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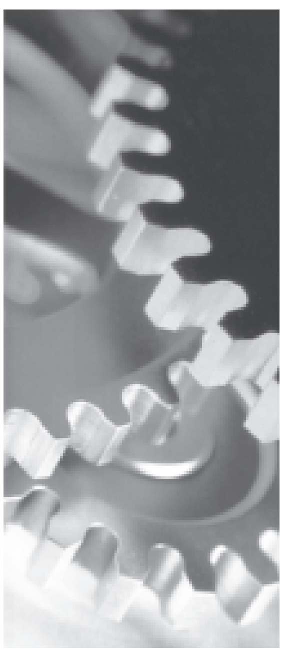

LEHMAN BROTHERS

A dynamic default correlation model

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We develop a dynamic model of default that matches single-name CDS spreads by construction and can be calibrated to standard CDO tranche spreads. We assume that single-name default intensities have a one-factor structure with an unobservable systematic factor. Our model formalizes an intuitive idea: while the current level of spreads is determined by the distribution of default times, spread volatility is determined by the process of resolution of uncertainty about default times. We model resolution of uncertainty as learning about the unobservable systematic factor. Our model is computationally tractable and has the appealing property that the learning process can be specified to match empirically observed spread volatility with no affect on the model-implied spread levels[1].

1. INTRODUCTION

We develop a framework for pricing multi-name credit products dependent on correlation dynamics, such as options on tranches, forward-starting tranches, etc. Our model is "bottom-up" in nature: we first fit single-name parameters to the observed CDS spread curves, and then derive the joint default distribution from a dynamic factor model. Our model formalizes an intuitive idea: while the current level of spreads is determined by the *distribution of default times*, spread volatility is determined by the process of *resolution of uncertainty about default times*. We model resolution of uncertainty as learning about the unobserved aggregate state of the market. This mechanism generates appealing features of spread and default dynamics, e.g., default-induced systematic spread widening, it is tractable and offers valuable flexibility in generating spread dynamics.

We explain the basic idea in the context of pricing of options on standard CDO tranche spreads. Standard static copula-based models allow one to match current spreads on single names and standard liquid tranches, but they are silent on the volatility (more generally, the dynamics) of tranche spreads, since they do not explicitly specify how uncertainty about the joint distribution of default times is resolved. In the Chapovsky, Rennie, and Tavares model (2006), defaults are modelled in a doubly-stochastic framework and are assumed to be conditionally independent, i.e., their arrivals are independent conditionally on realized intensities. Furthermore, default correlation is induced by the factor structure of default intensities, e.g., by a common stochastic factor driving single-name default intensities. Conceptually, this framework can be as tractable as the standard copula-based models and can be calibrated to match spreads on individual names and liquid tranches. However, calibrating their model to match the volatility of tranche spreads as well, particularly for senior tranches, may be quite difficult. The common stochastic factor driving default arrival intensities affects both the correlation of defaults and the volatility of tranche spreads. Thus, the model must be calibrated to match tranche spread levels and their volatility simultaneously, which is a challenging modelling and computational problem. Our approach sidesteps this obstacle. We specify the dynamics of default intensities such that the tranche spread volatility is essentially de-coupled from current spread levels. Thus, one can start by calibrating single-name CDS spreads and current spreads on standard tranches, and then specify the dynamics of the model to produce the desired spread volatility without affecting the quality of initial calibration. Initial parameter choices impose constraints on the maximum achievable tranche spread volatility produced by the model, but otherwise do not affect the level and dynamics of volatility produced by the model. Such quasi-separability of our model is convenient and the underlying mechanism behind our model is intuitive.

[1] *I would like to thank Hongwei Cheng, Yadong Li, Marco Naldi, and Lutz Schloegl for many helpful discussions.*

Consider a market in which single-name default intensities are driven by a common factor that in turn depends on an unobservable aggregate state. Market participants know the process driving the aggregate state, but do not observe the current state and instead form beliefs about it by observing public signals, such as news items and default events. The current level of spreads is based on current beliefs about the likelihood of all possible future values of default intensities of portfolio names. For senior tranches, this essentially reduces to the likelihood of various values of the common factor. However, the future volatility of spreads depends on how uncertainty about the future is resolved, i.e., it depends on the nature of the learning process. In the absence of news, senior tranche spreads change slowly and largely deterministically, while arrival of news leads to changes in investors' beliefs and tranche spreads. Thus, our model effectively separates current spread levels, which depend on the joint distribution of default currently perceived by the market, from spread volatility, which is affected by the news arrival process. This separation is not complete. To generate sufficiently high spread volatility, the model must give rise to sufficiently high dispersion in spread levels across the (unobservable) states of the economy; otherwise learning cannot have much quantitative impact. With this caveat, the possibility of learning about the aggregate state from sources other than defaults, and the effect of such learning on spread volatility, gives our approach significant flexibility.

