Internet Appendix for "Quantifying Liquidity and Default Risks of Corporate Bonds over the Business Cycle"

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1 Technical Details of Model Solution

First, we present the HJB equations for value functions for the 2-state case in the paper. Then, we present the HJB equations for the n-state case and present the solutions to the value functions, of which the $n = 2$ case is a special case. To this end, we split Proposition 1 in the main text into two propositions, one for the debt value function (Proposition 1), and one for the equity value function (Proposition 2).

Throughout, diag$(\cdot)$ is the diagonalization operator mapping any row or column vector into a diagonal matrix (in which all off-diagonal elements are identically zero).

1.1 State Transitions

As notational conventions, we use capitalized bold-faced letters (e.g., $\mathbf{X}$) to denote matrices, lower case bold face letters (e.g. $\mathbf{x}$) to denote vectors, and non-bold face letters denote scalars (e.g. x). The only exceptions are the value functions for debt and equity, $\mathbf{D}, \mathbf{E}$ respectively, which will be vectors, and the (diagonal) matrix of drifts, $\boldsymbol{\mu}$. Dimensions for most objects are given underneath the expression. While we focus on 2-aggregate-state case where $s \in \{G, B\}$, the Appendix presents general results for an arbitrary number of (Markov) aggregate states.

Given the aggregate state s, recall that we have assumed that the intensity of switching from state-H to state-L is ξ_s, and the L-state is absorbing, i.e., those L-type investors leave the market forever. However, an L-type bond holder meets a dealer with intensity λ_s and sells the bond for $B^s = \beta D_H^s + (1-\beta) D_L^s$ that he himself values at D_L^s. Then the L-type's *intensity-modulated surplus* when meeting the dealer can be rewritten as

$$\lambda_s (B^s - D_L^s) = \lambda_s \beta (D_H^s - D_L^s).$$

As a result, for the purpose of pricing, the *effective transitioning intensity* from L-type to H-type is $q_{Ls \to Hs} = \lambda_s \beta$ where λ_s is the state-dependent intermediation intensity and β is the investor's bargaining power.

1.2 Debt

Recall that the midpoint price is

$$P^s = \frac{A^s + B^s}{2} = \frac{(1+\beta) D_H^s + (1-\beta) D_L^s}{2} = w_H D_H^s + w_L D_L^s$$

with $w_H + w_L = 1$ and that holding costs are

$$hc^s = \chi_s [N - P^s] = \chi_s [N - w_H D_H^s - w_L D_L^s]$$

Stacking the holding cost function, we have

$$\mathbf{hc}(y) = \boldsymbol{\chi} [\mathbf{N} - \mathbf{W} \cdot \mathbf{D}(y)]$$

where $\boldsymbol{\chi} = \mathrm{diag}\left([\chi_G, \chi_G, \chi_B, \chi_B]\right)$, $\mathbf{N} = [0, N, 0, N]^\top$ and

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ w_H & w_L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w_H & w_L \end{bmatrix}$$

is the appropriate weighting matrix. Thus, $\mathbf{hc}\,(y)$ has identcially zero odd-numbered rows. Then, debt follows, on I_2,

$$\begin{aligned} \underbrace{\hat{\mathbf{R}} \cdot \mathbf{D}^{(2)}\,(y)}_{Discounting,4\times 1} &= \underbrace{\boldsymbol{\mu}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)'(y)}_{4\times 1} + \frac{1}{2}\underbrace{\boldsymbol{\Sigma}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)''(y)}_{4\times 1} + \underbrace{\hat{\mathbf{Q}} \cdot \mathbf{D}^{(2)}\,(y)}_{Transition,4\times 1} \\ &\quad + \underbrace{c\mathbf{1}_4}_{Coupon,4\times 1} + \underbrace{m\left[p\mathbf{1}_4 - \mathbf{D}^{(2)}\,(y)\right]}_{Maturity,4\times 1} - \underbrace{\mathbf{hc}\,(y)}_{Holding\,Cost,4\times 1}, \end{aligned} \tag{1}$$

where $\hat{\mathbf{Q}}$ is the effective transition matrix (accounting for the OTC market outcome via the recovery intensity $\lambda_s\beta$) for debt out on interval I_2, given by

