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working paper

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THE TRAVELLING SALESMAN PROBLEM AND RELATED PROBLEMS

by

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OR 078-78 JULY 1978

Research supported, in part, by the Office of Naval Research under Contract N00014-75-C-0556.

ABSTRACT

New formulations are presented for the Travelling Salesman problem, and their relationship to previous formulations is investigated. The new formulations are extended to include a variety of transportation scheduling problems, such as the Multi-Travelling Salesman problem, the Delivery problem, the School Bus problem and the Dial-a-Bus problem.

A Benders decomposition procedure is applied on the new formulations and the resulting computational procedure is seen to be identical to previous methods for solving the Travelling Salesman problem.

Based on the Lagrangean Relaxation method, a new procedure is suggested for generating lagrange multipliers for a subgradient optimization procedure. The effectiveness of the bounds obtained is demonstrated by computational test results.

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1. Introduction

The most compact methematical formulation to the Travelling Salesman problem known so far is the formulation given by Miller, et. al. [20] in 1960 as:

Problem - IP:*

Find variables X_{ij} and U_i i,j=1,2,...,n that minimize

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} \tag{1}$$

subject to:

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad j=1,2,\ldots,n, \tag{2}$$

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad i=1,2,\ldots,n, \tag{3}$$

$$U_i - U_j + n X_{ij} \leq n - 1 \qquad i,j=2,\ldots,n \quad i \neq j \tag{4}$$

$$X_{ij} = 0,1 \qquad \forall\ i,j \tag{5}$$

This is a mixed integer programming formulation with n^2 zero-one variables and n-1 continuous variables. In spite of the compactness of this formulation, no algorithms or computational test results have appeared in the open literature which have used this formulation as a basis for solving the Travelling Salesman problem. One of the major drawbacks of this formulation is the fact that it is

*We assume throughout that $C_{ii} = \infty$, i=1,2,...,n.

limited to the Travelling Salesman problem only and cannot be easily extended to other transportation scheduling problems which are related to the Travelling Salesman problems such as the Multi-Travelling Salesman problem, the Delivery problem, the School Bus problem, the Multi-Terminal Delivery problem, or the Static Dial-a-Bus problem. Mathematical formulations to some of those problems were given in Gavish and Shlifer [11].

In this paper, we shall investigate the Miller, et. al. [20] formulation and investigate its relationship to the highly successful Lagrangean Relaxation method developed by Held and Karp [15,16] for solving the Travelling Salesman Problem. In addition, alternative formulations of the Travelling Salesman Problem are developed which have the advantage of leading to relatively simple mathematical formulations to the above mentioned problems. These new formulations have the potential of leading to new algorithms based on Bender's decomposition or the combination of Lagrangean Relaxation and subgradient optimization. The remainder of the paper is organized as follows: In the next section, two new formulations of the Travelling Salesman problem are given. The third section extends this formulation to a general class of transportation scheduling problems. In Section 4, we show how to apply Bender's decomposition to the new formulations of the Travelling Salesman problem. The relationship between the Miller, et. al. [20] formulation and the Lagrangean Relaxation are investigated and an effective method for generating the initial Lagrange multipliers for a subgradient optimization procedure is developed and demonstrated in Section 5.

2. New Formulations for the Travelling Salesman Problem

In this section, we present and prove two new formulations for the Travelling Salesman problem. Those formulations are later used for formulating other transportation scheduling problems which are related to the Travelling Salesman problem. Both formulations use the same number of variables; however, they differ in their constraint set and lead to different decompositions.

Problem P1:

Find variables X_{ij}, y_{ij} $i,j=1,2,\ldots,n$ that minimize

$$Z = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} \right\} \qquad (6)$$

subject to:

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad j=1,2,\ldots,n \qquad (7)$$

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad i=1,2,\ldots,n \qquad (8)$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} y_{ij} - \sum_{\substack{j=2 \\ j \neq i}}^{n} y_{ji} = 1 \qquad i=2,\ldots,n \qquad (9)$$

$$y_{ij} \leq S X_{ij} \qquad i=2,\ldots,n;\ j=1,2,\ldots,n \quad i \neq j \qquad (10)$$

$$X_{ij} = 0,1 \ , \ y_{ij} \geq 0 \qquad (11)$$

where $S \geq n-1$

For fixed values of X, the constraints given in (9) and (10) form a network flow problem, and therefore the y_{ij} values will be integer.

Lemma: Problem P1 solves the Travelling Salesman problem.

Proof: The problem given by the constraints (6-8) is an assignment problem; it is well known that the positive variables in the extreme points of the assignment polytope form distinct loops of arcs in a graph that contains arc (i,j) iff $X_{ij} = 1$, and each node appears in only one of those loops. Therefore City 1 is contained in only one of those loops.