Our model learning generalizes Collin-Dufresne, Goldstein, and Helwege (2003). Like theirs, our model also generates contagion: default of a single name leads to spread widening on other names in the portfolio. We extend their analysis to cover a more general specification of single-name intensity processes necessary to reproduce single-name forward default probabilities.

The rest of the paper is organized as follows. Section 2 presents the theoretical model. Ssection 3 discusses the numerical solution method. Section 4 presents results of calibration and numerical experiments. Section 5 concludes.

2. THE MODEL

Our model has a single-factor structure, with a common factor driving the default intensities of all names in the portfolio. We first describe the dynamics of single-name default intensities and then derive the dynamics of beliefs about the common aggregate factor. All stochastic processes are defined under the risk-neutral probability measure. To simplify notation, we keep this implicit. Furthermore, we assume that interest rates are non-stochastic.

The single-name default intensity process

We model the default intensity process of name i, $\tilde{\lambda}_i(t)$, as follows. Let $\tilde{y}(t) \geq 0$ be a stochastic process representing the systematic default intensity factor. We assume that

$$\int_0^T \tilde{\lambda}_i(t)dt = \int_0^T \tilde{\alpha}_i(t)dt + \beta_i(T)\int_0^T \beta_A(t)\tilde{y}(t)dt$$

where $\beta_A(t)$ is a non-negative deterministic function of time and $\tilde{\alpha}_i(t) \geq 0$ is a stochastic process representing the idiosyncratic component of the single-name default intensity assumed to be independent across names and from the aggregate process $\tilde{y}(t)$. We separate $\beta_A(t)$ from $\tilde{y}(t)$ purely for convenience of calibration. We also set the current time to zero.

Let $P_i(t)$ denote the current survival probability curve of name i. Since we require the default intensity process to be consistent with the empirical single-name survival probability curves, the following equality must hold: $P_i(t) = E_0\left[\exp\left(-\int_0^t \tilde{\lambda}_i(s)\,ds\right)\right]$. As in Chapovsky, Rennie, and Tavares (2006), this is satisfied by construction.

Define two compensators

$$\Phi(t,b) = E_0\left[\exp\left(-b\int_0^t \beta_A(s)\tilde{y}(s)\,ds\right)\right]$$
$$\Psi(t,\tilde{\alpha}) = E_0\left[\exp\left(-\int_0^t \tilde{\alpha}(s)\,ds\right)\right]$$

Independence of the systematic and idiosyncratic components of the default intensity process imply that

$$P_i(t) = \Psi(t,\tilde{\alpha}_i)\Phi(t,\beta_i(t))$$

Without further restrictions, for any level of the single-name systematic risk exposure $\beta_i(t)$, one can construct the process $\tilde{\alpha}_i(t)$ so that the model reproduces the empirical single-name survival curves:

$$\tilde{\alpha}_i(t) = -\frac{d}{dt}\ln\left(\frac{P_i(t)}{\Psi(t,\tilde{z}_i)\Phi(t,\beta_i(t))}\right) + \tilde{z}_i(t)$$

where $\tilde{z}_i(t)$ is any non-negative process with support $[0,\infty)$. However, to make sure that the resulting default arrival intensity is always non-negative, we do need to restrict $\tilde{z}_i(t)$ to as set of admissible processes. First, to ensure non-negativity of $\tilde{\lambda}_i(t)$ for any value of the systematic factor $\tilde{y}(t)$, we require single-name systematic risk exposure $\beta_i(t)$ to be non-negative and non-decreasing. Second, we require the ratio $\frac{P_i(t)}{\Psi(t,\tilde{z}_i)\Phi(t,\beta_i(t))}$ to be non-increasing. These two constraints guarantee that default intensities are always non-negative. The above constraints are imposed in calibration.

The systematic component of default arrival intensities

We assume that the systematic component of single-name default intensities, $\tilde{y}$, depends on the state of the economy s: $s \in \{1,\ldots,J\}$. The state of the economy follows a continuous-time Markov chain moving between the J states with transition rates q_{jk}^s. Specifically, for two distinct states j and k,

$$\text{Prob}\left[s(t+dt) = k \mid s(t) = j\right] = q_{jk}^s dt\,,$$

and $q_{jj}^s = -\sum_{k:k\neq j} q_{jk}^s$.