$$\hat{\mathbf{Q}} = \begin{bmatrix} -\xi_G - \zeta_G & \xi_G & \zeta_G & 0 \\ \beta\lambda_G & -\beta\lambda_G - \zeta_G & 0 & \zeta_G \\ \zeta_B & 0 & -\xi_B - \zeta_B & -\xi_B \\ 0 & \zeta_B & \beta\lambda_B & -\beta\lambda_B - \zeta_B \end{bmatrix},$$

and where

$$\hat{\mathbf{R}} \equiv \mathrm{diag}\left([r_G, r_G, r_B, r_B]\right), \boldsymbol{\mu} \equiv \mathrm{diag}\left([\mu_G, \mu_G, \mu_B, \mu_B]\right), \boldsymbol{\Sigma} \equiv \mathrm{diag}\left(\left[\sigma_G^2, \sigma_G^2, \sigma_B^2, \sigma_B^2\right]\right).$$

Substituting it in, and noting that $\mathbf{X} = \mathbf{X}^{(n)}$ where n is the number of state, we have

$$\begin{aligned} \underbrace{\hat{\mathbf{R}}^{(2)} \cdot \mathbf{D}^{(2)}\,(y)}_{Discounting,4\times 1} &= \underbrace{\boldsymbol{\mu}^{(2)}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)'(y)}_{4\times 1} + \frac{1}{2}\underbrace{\boldsymbol{\Sigma}^{(2)}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)''(y)}_{4\times 1} + \underbrace{\hat{\mathbf{Q}}^{(2)} \cdot \mathbf{D}^{(2)}\,(y)}_{Transition,4\times 1} \\ &\quad + \underbrace{c\mathbf{1}_4}_{Coupon,4\times 1} + \underbrace{m\left[p\mathbf{1}_4 - \mathbf{D}^{(2)}\,(y)\right]}_{Maturity,4\times 1} - \underbrace{\left[\boldsymbol{\chi}^{(2)}\mathbf{N}^{(2)} - \boldsymbol{\chi}^{(2)}\mathbf{W} \cdot \mathbf{D}^{(2)}\,(y)\right]}_{Holding\,Cost,4\times 1}, \end{aligned} \tag{2}$$

For simulation purposes, we need to incorporate $\mathbf{W}$ in such a way into $\mathbf{R}$ and $\mathbf{Q}$ that there is a unique discount rate in each state (i, s), i.e. $\mathbf{R}$ stays a diagonal discount matrix, and that $\mathbf{Q}$ stays a true transition matrix.[1] Thus, decompose $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$ so that

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ w_H & -w_H & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w_H & -w_H \end{bmatrix}$$

First, we augement $\hat{\mathbf{Q}}$ to give the new transition matrix

$$\mathbf{Q} \equiv \hat{\mathbf{Q}} + \boldsymbol{\chi}\mathbf{W}_2 = \begin{bmatrix} -\xi_G - \zeta_G & \xi_G & \zeta_G & 0 \\ \beta\lambda_G + \chi_G w_H & -\beta\lambda_G - \zeta_G - \chi_G w_H & 0 & \zeta_G \\ \zeta_B & 0 & -\xi_B - \zeta_B & -\xi_B \\ 0 & \zeta_B & \beta\lambda_B + \chi_B w_H & -\beta\lambda_B - \zeta_B - \chi_B w_H \end{bmatrix}$$

[1] Let $\mathbf{Q}$ be a transition matrix. Then define, for a finited time-interval Δt, the discrete-time transition matirx is given by

$$\mathbf{Q}_{discrete} \equiv \exp(\mathbf{Q}\Delta t)$$

Note that $\Delta t = 0$ implies that $\mathbf{Q}_{discrete} = \mathbf{I}$ and thus there is total persistence of the states. Also, note that since $\mathbf{Q}$ is a transition matrix that we have $\mathbf{Q}_{discrete}\mathbf{1} = \mathbf{1}$. This comes from the fact that

$$\exp(\mathbf{Q}\Delta t)\,\mathbf{1} = \mathbf{I} \cdot \mathbf{1} + \sum_{k=1}^{\infty} \frac{\mathbf{Q}^k \cdot \mathbf{1}\,(\Delta t)^k}{k!} = \mathbf{1}$$

as $\mathbf{Q}^k\mathbf{1} = \mathbf{0}$ for all $k \geq 1$.