In order for the solution to Problem P1 to be the solution to the Travelling Salesman problem it must contain exactly one loop. Assume that the solution contains more than one loop, all of them distinct. Consider a loop which does not contain node 1. This loop is composed of $\{i_1,i_2,\ldots,i_r,i_1\}$. Now let $y_{i_1 i_2}=f$; from (9) it follows that $y_{i_2 i_3}=f+1$ and $y_{i_r i_1}=f+r-1$. Therefore, we have

$$\sum_{j=1}^{n} y_{i_1 j} - \sum_{j=1}^{n} y_{j i_1} = f - (f+r-1) = 1 - r$$

which contradicts with (9). Thus, no loops can exist that do not contain node 1; since node 1 is contained in exactly one loop, then at most one loop is generated.

To show that a feasible solution exists to Problem P1, assume that $\tau = \{1,i_1,i_2,\ldots,i_{n-1},1\}$ is the optimal tour; by assigning $y_{i_j,i_{j+1}}=j$ for $j=1,2,\ldots,n-2$, and $y_{i_{n-1},1} = n-1$ we satisfy the network flow constraints.

In the next formulation, we eliminate the assignment constraints (8) and replace them by extra constraints on the network flow problem.

Problem P2:

Find variables X_{ij}, y_{ij} $i,j=1,2,\ldots,n$ that minimize

$$Z = \left[\sum_{i=2}^{n}\sum_{j=2}^{n} C_{ij}X_{ij} + \sum_{j=2}^{n} C_{1j}y_{1j} + \sum_{i=2}^{n} \frac{C_{i1}}{n} y_{i1}\right] \qquad (12)$$

subject to:

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad j=1,2,\ldots,n \tag{13}$$

$$\sum_{j=2}^{n} y_{1j} = 1 \tag{14}$$

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} y_{ij} - \sum_{\substack{j=1 \\ j\neq i}}^{n} y_{ji} = 1 \qquad i=2,3,\ldots,n \tag{15}$$

$$y_{ij} \leq (n-1)X_{ij} \qquad i=1,2,\ldots,n,\; j=2,3,\ldots,n,\; i\neq j \tag{16}$$

$$y_{i1} = n\, X_{i1} \qquad i=2,3,\ldots,n \tag{17}$$

$$X_{ij} = 0,1 \quad y_{ij} \geq 0 \qquad \forall\, i,j \tag{18}$$

Lemma: The optimal solution to problem P2 solves the Travelling Salesman problem.

Proof: From (13) it follows that exactly one arc leads into each node; from (14) and (15) it follows that at least one arc leads out from each one of the nodes (2,3,...,n), and that all loops must include node 1.

Since from (13) and (17) there can be only one arc with flow of n units leading into node 1, there is only one loop and that loop must contain all of the nodes. Hence, any feasible solution to the constraint set (13)-(18) is a feasible tour for the Travelling Salesman problem; furthermore it is easy to see that any feasible tour for the Travelling Salesman problem will satisfy (13)-(18).

We have presented two new formulations for the Travelling Salesman problem. Now, we will show a strong relationship between problem P1 and the original Miller, et. al. [20] formulation.

Without loss in generality, we may rewrite (1-5) as:

$$\text{Min}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij} + \text{Min}\left\{\sum_{i=1}^{n} U_i\right\}\right] \tag{19}$$

subject to:

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad j=1,2,\ldots,n \tag{20}$$

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad i=1,2,\ldots,n \tag{21}$$

$$U_i - U_j + nX_{ij} \le n - 1 \qquad i=1,\ldots,n,\ j=2,\ldots,n, i\neq j \tag{22}$$

$$X_{ij} = 0,1 \quad U_i \ge 0 \qquad \forall\ i,j \tag{23}$$

The optimal value for the inner-minimization problem is obtained for $U_{i_k} = k$, $k=0,1,2,\ldots,n-1$ where $\{i_o = 1, i_1, i_2, \ldots, i_{n-1}, 1\}$ is the optimal Travelling Salesman tour. Thus, the optimal objective function value is equal to a constant $n(n-1)/2$ and therefore will not have any effect on the optimization over the assignment values.

The dual problem to the inner-minimization problem is given by:

$$\text{Max}\left\{\sum_{i=1}^{n}\sum_{\substack{j=2\\ j\neq i}}^{n} y_{ij}[nX_{ij}-n+1]\right\} \tag{24}$$

subject to:

$$\sum_{\substack{j=1\\ j\neq i}}^{n} y_{ji} - \sum_{\substack{j=2\\ j\neq i}}^{n} y_{ij} \le 1 \qquad \forall\ i=2,\ldots,n \tag{25}$$

$$-\sum_{j=2}^{n} y_{1j} \le 1 \tag{25a}$$

$$y_{ij} \geq 0 \qquad i=1,\ldots,n \quad j=2,\ldots,n \quad j\neq i \tag{26}$$

Due to (27) the constraint (25a) is redundant and therefore is eliminated from further consideration. Consider any feasible extreme point for the assignment constraints (20), (21); it is easy to show that (24)-(26) is unbounded unless the extreme point is a feasible Travelling Salesman tour. Hence, we can restrict our attention to those extreme points which denote a feasible tour. For any feasible Travelling Salesman tour, the optimal values $\tilde{y}_{ij}$ for (24)-(26) satisfy the following relations:

$$\tilde{y}_{ij} \cdot (nX_{ij} - n+1) = \begin{cases} \tilde{y}_{ij} & \text{for} \quad X_{ij} = 1 \\ 0 & \text{for} \quad X_{ij} = 0 \end{cases} \tag{27}$$

and

$$0 \leq \tilde{y}_{ij} \leq n - 1 \tag{28}$$

Therefore, the objective function (24) can be replaced by a new objective function and an extra set of constraints:

$$\text{Max} \left\{ \sum_{i=1}^{n} \sum_{\substack{j=2 \\ j\neq i}}^{n} y_{ij} \right\} \tag{29}$$

$$y_{ij} \leq SX_{ij} \qquad i=1,\ldots,n, j=2,\ldots,n, \; i\neq j \tag{30}$$

and

$$S \geq n-1$$

Replacing the inner minimization problem in (19)-(23) with the equivalent problem given by (25,26,29,30) yields a problem nearly identical to P1. The only difference is that in problem P1 the y_{ij} flows are strictly increasing while here they are strictly decreasing. In order to get from the original formulation a dual which resembles our problem P1, (4) has to be replaced by

$$U_j - U_i + n\,X_{ij} \le n - 1 \qquad i=1,2,\dots,n,\ j=2,\dots,n,\ i \ne j \qquad (31)$$

which has the same effect as (4) in preventing subtours.

3. Formulations of Transportation Scheduling Problems

The formulations which were given in Section 2 can be used as a basis for formulating a variety of transportation scheduling problems. In this section, we present these problems and their formulations.

3.1 The Multi-Travelling Salesman Problem

The Travelling Salesman problem as formulated by Miller, et. al. [20] was extended by Gavish [9] to the Multi-Travelling Salesman problem. For this problem, we have to find M tours (one for each salesman) such that each tour originates and ends at the depot at node 0. Each node (1,2,...,n) is visited exactly once, and total travel costs are minimized. Based on the formulation given in Problem P1, the formulation to the Multi-Travelling Salesman problem is:

Find variables X_{ij}, y_{ij} $i,j=0,1,2,\ldots,n$ that minimizes:

$$Z = \sum_{i=o}^{n} \sum_{j=o}^{n} C_{ij} X_{ij} \tag{32}$$

subject to:

$$\sum_{i=o}^{n} X_{ij} = 1 \qquad j=1,2,\ldots,n \tag{33}$$

$$\sum_{j=o}^{n} X_{ij} = 1 \qquad i=1,2,\ldots,n \tag{34}$$

$$\sum_{i=o}^{n} X_{io} = M \tag{35}$$

$$\sum_{j=o}^{n} X_{oj} = M \tag{36}$$

$$\sum_{j=o}^{n} y_{ij} - \sum_{j=o}^{n} y_{ji} = 1 \qquad i=1,2,\ldots,n \qquad (37)$$

$$y_{ij} \le (n-M+1)X_{ij} \qquad i,j=0,1,2,\ldots,n \qquad (38)$$

$$X_{ij} = 0,1 \;,\; y_{ij} \ge 0 \qquad \forall\, i,j \qquad (39)$$

By adding the equality $X_{oj} = y_{oj}$ $\forall j = 1,2,\ldots,n$, we assure that the y_{ij} values will also determine the arcs location within its tour. The formulation given above assumes that all the salesmen are identical; however, in reality they may have different qualifications and we would prefer to assign different numbers of cities to different salesmen. Moreover, in certain cases we would like to have a certain load balancing; i.e., that the variation in the number of cities assigned to the different salesmen will be within given limits. To model this, we need only replace the corresponding constraints in (38) with

$$X_{io}L \le y_{io} \le X_{io}U \qquad i=1,2,\ldots,n \qquad (40)$$

where L and U are the lower and upper bounds on the number of cities visited by a salesman.

3.2 The Delivery Problem

This problem is described (see references 5,7,8) as follows: Given an n by n matrix $\{C_{ij}\}$ of travelling costs between n nodes, M trucks and a non-negative load d_i, $i=1,2,\ldots,n$ associated with each node i, find M tours of minimum total cost that leave a depot 0, visit each node only once, and return to the depot. In each stop j the truck is loaded (or unloaded) by the extra load d_j. There is a limit Q on truck capacity such that the amount collected in each tour cannot exceed this limit.

Given that the loads d_i are integer for every i, the Delivery problem is formulated as:

Problem D:

Find variables X_{ij}, y_{ij} i,j=0,1,2,...,n which minimize

$$Z = \sum_{i=o}^{n} \sum_{j=o}^{n} C_{ij} X_{ij} \tag{41}$$

subject to (33-36), and

$$\sum_{j=o}^{n} y_{ij} - \sum_{j=o}^{n} y_{ji} = d_i \qquad i=1,2,\ldots,n \tag{42}$$

$$y_{ij} \le Q\, X_{ij} \qquad i,j=0,1,2,\ldots,n \tag{43}$$

$$X_{ij} = 0,1 \; , \; y_{ij} \ge 0 \qquad \forall\, i,j \tag{44}$$

The constraints (33-36) ensure that the X_{ij} values will form tours, while the constraints in (42) ensure that all tours will contain the depot (node 0); the constraints in (43) assure that the total load collected in a single tour will not exceed the truck capacity.

