^{kl} \leq r\sqrt{B_{ij}^{ij} B_{kl}^{kl}}$.

Hence, we can write:

$$\begin{aligned}
\|\mathbf{d}\|_{\mathbf{B}}^2 &\leq \mathbf{d}^T \begin{pmatrix} B_{11}^{11} & \dots & r\sqrt{B_{ij}^{ij} B_{kl}^{kl}} \\ \vdots & \ddots & \vdots \\ r\sqrt{B_{ij}^{ij} B_{kl}^{kl}} & \dots & B_{nm}^{nm} \end{pmatrix} \mathbf{d} \\
&= \mathbf{d}^T \begin{pmatrix} rB_{11}^{11} & \dots & r\sqrt{B_{ij}^{ij} B_{kl}^{kl}} \\ \vdots & \ddots & \vdots \\ r\sqrt{B_{ij}^{ij} B_{kl}^{kl}} & \dots & rB_{nm}^{nm} \end{pmatrix} \mathbf{d} + \mathbf{d}^T \begin{pmatrix} (1-r)B_{11}^{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (1-r)B_{nm}^{nm} \end{pmatrix} \mathbf{d}
\end{aligned}$$

We denote the diagonal matrix corresponding to the diagonal of matrix $\mathbf{B}$ by:

$$\mathbf{\Gamma} = \text{diag}(B_{11}^{11}, \cdots, B_{nm}^{nm})$$

We obtain:

$$\|\mathbf{d}\|_{\mathbf{B}}^2 \leq r\ \mathbf{d}^T \sqrt{\mathbf{\Gamma}} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \sqrt{\mathbf{\Gamma}}\ \mathbf{d} + (1-r)\ \mathbf{d}^T \mathbf{\Gamma} \mathbf{d}$$

Since $\mathbf{H} = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$ has two eigenvalues 0 and nm, we have $\mathbf{d}^T \mathbf{H} \mathbf{d} \leq nm\ \|\mathbf{d}\|^2$ for all $\mathbf{d}$.

$$\begin{aligned}
\|\mathbf{d}\|_{\mathbf{B}}^2 &\leq r \cdot nm\ \mathbf{d}^T \mathbf{\Gamma} \mathbf{d} + (1-r)\ \mathbf{d}^T \mathbf{\Gamma} \mathbf{d} \\
&\leq (1 + r \cdot (nm-1))\ \|\mathbf{d}\|_{\mathbf{B^{Bdiag}}}^2
\end{aligned}$$

□

E Derivation of oligopoly variational inequality

At a Nash equilibrium solution, the optimization problem facing a single firm is:

$$\max_{\mathbf{d}_i} \quad \mathbf{d}_i \cdot \left\{ \bar{\mathbf{p}}_i - \begin{pmatrix} B_{i1} \\ \vdots \\ B_{im} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{d}_i \\ \mathbf{d}_{-i}^{OP} \end{pmatrix} \right\} \tag{8}$$

$$\text{s.t.} \qquad \mathbf{d}_i \in K_i$$

This problem is a maximization of a concave objective function over a convex set, it is a convex problem. A general convex problem of the form:

$$\begin{aligned} \max_{x} \quad & F(x) \\ \text{s.t.} \quad & x \in K \end{aligned}$$

with a concave objective $F(x)$ is equivalent (see [2], [4]) to the variational inequality problem:

$$\text{Find } x_0 \in K: \quad -\nabla F(x_0)\ (x - x_0) \geq 0 \qquad \forall x \in K$$

Applying this to (8), we obtain for each firm i:

$$\text{Find } \mathbf{d}_i^{OP} \in K_i: \quad \left\{-\bar{\mathbf{p}}_i + \mathbf{B}_i \cdot \mathbf{d}^{OP} + \mathbf{B}_i^i \cdot \mathbf{d}_i^{OP}\right\}^T (\mathbf{d}_i - \mathbf{d}_i^{OP}) \geq 0 \qquad \forall \mathbf{d}_i \in K_i$$

where $\mathbf{B}_i$ denotes the rows of matrix $\mathbf{B}$ corresponding to firm i.

Now, since the constraint set of each firm i is independent of the quantities chosen by other firms, it is equivalent to satisfy every one of these variational inequalitites (for firm i) or to satisfy the sum of these inequalities. Clearly, if $\mathbf{d}^{OP}$ satisfies all these inequalities it satisfies the sum of the inequalities. On the other hand if $\mathbf{d}^{OP}$ satisfies the sum of the inequalities, by choosing $\mathbf{d} = (\mathbf{d}_i, \mathbf{d}_{-i}^{OP})$ for all $\mathbf{d}_i \in K_i$, it is easy to check that it will satisfy every variational inequality separately as well. The sum of these inequalities is exactly the variational inequality used in this paper:

$$\text{Find } \mathbf{d}^{OP} \in K: \quad \left\{-\bar{\mathbf{p}} + \mathbf{B} \cdot \mathbf{d}^{OP} + \mathbf{B}^{\mathbf{Bdiag}} \cdot \mathbf{d}^{OP}\right\}^T (\mathbf{d} - \mathbf{d}^{OP}) \geq 0 \qquad \forall \mathbf{d} \in K$$

References

[1] C. R. Johnson, "Inverse M-Matrices", *Linear Algebra and its Applications*, vol. 47, 195-216, 1982.

[2] O. Mancino and G. Stampacchia, "Convex programming and variational inequalities", *Journal of Optimization Theory and Application*, vol. 9, 3-23, 1972.

[3] A. M. Ostrowski, "Note on Bounds for Determinants with Dominant Principal Diagonal", *Proceedings of the American Mathematical Society*, vol. 3, No. 1, pp. 26-30, 1952.

[4] R. M. Rockafellar, "Convex Functions, Monotone Operators, and Variational Inequalities", *Theory and Applications of Monotone Operators, Proceedings of the NATO Advanced Study Institute, Venice, Italy*, 1968 (Edizioni Oderisi, Gubbio, Italy, 1968).