Note that $\mathbf{Q}$ is still a **true** transition matrix. Next, let us define the diagonal discount matrix $\mathbf{R}^{(2)}$ as

$$\mathbf{R} \equiv \hat{\mathbf{R}} - \boldsymbol{\chi}\mathbf{W}_1 = \begin{bmatrix} r_G & 0 & 0 & 0 \\ 0 & r_G - \chi_G & 0 & 0 \\ 0 & 0 & r_B & 0 \\ 0 & 0 & 0 & r_B - \chi_B \end{bmatrix}$$

and we acknowledge the possibility of an indiviual entry being negative. Then, we have

$$\begin{aligned} \underbrace{\left[\mathbf{R}^{(2)} + m\mathbf{I}_4 - \mathbf{Q}^{(2)}\right]}_{4\times 4}\underbrace{\mathbf{D}^{(2)}(y)}_{4\times 1} &= \underbrace{\boldsymbol{\mu}^{(2)}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)'(y)}_{4\times 1} + \frac{1}{2}\underbrace{\boldsymbol{\Sigma}^{(2)}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)''(y)}_{4\times 1} \\ &+ \underbrace{c\mathbf{1}_4}_{Coupon, 4\times 1} - \underbrace{\boldsymbol{\chi}^{(2)}\mathbf{N}^{(2)}}_{Holding\,Cost, 4\times 1} + \underbrace{m\cdot p\mathbf{1}_4}_{Maturity, 4\times 1}, \end{aligned} \tag{3}$$

the final form of the second-order ODE we are interested in.

Define $\mathbf{R}^{(1)} = \begin{bmatrix} r_G + \zeta_G & 0 \\ 0 & r_G - \chi_G + \zeta_G \end{bmatrix}$, $\mathbf{Q}^{(1)} \equiv \begin{bmatrix} -\xi_G & \xi_G \\ \beta\lambda_G + \chi_G w_H & -\beta\lambda_G - \chi_G w_H \end{bmatrix}$ and $\tilde{\mathbf{Q}}^{(1)} \equiv \begin{bmatrix} \zeta_G & 0 \\ 0 & \zeta_G \end{bmatrix} = \zeta_G \mathbf{I}_2$. On interval $I_1 = [y_{def}(G), y_{def}(B)]$, the bond is "dead" in state B, and the alive bonds $\mathbf{D}^{(1)} = \left[D_H^{(G,1)}, D_L^{(G,1)}\right]^\top$ solve

$$\begin{aligned} \left[\mathbf{R}^{(1)} + m\mathbf{I}_2 - \mathbf{Q}^{(1)}\right]\mathbf{D}^{(1)} &= \boldsymbol{\mu}^{(1)}\left(\mathbf{D}^{(1)}\right)' + \frac{1}{2}\boldsymbol{\Sigma}^{(1)}\left(\mathbf{D}^{(1)}\right)'' + \tilde{\mathbf{Q}}^{(1)}\begin{bmatrix} \alpha_H^B \\ \alpha_L^B \end{bmatrix} v_U^B(y) \\ &+ \left(c\mathbf{1}_2 - \boldsymbol{\chi}^{(1)}\mathbf{N}^{(1)}\right) + m\cdot p\mathbf{1}_2 \end{aligned} \tag{4}$$

for

$$y \in I_1 = [y_{def}(G), y_{def}(B)],$$

where the last term is the recovery value in case of a jump to default brought about by a state jump.

The boundary conditions at $y = \infty$ and $y = y_{def}(G)$ are standard:

$$\lim_{y\to\infty} \left|\mathbf{D}^{(2)}(y)\right| < \infty, \text{ and } \mathbf{D}^{(1)}\left(y_{def}^{G}\right) = \begin{bmatrix} \alpha_H^G \\ \alpha_L^G \end{bmatrix} v_U^G\left(y_{def}^G\right) \tag{5}$$

For the boundary y_{def}^B, we must have value matching conditions for all functions across $y_{def}(B)$:

$$\mathbf{D}^{(2)}\left(y_{def}^B\right) = \begin{bmatrix} \mathbf{D}^{(1)}\left(y_{def}^B\right) \\ \begin{bmatrix} \alpha_H^B \\ \alpha_L^B \end{bmatrix} v_U^B\left(y_{def}^B\right) \end{bmatrix} \tag{6}$$

and smooth pasting conditions for functions that are alive across y_{def}^B ($\mathbf{x}_{[1:2]}$ selects the first 2 rows of vector $\mathbf{x}$):

$$\left(\mathbf{D}^{(2)}\right)'\left(y_{def}^B\right)_{[1:2]} = \left(\mathbf{D}^{(1)}\right)'\left(y_{def}^B\right). \tag{7}$$