Another extension of the Delivery problem is the case in which the number of trucks is not given beforehand, and there is an extra fixed cost P associated with each additional truck used for delivery. This case may be formulated as:

$$\text{Min} \left\{ \sum_{i=1}^{n} \sum_{j=o}^{n} C_{ij} X_{ij} + \sum_{j=o}^{n} (C_{oj} + P) X_{oj} \right\} \tag{45}$$

subject to (33), (34), (42), (43), (44), and

$$\sum_{j=o}^{n} X_{oj} = \sum_{j=o}^{n} X_{jo} \tag{46}$$

3.3 The Multi-Terminal Delivery Problem

The Multi-Terminal Delivery problem is an extension of the Delivery problem in which we have K depots which may be used as starting points for tours. There exists an extra restriction that a tour will always return to the same depot from which it started. Different types of trucks may be used for performing the deliveries. Truck type h has a capacity Q_h, and a fixed cost P_{kh} for using truck type h from the k-th depot; there exist a limit M_{kh} on the number of trucks type h which may originate from the k-th depot; C_{ijh} is the travelling cost from node i to node j using truck type h. We assume that a node is serviced by just one truck.

Let H be the index set of truck types (i.e., $h \epsilon H$), and use the following indexing scheme the depots are indexed as $i=1,2,\ldots,K$, while the nodes are indexed as $i=K+1,\ldots,K+n$.

The Multi-Terminal Delivery Problem is formulated as:

$$Z = \text{Min} \left\{ \sum_{i=K+1}^{K+n} \sum_{j=1}^{K+n} \sum_{h \epsilon H} C_{ijh} X_{ijh} + \sum_{i=1}^{K} \sum_{j=K+1}^{K+n} \sum_{h \epsilon H} (C_{ijh}+P_{ih}) X_{ijh} \right\}$$

subject to:

$$\sum_{i=1}^{K+n} \sum_{h \epsilon H} X_{ijh} = 1 \qquad j=K+1,\ldots,K+n \tag{47}$$

$$\sum_{j=1}^{K+n} X_{ijh} - \sum_{j=1}^{K+n} X_{jih} = 0 \qquad \forall\ h \epsilon H,\ i=1,2,\ldots,K+n \tag{48}$$

$$\sum_{j=K+1}^{K+n} X_{ijh} \le M_{ih} \qquad \forall\, h\epsilon H,\ i=1,2,\ldots,K \tag{49}$$

$$\sum_{j=1}^{K+n} y_{ij} - \sum_{j=K+1}^{K+n} y_{ji} = d_i \qquad i=K+1,\ldots,K+n \tag{50}$$

$$y_{ij} \le \sum_{h\epsilon H} Q_h X_{ijh} \qquad \begin{matrix} i=K+1,\ldots,K+n \\ j=1,2,\ldots,K+n \end{matrix} \tag{51}$$

$$X_{ijh} = 0,1 \qquad y_{ij} \ge 0 \qquad \forall\, i,j,h \tag{52}$$

This formulation is more complicated than that for the delivery problem due to the constraints on the X variables. Here, we need standard network flow constraints (48), rather than the assignment constraints, to ensure that if a truck of a given type enters a city, the same truck will also leave the city. (49) limits the number of trucks type h that may originate at the i-th depot.

3.4 The Deterministic Dial-a-Bus Problem

The Dial-a-Bus problem arises in the following situation. A bus driver who is initially located in location 0, is given a set of n deliveries to perform. Each delivery i consists of two locations, a_i and b_i. Location b_i can be visited only after location a_i has been visited. We have to find a tour in which all deliveries will be performed while minimizing the travelling cost. The set of feasible tours for a problem with two deliveries (a_1,b_1), (a_2,b_2) is therefore

$\{0,a_1,a_2,b_1,b_2,0\}$
$\{0,a_1,a_2,b_2,b_1,0\}$
$\{0,a_2,a_1,b_1,b_2,0\}$
$\{0,a_2,a_1,b_2,b_1,0\}$
$\{0,a_1,b_1,a_2,b_2,0\}$
$\{0,a_2,b_2,a_1,b_2,0\}$

revealing that an unloading point is allowed to be visited only after the appropriate loading point has been visited earlier in the tour.

This problem arises in a Dial-a-Bus system; it may also characterize an delivery air-service or cargo ship service which has to schedule airport landings or port visits in order to satisfy all deliveries without violating the delivery loading/unloading constraints.