We model the common (systematic) factor of default intensities as a deterministic function of time and state, $y_j(t)$. The systematic factor is given by $\tilde{y}(t) = y_{s(t)}(t)$. The Markov-chain structure of the state process is analytically and numerically tractable and has been used by Graziano and Rogers (2006) to model CDO tranche spreads. However, their model does not fit single-name CDS spreads, and they do not model spread dynamics because of learning.

Evolution of beliefs

The state of the economy is not observable and market participants must form beliefs about the current state. Given the current beliefs, one can compute prices of various default-contingent claims. In particular, define a cumulative process $\tilde{X}(T)=\int_0^T \beta_A(t)\tilde{y}(t)dt$.

Given our specification of the single-name default intensity process, and the assumption of conditional independence, the joint distribution of survival probabilities of all names can be easily summarized by conditioning on $\tilde{X}(t)$. Survival probabilities conditional on the state of other credits in the portfolio can be expressed as functions of $\tilde{X}(t)$:

$$P_i\left(t \mid \tilde{X}(t)\right)=P_i\left(t,\beta_i(t)\right)\Phi\left(t,\beta_i(t)\right)^{-1}e^{-\beta_i(t)\tilde{X}(t)}$$

The conditional expected payoff of a multi-name credit derivative, $F\left(\tilde{X}(t)\right)$, is then easy to compute. For example, under the assumption of deterministic recovery rate that we maintain in our numerical experiments, conditional expected loss on CDO tranches can be computed using a standard recursion. One can then integrate the conditional expected loss over the distribution of future values of $\tilde{X}(t)$.

Given the Markovian structure of the state process, the joint process $\left(s(t),\tilde{X}(t)\right)$ is Markov, and therefore the distribution of $\tilde{X}(t)$ is completely characterized by the initial distribution over the starting values $\left(s(0),\tilde{X}(0)\right)$ and the probability law of the process $\left(s(t),\tilde{X}(t)\right)$. Note that we do not need to know how beliefs about the unobservable values of $\left(s(t),\tilde{X}(t)\right)$ evolve over time. To price CDO tranches, it suffices to know the distribution of future values of $\left(s(t),\tilde{X}(t)\right)$ given the initial information set. However, the dynamics of tranche spreads depends on how beliefs change. More formally, current tranche spreads depend only on the conditional distribution of future values of default arrival intensities (current information), and not on the filtration (the process of information revelation). Thus, once the conditional distribution of future intensities has been calibrated to match tranche spreads, one can capture spread volatility by constructing the appropriate filtration. This is a formal counterpart of our earlier discussion of separation between calibration of standard tranche spreads and spread dynamics.

We now describe how beliefs about the pair $\left(s(t),\tilde{X}(t)\right)$ evolve. Since the process $\tilde{X}(t)$ can take a continuum of values, the probability distribution (beliefs) over $\left(s(t),\tilde{X}(t)\right)$ are characterized by a stochastic PDE. To avoid unnecessary technicalities, and since the stochastic PDE will eventually need to be discretized and solved numerically, we instead construct a continuous-time Markov chain approximating the process $\left(s(t),\tilde{X}(t)\right)$ and derive the evolution of beliefs about its state, which is a well-known filtering problem. Finally, we use a discrete-time Markov chain approximation for numerical computations. The main reason we use a continuous-time Markov chain approximation as an intermediate step is to derive intuitive analytical expressions for impact of news arrival on beliefs. This step is not necessary for numerical computations, which one can base directly on the discrete-time Markov chain approximation of the original process.

We approximate the process $\tilde{X}(t)$ by a continuous-time Markov chain $X^{MC}(t)$ moving between the grid points X_k. Let q^X denote the transition rate matrix of this Markov chain. We specify the transition rates in a standard manner to obtain a consistent approximation of the original process (see Kushner and Dupuis, 2001, Chs. 5, 12). Specifically, we assume that the process either remains at the same grid point or moves up to the next grid point, and that transition probabilities satisfy

$$q_{jk}^X(t)\left(X_{k+1} - X_k\right) = \beta_A(t)\tilde{y}(t)$$

which means that the expected rate of change of the Markov chain coincides with that of the original process $\tilde{X}(t)$. Transitions of the Markov chain process for X are independent from those for the state s, which completely specifies the dynamics of the pair $\left(s(t), X^{MC}(t)\right)$.

Let $p_{jk}(t)$ denote the probability distribution capturing current beliefs about the value of the pair $\left(s(t), X^{MC}(t)\right)$. Beliefs are updated as a result of the arrival of information from multiple sources. We model two types of information: defaults and general ("macro-economic") news. Assume for simplicity that there are two types of general news processes: good and bad news items. Then the information set available to investors at time T consists of:

1. The history from 0 to T of the default status $1_{\{\tau_i > t\}}$ of the portfolio names, where τ_i denotes the default time of name i.
2. The history from 0 to T of the non-decreasing counting processes $N^{(G)}(t)$ and $N^{(B)}(t)$ which serve to capture good and bad news arrival in the economy.