Using the indexing scheme such that all loading points are numbered from $\{1,2,\ldots,n\}$, and all unloading points from $\{n+1,\ldots,2n\}$, where unloading point $n+i$ corresponds to loading point i, then the Deterministic Dial-a-Bus problem is formulated as:

$$Z = \text{Min}\left\{\sum_{i=o}^{2n}\sum_{j=o}^{2n} C_{ij}X_{ij}\right\}$$

subject to:

$$\sum_{i=o}^{2n} X_{ij} = 1 \qquad j=0,1,\ldots,2n \qquad (53)$$

$$\sum_{j=o}^{2n} X_{ij} = 1 \qquad i=0,1,\ldots,2n \qquad (54)$$

$$\sum_{j=o}^{2n} y_{ij} - \sum_{j=o}^{2n} y_{ji} = 1 \qquad i=1,2,\ldots,2n \qquad (55)$$

$$y_{ij} \le 2(n+1)X_{ij} \qquad \forall\ i,j \qquad (56)$$

$$\sum_{j=o}^{2n} y_{n+i,j} \ge \sum_{j=o}^{2n} y_{ij} + 1 \qquad i=1,2,\ldots,n \qquad (57)$$

$$X_{ij} = 0,1\ ,\ y_{ij} \ge 0 \qquad \forall\ i,j \qquad (58)$$

The constraints in (57) are needed to assure that node n+i will be visited only after node i has been visited.

This simplified formulation of the Dial-a-Bus problem has been extended by Gavish and Srikanth [12] to handle due dates/times that are specified by the passengers and multiple buses.

3.5 The School Bus Problem

School buses which are initially located at school (node 0), have to collect students waiting at n pick-up points (nodes 1,2,...,n), and deliver them to school. The capacity of each bus is limited to Q students. The number of students waiting at the i-th pick-up point is equal to d_i, $0 < d_i \leq Q$, $i=1,2,\ldots,n$. t_{ij} is the travel time from pick-up point i to point j. Security and operational considerations limit the time that students at pick-up point i are allowed to spend on the bus to T_i time units, $t_{io} \leq T_i \; \forall \; i$. Only one bus is allowed to stop at a pick-up point. To simplify the presentation, we assume a negligible loading time in each station. The relaxation of this assumption can be handled by minor modification to the following formulation.

Let P be the cost of using an extra bus for the schedule, C_{ij} be the operational cost of travelling from point i to j, X_{ij} be a binary variable denoting travel from point i to point j, y_{ij} be the number of students on the bus between points i and j, while z_{ij} is the travel time from point i to school assuming that the next bus stop is at point j.

The School Bus problem can be formulated as:

Find variables X_{ij}, y_{ij}, z_{ij}, $i,j=0,1,2,\ldots,n$ that minimizes:

$$Z = \sum_{i=1}^{n} \sum_{j=o}^{n} C_{ij} X_{ij} + \sum_{j=o}^{n} (C_{oj}+P) X_{oj} \tag{59}$$

subject to:

$$\sum_{i=o}^{n} X_{ij} = 1 \qquad j=1,2,\ldots,n \qquad (60)$$

$$\sum_{j=o}^{n} X_{ij} = 1 \qquad i=1,2,\ldots,n \qquad (61)$$

$$\sum_{j=o}^{n} y_{ij} - \sum_{j=o}^{n} y_{ji} = d_i \qquad i=1,2,\ldots,n \qquad (62)$$

$$y_{ij} \le Q\, X_{ij} \qquad i,j=0,1,2,\ldots,n \qquad (63)$$

$$\sum_{k=o}^{n} z_{ki} - \sum_{j=o}^{n} z_{ij} = \sum_{k=o}^{n} t_{ki} X_{ki} \qquad i=1,2,\ldots,n \qquad (64)$$

$$z_{ij} \le T_i X_{ij} \qquad i,j=0,1,2,\ldots,n \qquad (65)$$

$$X_{ij} = 0,1 \quad z_{ij},\ y_{ij} \ge 0 \qquad \forall\ i,j \qquad (66)$$

The constraints in (60-61) assure that each pick-up point will be visited by exactly one bus, (62-63) prevent subtour formation and limit the number of students in the bus to the bus capacity, while (64-65) assure that the routes will meet the travel time constraints.

4. Application of Benders Decomposition

This section applies Benders Decomposition [4] to the Travelling Salesman formulations given by (P1) and (P2). The decomposition of (P1) is straightforward. Given values for the assignment variables X_{ij}, the subproblem (SP1) is as follows:

$$\text{(SP1)} \quad \min w \tag{67}$$

s.t.

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} y_{ij} - \sum_{\substack{j=2 \\ j\neq i}}^{n} y_{ji} = 1 \qquad i=2,\ldots,n \tag{68}$$

$$0 \le y_{ij} \le SX_{ij} \qquad i=2,\ldots,n;\ j=1,2,\ldots,n;\ i\neq j \tag{69}$$

The master problem for this decomposition is the standard assignment problem given by (6)-(8) supplemented by the set of cuts generated by (SP1). It will be shown that the generated constraints are just the subtour breaking constraints identified by Dantzig, Fulkerson and Johnson [6].

The dual of SP1 is

$$\text{(SP1')} \quad \max v = -\sum_{i=2}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} SX_{ij}\gamma_{ij} + \sum_{i=2}^{n} \mu_i \tag{70}$$

s.t.

$$-\gamma_{ij} + \mu_i - \mu_j \le 0 \qquad i=2,\ldots,n;\ j=2,\ldots,n;\ i\neq j \tag{71}$$

$$-\gamma_{i1} + \mu_i \le 0 \qquad i=2,\ldots,n \tag{72}$$

$$\gamma_{ij} \ge 0 \tag{73}$$

The solution of the dual problem (and hence, the Benders cut) depends upon the given assignment variables $\{X_{ij}\}$. Since $\{X_{ij}\}$ must satisfy the assignment constraints (7)-(8), the values must identify a set of disjoint subtours. The set of nodes (cities) can be divided into two sets N_1, N_2 depending on whether the node's subtour contains the depot (node 1). That is, $i \epsilon N_1$ if node i is contained in the subtour which includes node 1; otherwise, $i \epsilon N_2$ and its subtour does not contain the depot.

If N_2 is empty, $\{X_{ij}\}$ defines a feasible tour, and an optimal solution to (SP1') is $\gamma_{ij} = \mu_i = \mu_j = 0$ for all i,j. Provided that $\{X_{ij}\}$ solves the current master problem, then it will be an optimal solution to the Travelling Salesman problem.

If N_2 is not empty, (SP1') is unbounded and will generate a constraint for the master problem. An extreme ray for (SP1') is given by

$$\mu_i = \begin{cases} 0 & i \epsilon N_1 \\ 1 & i \epsilon N_2 \end{cases} \tag{74}$$

$$\gamma_{ij} = [\mu_i - \mu_j]^+ \qquad i=2,\ldots,n \quad j=2,\ldots,n \quad i \neq j \tag{75}$$

$$\gamma_{i1} = \mu_i \qquad i=2,\ldots,n \tag{76}$$

The Benders cut generated by this ray is

$$-\sum_{i=2}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} SX_{ij}\gamma_{ij} + \sum_{j=2}^{n} \mu_j \leq 0 \tag{77}$$

Using (75)-(77), the constraint may be restated as

$$\sum_{i \epsilon N_2} \sum_{j \epsilon N_1} X_{ij} \geq \frac{|N_2|}{S} \tag{78}$$

where $|N_2|$ is the cardinality of set N_2. Since $|N_2| \leq S$, and $X_{ij} = 0,1$, we have

$$\sum_{i \in N_2} \sum_{j \in N_1} X_{ij} \geq 1 \tag{79}$$

But this is just the subtour-breaking constraint proposed in [6], which requires the use of at least one arc going from set N_2 to N_1. Hence, the application of Benders Method to formulation (P1) is identical to starting with the assignment constraints and sequentially generating subtour breaking constraints as given in (79). Clearly, this application offers no new computational breakthroughs.

The decomposition for the formulation (P2) is similar to that for (P1). To simplify the presentation of the method, it will be helpful to restate the objective function (12) of (P2) as

$$\min\left\{\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}X_{ij}\right\} \tag{80}$$

Now, the master problem is given by (80) subject to the assignment constraint (13), the zero-one restriction on X_{ij}, plus the generated constraints from the subproblem. Given values for $\{X_{ij}\}$ which satisfy the master problem, the subproblem (SP2) is given by

$$\text{(SP2)} \quad \min \{w\} \tag{81}$$

s.t.

$$\sum_{j=2}^{n} y_{1j} = 1 \tag{82}$$

$$\sum_{\substack{j=1\\j\neq i}}^{n} y_{ij} - \sum_{\substack{j=1\\j\neq i}}^{n} y_{ji} = 1 \qquad i=2,\ldots,n \tag{83}$$

$$0 \le y_{ij} \le (n-1)X_{ij} \qquad i=1,2,\ldots,n \quad j=2,\ldots,n \quad i\neq j \tag{84}$$

$$y_{i1} = n\,X_{i1} \qquad i=2,\ldots,n \tag{85}$$

The dual of (SP2) is as follows:

$$(\text{SP2}') \quad \max\{w\} = -\sum_{i=1}^{n}\sum_{\substack{j=2\\j\neq i}}^{n} (n-1)X_{ij}\gamma_{ij} - \sum_{i=2}^{n} nX_{i1}\gamma_{i1} + \sum_{i=1}^{n} \mu_i \tag{86}$$

s.t.

$$-\gamma_{ij} + \mu_i - \mu_j \le 0 \qquad i=1,\ldots,n \quad j=2,\ldots,n \quad i\neq j \tag{87}$$

$$-\gamma_{i1} + \mu_i \le 0 \qquad i=2,\ldots,n \tag{88}$$

$$\gamma_{ij} \ge 0 \qquad i=1,\ldots,n \quad j=2,\ldots,n \quad i\neq j \tag{89}$$