We model news arrival as counting processes, so the same analytical results apply to updating of beliefs based on defaults and news arrival. It is straightforward to model news arrival as Brownian diffusions.

We derive the formulas for belief dynamics using a heuristic application of the Bayes' rule to emphasize the economic intuition. Lipster and Shiryaev (Th. 19.6) offer a formal justification of our result.

Consider a counting process $N(t)$ (a news process) with the current arrival rate $\lambda_{jk}(t)$, which depends on the current state $\left(s_j, X_k^{MC}\right)$. We temporarily ignore the regular transitions of the chain and focus on updating of beliefs based on observations of $N(t)$. Using the Bayes' rule,

$$\text{Prob}_{\text{t+dt}}\left(s_j, X_k^{MC} \mid \text{Arrival during } (t, t+dt]\right) =$$

$$\frac{\text{Prob}_{\text{t}}\left(s_j, X_k^{MC}\right)\text{Prob}_{\text{t}}\left(\text{Arrival during } (t, t+dt] \mid s_j, X_k^{MC}\right)}{\text{Prob}_{\text{t}}\left(\text{Arrival during } (t, t+dt]\right)} =$$

$$\frac{p_{jk}(t)\lambda_{jk}(t)dt}{\sum_{j',k'} p_{j'k'}(t)\lambda_{j'k'}(t)dt} = \frac{p_{jk}(t)\lambda_{jk}(t)}{\sum_{j',k'} p_{j'k'}(t)\lambda_{j'k'}(t)}$$

This is an intuitive relationship. If news arrival is more likely in a particular state $\left(s_j, X_k^{MC}\right)$ (relatively high value of λ_{jk}), then after observing an arrival one raises the probability of this being the current state. Similarly, conditioning on no arrival,

$$\text{Prob}_{\text{t+dt}}\left(s_j, X_k^{MC} \mid \text{No arrival during } (t, t+dt]\right) =$$

$$\frac{\text{Prob}_{\text{t}}\left(s_j, X_k^{MC}\right)\text{Prob}_{\text{t}}\left(\text{No arrival during } (t, t+dt] \mid s_j, X_k^{MC}\right)}{\text{Prob}_{\text{t}}\left(\text{No arrival during } (t, t+dt]\right)} =$$

$$\frac{p_{jk}(t)\left(1-\lambda_{jk}(t)dt\right)}{1-\sum_{j',k'} p_{j'k'}(t)\lambda_{j'k'}(t)dt} =$$

$$p_{jk}(t)\left(1-\lambda_{jk}(t)dt + \sum_{j',k'} p_{j'k'}(t)\lambda_{j'k'}(t)dt\right)$$

Again, we find an intuitive relationship. If the arrival rate of the news item is more likely in a particular state $\left(s_j, X_k^{MC}\right)$, then after observing no arrival one lowers the probability attached to this state. For example, while arrival of bad news leads one to revise one's beliefs towards the relatively bad states, lack of bad news has the opposite effect. The difference is that the affect of news arrival on beliefs is immediate and discontinuous, while the affect of the absence of news is continuous and gradual.

We now state the above results in more compact form. Define the average arrival rate of the counting process

$$\bar{\lambda}(t) = \sum_{j,k} p_{jk}(t)\lambda_{jk}(t)$$

Consider a martingale process $M(t)$ defined by

$$dM(t) = dN(t) - \bar{\lambda}(t-)dt$$

where $x(t-)$ denotes the value of process x immediately before time t. Then,

$$\text{d}p_{jk}(t) = p_{jk}(t-)\left(\frac{\lambda_{jk}(t-)}{\bar{\lambda}(t-)} - 1\right)dM(t)$$

We now combine the updating due to different sources of information with the underlying dynamics of the Markov chain. First, note that the default rate of name i is given by

$$\tilde{\lambda}_i(t) = \tilde{\alpha}_i(t) + \dot{\beta}(t)\tilde{X}(t) + \beta_i(t)\beta_A(t)\tilde{y}(t).$$