The dual problem is unbounded if the given assignment variables $\{X_{ij}\}$ do not form a tour. For a given set of values $\{X_{ij}\}$ that solve the master problem, the node set N can be divided into two sets N_1, N_2. The set N_1 contains all nodes for which in the given assignment there is a directed path from that node to the depot; set N_2 is just $N-N_1$. If N_2 is non-empty, the given assignment is not a tour, and an extreme ray for (SP2') is given by

$$\mu_i = \begin{cases} 0 & i\in N_1 \\ 1 & i\in N_2 \end{cases} \tag{90}$$

$$\gamma_{ij} = [\mu_i - \mu_j]^+ \quad \forall\ i,j \tag{91}$$

The constraint generated by this extreme ray is, after simplification,

$$\sum_{i \in N_2} \sum_{j \in N_1} X_{ij} \geq 1, \tag{92}$$

which is identical in form to the constraints found for (P1). Now, however, the constraint requires at least one arc connecting nodes on a directed path to the depot with all other nodes. Hence, again the application of Benders Method to (P2) results in nothing new; rather, the Benders Method applied to the new formulations will result in the generation of the pure integer formulation in [6].

5. Initial Lagrange Multipliers for the Subgradient Optimization

The most successful algorithm for solving the Travelling Salesman problem was originally developed by Held and Karp [15,16]. The algorithm uses a Lagrangean Relaxation technique combined with a subgradient optimization procedure for obtaining tight lower bounds on the objective function value, and a branch and bound procedure for closing the integer gap in cases that such a gap was detected. Later modifications to this basic procedure are due to Held, Wolfe and Crowder [17], Hansen and Krarup [14], and Smith and Thompson [21]. Empirical observations which are based on the computational experience gained in those experiments reveal that the bounds obtained through this relaxation procedure are tight, the depth and number of nodes generated by this branch and bound procedure is quite limited (less than a thousand) and most of the computer time was spent in finding the best multipliers for the subgradient optimization procedure. A careful examination of the Miller, Tucker, Zemlin formulation (MTZF) of the Travelling Salesman problem reveals a strong relationship between this formulation and Held and Karp's procedure.

First, we will add to the problem given by (1-5) the redundent set of constraints:

$$X_{1j} + X_{j1} \leq 1 \qquad \forall\ j=2,\ldots,n \tag{93}$$

Multiplying the constraints in (3) by a vector of Lagrange multipliers $\overline{\Psi} = \{\psi_i,\ i=2,\ldots,n\}$ and adding them to the objective function we obtain the following problem:

$$L_{IP}(\psi) = \underset{X}{\text{Min}} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}X_{ij} + \sum_{i=2}^{n} \psi_i \left(1 - \sum_{j=1}^{n} X_{ij}\right) \right\} \tag{94}$$

$$
\begin{cases}
\text{s.t.} \quad \sum_{i=1}^{n} X_{ij} = 1 & j=1,2,\ldots,n, \\
\sum_{j=1}^{n} X_{1j} = 1 & \\
X_{1j} + X_{j1} \le 1 & \forall\ j=2,\ldots,n \\
U_i - U_j + nX_{ij} \le n-1 & i,j=2,\ldots,n,\ i \ne j \\
X_{ij} = 0,1 & \forall\ i,j
\end{cases} \qquad (95)
$$

It is easy to see that for a fixed vector $\overline{\psi}$, the internal optimization problem over X generates a 1-tree whose root is node 1, and the multipliers in the Held and Karp's procedure correspond to the appropriate constraints in (3). Letting Z_{IP} be the optimal objective function value for the problem given by (1-5,95) and Z_{LP} for its linear programming relaxation, the following relations hold between Z_{IP}, Z_{LP} and $L(\psi)$;

$$Z_{IP} \ge L(\psi)$$

$$Z_{IP} \ge Z_{LP}$$

$L(\psi)$ could, therefore, be used as a lower bound on Z_{IP}. Since it is desired to get tight lower bounds on Z_{IP}, we are looking for the multipliers ψ^* that will satisfy

$$L(\psi^*) = \underset{\psi}{\text{Max}}\{L(\psi)\}$$

i.e., minimize the integer gap between Z_{IP} and its Lagrangean Relaxation. Geoffrion [13] has proved the following relation

$$Z_{IP} \geq L(\psi^*) \geq Z_{LP}$$

In many computational tests, it was found that $L(\psi^*)$ is a tight bound. However, no efficient universal methods exist for computing the optimal multipliers.

A reasonable procedure for getting initial values for the lagrange multipliers is to relax the integrality constraints on the problem given by Z_{IP}, solve the linear program and use the values of the dual variables to the appropriate constraints in (3) for starting the subgradient optimization procedure. Gavish [10] has successfully applied this procedure for solving Interval Bounded Knapsack problems. Let $L_{LP}(\psi)$ be the relaxed version of $L_{IP}(\psi)$ in which the integrality constraints have been relaxed and ψ_{LP} be the dual variables of the appropriate constraints (3) in Z_{LP}.

Lemma: The following relation holds

$$L_{IP}(\psi_{LP}) \geq Z_{LP}$$

Proof: Since the problem associated with $L_{LP}(\psi)$ is a relaxed version of $L_{IP}(\psi)$ the following holds:

$$L_{IP}(\psi) \geq L_{LP}(\psi)$$

Since $L_{LP}(\psi)$ and Z_{LP} are pure linear programs, they satisfy

$$L_{LP}(\psi_{LP}) = \max_{\psi}\{L_{LP}(\psi)\} = Z_{LP}$$

thus

$$L_{IP}(\psi_{LP}) \geq L_{LP}(\psi_{LP}) = Z_{LP} \quad .$$

The lemma implies that using ψ_{LP} in the lagrangean relaxation results in a bound on the optimal travelling salesman tour which is at least as tight as the bound obtained from the linear programming relaxation. Moreover, a subgradient optimization procedure could be used to update the multiplier values leading to tighter bounds.

In order to investigate the quality of the bounds, we have conducted a set of computational tests in which four methods for generating bounds to the travelling salesman problem have been examined. Two of the methods are linear programming relaxations, while the other two are based on a lagrangean relaxation in which the integrality constraints are conserved.

The first linear programming relaxation is the solution to the assignment problem given by (1-3). The second linear programming relaxation is obtained by relaxing the integrality constraints in the Miller et. al. [20] formulation with the additional constraints (95) on arcs leading from and into city 1. The second relaxation method is constrained relative to the first and will produce tighter bounds.

The lagrangean relaxations are based on the 1-tree formulation and its relaxation as given in (94). The two methods which were tested differ in the initial lagrange multipliers ψ which are applied in (94). Held and Karp [15,16] have suggested to use the dual variables to (3) which are obtained from solving the assignment problem as initial multipliers for a subgradient optimization procedure. The fourth method uses the dual variables to (3) in the second linear programming relaxation as multipliers for the lagrangean relaxation. Since we were interested in the quality of the initial "guess" for the multiplier values, no subgradient optimization procedure was used and the multiplier values were not updated.

The computational tests were performed on problems with known optimal solutions that were used as reference points for testing the quality of the bounds. The results are summarized in table 1. As can be seen from those limited computational experiments the bounds obtained from a lagrangean relaxation based on the 1-tree formulation are superior to those obtained from relaxing the integrality constraints. In the computational tests the dual variables from the linear programming relaxation of the 1-tree formulation produce better multipliers and bounds then multipliers obtained from the assignment problem. The bounds given in table 1 are based on the initial multipliers which were provided from the linear programming relaxation, no attempt was made to improve those bounds by using a subgradient optimization procedure in which the multipliers are updated.

The computational tests were limited to 42 cities due to the excessive computer time needed to solve the large linear programs. However, even those limited experiments clearly demonstrate the effectiveness of this approach in generating good initial multipliers for the subgradient optimization procedure. Moreover, the bounds obtained by using a lagrangean relaxation approach in which the integrality constraints are preserved, clearly dominate the bounds obtained through relaxing the integrality constraints and solving the resulting linear program. A nonefficient general purpose linear programming package (MPS/360) was used during those experiments to solve the linear program. In the future, we plan to develop a specialized linear programming code which will use the special structure of the problem to reduce the computer time required to solve it and enable the solution of larger problems.

Table 1 - The Bounds Obtained by Different Bounding Methods

Number of Cities	The Value Obtained by the					Problem Source [reference]
	Linear Programming Relaxation of the		Lagrangean Relaxation Using Multipliers From the		Travelling Salesman Optimal Solution	
	Assignment Problem	1-Tree Formulation	Assignment Problem	1-Tree Formulation		
5	140	143.2	140	148	148	[18]
10	326	353.6	356	371	378	[2]
33	9948	9986.5	10496	10803	10861	[18]
42	9217	9299.2	11019	11826.2	12345	[6]

Summary

This paper has presented a new formulation for the Travelling Salesman problem. The formulation differs from earlier formulations in that two classes of variables are used: tour assignment variables and tour flow variables. This formulation has been shown to have a dual relationship with Miller et.al. [20] formulation. By decomposing the problem using Benders Method, the tour-breaking constraints given in Dantzig, Fulkerson and Johnson [6] are rederived.

The new formulation was extended to include a variety of related transportation scheduling problems. Finally, preliminary computational results are reported on generating lower bounds from a Lagrangean relaxation of the Travelling Salesman problem. These results suggest that tight bounds for the Travelling Salesman problem may be obtained by using the dual variables from the linear programming relaxation of the new formulation, as initial Lagrange multipliers in a subgradient optimization procedure.

Future research will fully explore this and the bounding procedure for developing optimization procedures and heuristics for solving the travelling salesman problem.

Acknowledgment

The authors are indebted to Professor E. Balas and T. Magnanti for helpful suggestions and comments on an earlier draft of the paper.

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