Thus, $\tilde{\lambda}_i(t)$ depends on both state variables, $\tilde{y}(t)$ and $\tilde{X}(t)$. We denote the default arrival rate of name i in state $\left(s_j, X_k^{MC}\right)$ by $\lambda_{i,jk}(t)$. Next, we define the dynamics of the news processes. Let $N^{(G)}(t)$ and $N^{(B)}(t)$ have arrival intensities $\gamma_j^{(G)}(t)$ and $\gamma_j^{(B)}(t)$

conditional on being in state $s = j$. We assume for simplicity that there are only two news processes and that the arrival rate of these processes depends only on the state s and does not depend on $\tilde{X}$. Because arrival intensities of the news processes depend on the state, investors can learn something about the current state by observing these processes. We define the corresponding martingales

$$dM^{\text{NEWS}}(t) = dN^{\text{NEWS}}(t) - \bar{\gamma}^{\text{NEWS}}(t-)dt, \quad \text{NEWS} \in \{\text{G}, \text{B}\}$$

Where

$$\bar{\gamma}^{\text{NEWS}}(t) = \sum_{j,k} p_{jk}(t)\bar{\gamma}_j^{\text{NEWS}}(t)$$

is the average arrival rate of the news process. We thus obtain the dynamics of beliefs

$$dp_{jk}(t) = \sum_{j'k'} p_{j'k'}(t-)q^{s}_{j'j}q^{X}_{k'k}dt + p_{jk}(t-)\sum_i \left(\frac{\lambda_{i,jk}(t-)}{\bar{\lambda}_i(t-)} - 1 \right) dM_i(t) +$$

$$p_{jk}(t-) \sum_{NEWS \in \{G,B\}} \left(\frac{\gamma_j^{NEWS}(t-)}{\bar{\gamma}^{NEWS}(t-)} - 1 \right) dM^{NEWS}(t)$$

We have already seen the intuition behind the updating of beliefs in response to news arrival. Next, consider how contagion arises in this model (the mechanism is qualitatively the same as in Collin-Dufresne, Goldstein and Helwege, 2003). Suppose that the first name defaults. Define the current expected values of the state variables

$$\bar{y}(t) = \sum_{jk} p_{jk}(t)\tilde{y}_j(t)$$

And

$$\bar{X}(t) = \sum_{jk} p_{jk}(t)X_k(t)$$

At the instant of default, the martingale corresponding to the first name, $M_1(t)$, increases by 1. Since

$$\lambda_{1,jk}(t) = \tilde{\alpha}_1(t) + \beta_1(t)\beta_A(t)y_j(t) + \dot{\beta}_1(t)X_k$$

and

$$\bar{\lambda}_1(t) = \tilde{\alpha}_1(t) + \beta_1(t)\beta_A(t)\bar{y}(t) + \dot{\beta}_1(t)\bar{X}(t)$$

we find that, following the default of the first name, the probability assigned to being in state $\left(s_j, X_k^{MC}\right)$ changes by

$$p_{jk} \frac{\beta_1\beta_A(y_j - \bar{y}) + \dot{\beta}_1(X_k - \bar{X})}{\tilde{\alpha}_1 + \beta_1\beta_A\bar{y} + \dot{\beta}_1\bar{X}}$$

where all variables are evaluated at the instance preceding the default.

A few qualitative results follow from the above formula. First, changes in probabilities assigned to different states are proportional, i.e., one's belief about the likelihood of a particular state following an event is proportional to the prior belief immediately before the event. Second, everything else being equal, a default of a name with high idiosyncratic component of default intensity $\tilde{\alpha}_i$ leads to a smaller revision of the probabilities assigned to different states of the systematic factor. The reason is that a default by a name with high idiosyncratic risk is not very informative about the systematic factor. Finally, a default of a name with high systematic risk loading, e.g., high β_i, increases the perceived likelihood of states with high systematic default risk.

In summary, we have characterized theoretically both the distribution of future default intensities conditional on the current information set, which determines current tranche spreads, and the process of belief updating, which determines spread evolution.

3. COMPUTATION

It is straightforward to compute the distribution of future values of the continuous-time Markov chain $\left(s(t), X^{MC}(t)\right)$. We have derived the transition rates in the previous subsection. We now use a standard algorithm (see Kushner and Dupuis, 2001, Chs. 5, 12) to approximate this continuous-time process using a discrete-time Markov chain. We index the states of the chain by (j,k). Transitions of the chain in each direction are independent, therefore we derive transition probabilities for the discrete-time approximation to $s(t)$ and $X^{MC}(t)$ separately. Let $\hat{q}_{((j,k),(j',k'))}(t)$ denote the time-t probability of the discrete-time Markov chain moving from state (j,k) at time t to state (j',k') at time $t+\Delta t$. Similarly, let $\hat{q}^s_{jj'}(t)$ and $\hat{q}^X_{kk'}(t)$ denote the transition probability alone each dimension. Then, $\hat{q}_{((j,k),(j',k'))}(t) = \hat{q}^s_{jj'}(t)\hat{q}^X_{kk'}(t)$.

Let Δt denote the time step of the discrete-time Markov chain. Then,

$$\hat{q}^s_{jj'}(t) = \left(1 - e^{q^s_{jj}(t)\Delta t}\right)\frac{q^s_{jj'}(t)}{\sum_{n:n\neq j} q^s_{jn}(t)}$$

is the transition probability of the approximating discrete-time Markov chain from state j to state $j' \neq j$. As before, $q^s_{jj'}(t)$ denotes the transition rates of the continuous-time chain we are approximating. The same approximation can be used for transitions in the X-dimension.

With the transition probabilities established, one can now compute the distribution over the future values of the chain simply by iterating forward the following one-step calculation, starting from the initial probability distribution over the states (beliefs):

$$p_{j'k'}(t+\Delta t) = \sum_{j,k} p_{jk}(t)\hat{q}^s_{jj'}(t)\hat{q}^X_{kk'}(t)$$

4. CALIBRATION

We calibrate our model to match prices of standard liquid CDO tranches. For each choice of the parameters governing the state variable process $s(t)$, we calibrate single-name systematic risk exposure as follows.

First, we calibrate the function $\beta_A(t)$ so that the compensator $\Phi(t,1)$ mimics the shape of the "representative" survival curve. This initial step is not required, but it helps maintain sufficiently high single-name loadings on the systematic source of default risk while ensuring that single-name default arrival rates are non-negative, i.e., that the condition

$$\frac{d}{dt}\ln\left(\frac{P_i(t)}{\Phi(t,\beta_i(t))}\right)<0$$

is satisfied.

Next, we calibrate single-name loadings on the systematic source of default risk $\beta_i(t)$. In particular, our aim is to allow for a sufficiently high contribution of systematic risk loading to the total variance of future default arrival rates, so that movements of the systematic factor could generate sufficiently high correlation of default arrivals across names. We do this by requiring that, for any two tenors of CDO tranches $T_1 < T_2$,

$$\ln(\Phi(T_2,\beta_i(T_2)))-\ln(\Phi(T_1,\beta_i(T_1)))\approx\theta_i^{sys}(\ln(P_i(T_2))-\ln(P_i(T_1)))$$

where the parameter $0<\theta_i^{sys}<1$ controls the systematic risk exposure of name i. Moreover, we require the function $\beta_i(t)$ to be non-decreasing. We aim to calibrate all single-name survival probability curves using the same value of the parameter θ_i^{sys}. However, it may be necessary to reduce the value of this parameter for certain names to make sure that the condition $\frac{d}{dt}\ln\left(\frac{P_i(t)}{\Phi(t,\beta_i(t))}\right)<0$ is satisfied by a non-decreasing curve $\beta_i(t)$. This may happen if, for instance, most names in the portfolio have upward-sloping forward default arrival rate curves, but a particular name has a steep downward-sloping forward default arrival rate curve. Then, for this name it may be impossible to achieve the desired level of systematic risk exposure with a non-decreasing curve $\beta_i(t)$.

Using the above procedure for matching single-name survival probability curves, we calibrate parameters of the systematic factor process, which include the initial probabilities attached to various states, the state-specific systematic factor values $y_j(t)$, and the transition rates between states, which we also allow to be deterministic functions of time. In particular, in our numerical experiments we found that sufficient flexibility is achieved by allowing these time-dependent functions to be piece-wise-linear between the CDO tranche tenors.

When choosing parameter values, we aim to minimize a measure of distance between model-predicted spreads and the observed bid-ask interval for tranche spreads. Depending on the application, an additional term can be added to the objective function to penalize day-to-day variation of calibrated model parameters, thus increasing parameter stability.

5. NUMERICAL EXPERIMENTS

In this section, we calibrate our model to tranche prices of the CDX IG9 index on 05/05/2008 and illustrate the impact of learning on tranche prices.

We assume a deterministic recovery rate of 40%. It is well known that, under recent market conditions, constant-recovery models have difficulty matching prices of senior tranches. However, we opt for the simplicity afforded by the constant-recovery assumption. Our basic framework can be extended to allow for stochastic recovery rates.

We use a four-state specification of the model and order the states so that the systematic factor values $y_j(t)$ are increasing with the state index. We report the key parameter values in Figure 1. The matrix of transition rates between states, $q^s(t)$, and the state-dependent systematic factor values $y_j(t)$ are assumed to be piece-wise constant between tranche tenors. Since we are calibrating to prices of five-, seven-, and 10-year tranches, we end up with three sets of values for q^s and y_j. We also report the initial probabilities attached to the four states. We calibrate the model under the assumption that the initial value of the cumulative process, $X(0)$, almost surely equals zero, while the initial value of the state $s(0)$ is not observable.

Figure 1. Key parameters of the four-state model

Initial probabilities

Time interval	1	2	3	4
	81.7%	14.5%	3.8%	0.0%

Transition rate matrix q^s

Time interval	State	1	2	3	4
0-5Y	1	-0.220	0.201	0.019	0.000
	2	13.788	-14.799	1.000	0.011
	3	0.000	0.122	-0.167	0.045
	4	0.000	0.010	0.000	-0.010
5-7Y	1	-0.284	0.184	0.100	0.000
	2	13.845	-14.842	0.997	0.000
	3	0.000	1.510	-1.513	0.003
	4	0.000	0.010	0.000	-0.010
7-10Y	1	-0.862	0.762	0.100	0.000
	2	13.800	-14.800	1.000	0.000
	3	0.000	1.610	-1.650	0.045
	4	0.000	0.010	0.000	-0.010

Systematic factor values y

Time interval	1	2	3	4
0-5Y	0.000	0.062	0.191	10.000
5-7Y	0.015	0.198	1.266	10.000
7-10Y	0.000	0.000	0.011	10.000

Source: Lehman Brothers

Calibrated parameter values have an intuitive interpretation. By construction, the systematic factor is increasing with the state index, thus lower states are interpreted as relatively good (low likelihood of defaults) and high states as relatively bad (high likelihood of defaults). Since the values of the factor are modulated by the time-dependent function $\beta_A(t)$, only the relative values of the factor have economic meaning. We observe that the fourth state is significantly worse than any other state. We think of this as a disaster state.

At time zero, the model attaches the highest likelihood to state 1 (82%), with state 2 being the second most likely state (14%). Transition rates between states imply that the economy is usually in state 1, with infrequent transitions to state 2 and even more rare transition to states 3 and 4. Once in state 2 or 3, the economy tends to revert to a better state relatively quickly. This is not the case for the disaster state, 4: we set the transition probability out of this state to a very low value, which means that once the economy is in the disaster state, multiple defaults are practically inevitable. Current probability attached to states 2 and 3 is relatively high given the transition probabilities. This is consistent with the increase in risk observed during the past few months. In the framework of our model, one would attribute the relatively high likelihood of bad states to the recent arrival of negative news.

Figure 2 compares the tranche spreads implied by the model to the market data. We see that the four-state version of the model offers sufficient flexibility to match spreads of all tranches at the five-year tenor, but has difficulty matching spreads on tranches with longer tenors, particularly those of senior tranches.

Figure 2. Model-implied and market tranche spreads

Term	Tranche	Upfront or Spread	
		Data	Model
5Y	0-3%	60.91	60.83
	3-7%	383.23	385.27
	7-10%	197.07	194.76
	10-15%	96.31	95.89
	15-30%	41.03	40.44
7Y	0-3%	65.87	67.52
	3-7%	464.37	479.24
	7-10%	229.15	224.55
	10-15%	118.65	143.42
	15-30%	45.66	49.05
10Y	0-3%	69.31	69.70
	3-7%	550.57	561.60
	7-10%	278.36	264.76
	10-15%	133.18	159.10
	15-30%	54.57	65.80

CDX IG9 tranche quotes at close of business on 5 May 2008 and the corresponding model-implied spreads.
Source: Lehman Brothers

To illustrate the impact of learning on spread dynamics, suppose that we have just observed the arrival of a news process with arrival rates dependent on the state s and equal to 1.5, 1.0, 0.75, and 0.0001 in states 1, 2, 3, and 4, respectively. This process has relatively high arrival rates in good states compared to bad states, and can thus be interpreted as the good-news process. Using the general formula for updating beliefs, we find that the probabilities assigned to different states change from the calibrated values (81.7%, 14.5%, 3.8%, 0%) to (87.6%, 10.3%, 2.0%, 0%). As expected, the perceived likelihood of bad states has fallen, while the likelihood of the best state (state 1) has risen. Figure 3 reports the updated tranche spreads.

Figure 3. Model-implied tranche spread change following news arrival

Term	Tranche	Upfront or Spread	
		Before News	After News
5Y	0-3%	60.83	60.01
	3-7%	385.27	342.00
	7-10%	194.76	160.80
	10-15%	95.89	76.34
	15-30%	40.44	33.92
7Y	0-3%	67.52	65.29
	3-7%	479.24	382.20
	7-10%	224.55	148.69
	10-15%	143.42	72.59
	15-30%	49.05	27.55
10Y	0-3%	69.70	68.24
	3-7%	561.60	517.43
	7-10%	264.76	240.21
	10-15%	159.10	140.52
	15-30%	65.80	56.00

Model-implied tranche spreads before and after and arrival of positive news.
Source: Lehman Brothers

Positive news reduces tranche spreads, particularly for senior tranches. This is intuitive. Positive news affects the distribution of future values of the systematic factor, shifting probability mass towards lower values of the factor. This reduces individual CDS spreads and default correlation. Because news about the systematic factor does not affect idiosyncratic default intensities, it is not surprising that it has relatively weak impact on the equity tranche spread. Lower single-name spreads reduce the equity tranche spread, while reduced default correlation has the opposite affect. The net result is a small reduction in the equity tranche spread. In contrast, senior tranche spreads are much more affected, mostly because of reduced default correlation. Thus, senior tranche spreads show a lot more sensitivity to the change in beliefs about the initial value of the aggregate state. In contrast, junior tranche spreads are more sensitive to changes in idiosyncratic single-name default intensities (shocks to $\tilde{z}_i(t)$).

Next, we illustrate our model's ability to generate contagion. We assume that a default is observed at time zero. Since, mathematically, a default has the same impact as a negative news arrival, all we must do is calibrate a negative news process to mimic default of one credit.

We consider two cases: default by a high-yield name, and default by a relatively safe high-grade name. Based on the range of calibrated model parameters, we set $\beta_A(0) = 0.7$ and consider name i with $\beta_i(0) = 0.35$. We assume that credit i is not in the portfolio, so its default affects tranche spreads only as a result of its informational content. For a high-yield name, we set the current value of the idiosyncratic default arrival component to $\tilde{\alpha}_i(0) = 0.15$. Based on the general expression for the single-name default arrival rate, which simplifies because $\tilde{X}(0) = 0$, a default by such a name is equivalent to an arrival of the news process with state-contingent arrival-rates $\gamma_j = 0.15 + 0.245 * y_j(0)$. Immediately following the default, beliefs about the current aggregate state change from (81.7%, 14.5%, 3.8%, 0%) to (79.6%, 15.5%, 4.9%, 0%). In contrast, consider a high-grade name with a much lower idiosyncratic default arrival rate $\tilde{\alpha}_i(0) = 0.02$, so that

$\gamma_j = 0.02 + 0.245 * y_j(0)$. In the event of default by such a name, beliefs are revised to (68.1%, 21.2%, 10.6%, 0%). We see that a default by a high-grade name has much larger impact on the beliefs about the aggregate state. The reason is that the idiosyncratic component of the default arrival rate on such a name is relatively low, thus its default arrival rate is much more sensitive to the current state than that of a high-yield name. Accordingly, default by a high-grade name is much more informative about the aggregate state than default by a high-yield name.

Figure 4 compares post-default five-year tranche spreads under the two scenarios.

Figure 4. Model-implied tranche spreads following a default

Term	Tranche	Upfront or Spread		
		Before News	After News	
			Default by High-Yield	Default by High-Grade
5Y	0-3%	60.83	61.26	63.60
	3-7%	385.27	409.48	546.84
	7-10%	194.76	213.72	319.26
	10-15%	95.89	106.74	166.22
	15-30%	40.44	44.04	63.63

Model-implied tranche spreads for the five-year tenor before and after a default. The defaulted name is either high-yield, or high-grade.
Source: Lehman Brothers

Our framework has further implications for post-default spread dynamics. If the true state of the world in which the default took place is relatively good, then this default is not likely to be followed by further negative news or additional defaults. The lack of negative events is interpreted by the Bayesian observer as evidence against being in one of the bad states, thus, following the immediate post-default widening, spreads tend to drift back to normal levels

CONCLUSION

We have presented a framework for modeling default correlation and spread dynamics. Our numerical experiments illustrate the main features of the model and show that even a basic four-state version of the model can reproduce recently observed tranche spreads with reasonable accuracy.

Our analysis suggests several natural topics for further research. We calibrate our model to a single snapshot of market prices, and do not investigate parameter stability over time. Ideally, a well-specified model should allow one to fit a time series of tranche spreads with relatively stable parameters. Since increased model flexibility (e.g., more aggregate states) may be needed to match prices under various market conditions, one must deal with parameter proliferation, perhaps using some kind of regularization technique. One must also develop an effective algorithm for calibrating the news processes to match the empirical data on spread dynamics. Finally, it would be natural to introduce stochastic recovery in case of default, perhaps by assuming that the recovery ratio is lower in bad aggregate states. This may be necessary to replicate the recently observed high spreads on senior tranches.

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