1.3 Equity

For equity, we have

$$\mathbf{RR} = \operatorname{diag}\left([r_G, r_B]\right), \boldsymbol{\mu\mu} = \operatorname{diag}\left([\mu_G, \mu_B]\right), \boldsymbol{\Sigma\Sigma} = \operatorname{diag}\left(\left[\sigma_G^2, \sigma_B^2\right]\right), \tag{8}$$

and

$$\mathbf{QQ}^{(2)} = \begin{bmatrix} -\zeta_G & \zeta_G \\ \zeta_B & -\zeta_B \end{bmatrix}$$

so that

$$\begin{aligned} \underbrace{\mathbf{RR}^{(2)}\mathbf{E}^{(2)}(y)}_{Discounting,2\times1} &= \underbrace{\boldsymbol{\mu\mu}^{(2)}}_{2\times2}\underbrace{\left(\mathbf{E}^{(2)}\right)'(y)}_{2\times1} + \frac{1}{2}\underbrace{\boldsymbol{\Sigma\Sigma}^{(2)}}_{2\times2}\underbrace{\left(\mathbf{E}^{(2)}\right)''(y)}_{2\times1} + \underbrace{\mathbf{QQ}^{(2)}\mathbf{E}^{(2)}(y)}_{Transition,2\times1} \\ &+ \underbrace{\mathbf{1}_2\exp(y)}_{Cashflow,2\times1} - \underbrace{(1-\pi)\,c\mathbf{1}_2}_{Coupon,2\times1} + \underbrace{m\left[\mathbf{S}^{(2)}\cdot\mathbf{D}^{(2)}(y) - p\mathbf{1}_2\right]}_{Rollover,2\times1} \end{aligned} \tag{9}$$

where $\mathbf{QQ}^{(2)}$ is the effective transition matrix for equity out on interval I_2 and $\mathbf{RR}^{(2)}$ is the effective discount matrix.

For I_1, define $\mathbf{RR}^{(1)} = r_G + \zeta_G, \boldsymbol{\mu\mu}^{(1)} = \mu_G, \mathbf{\Sigma\Sigma}^{(1)} = \sigma_G^2, \mathbf{QQ}^{(1)} = 0, \widetilde{\mathbf{QQ}}^{(1)} = \zeta_G$. Then

$$\begin{aligned}\left[\mathbf{RR}^{(1)} - \mathbf{QQ}^{(1)}\right] \mathbf{E}^{(1)}(y) &= \boldsymbol{\mu\mu}^{(1)} \left(\mathbf{E}^{(1)}\right)'(y) + \frac{1}{2}\mathbf{\Sigma\Sigma}^{(1)} \left(\mathbf{E}^{(1)}\right)''(y) + \widetilde{\mathbf{QQ}}^{(1)} \mathbf{0}_1 \\ &\quad + \mathbf{1}_1 \exp(y) - (1-\pi) c \mathbf{1}_1 + m \left[\mathbf{S}^{(1)} \cdot \mathbf{D}^{(1)}(y) - p \mathbf{1}_1\right] \end{aligned} \quad (10)$$

The particular solution is

$$\begin{aligned}\underbrace{\mathbf{E}^{(2)}(y)}_{2\times 1} &= \underbrace{\mathbf{GG}^{(2)}}_{2\times 4} \cdot \underbrace{\exp\left(\mathbf{\Gamma\Gamma}^{(2)} y\right)}_{4\times 4} \cdot \underbrace{\mathbf{bb}^{(2)}}_{4\times 1} + \underbrace{\mathbf{KK}^{(2)}}_{2\times 8} \underbrace{\exp\left(\mathbf{\Gamma}^{(2)} y\right)}_{8\times 8} \underbrace{\mathbf{b}^{(2)}}_{4\times 2} + \\ &\quad \underbrace{\mathbf{kk}_0^{(2)}}_{2\times 1} + \underbrace{\mathbf{kk}_1^{(2)}}_{2\times 1} \exp(y) \text{ for } y \in I_2 \\ \underbrace{\mathbf{E}^{(1)}(y)}_{1\times 1} &= \underbrace{\mathbf{GG}^{(1)}}_{1\times 2} \cdot \underbrace{\exp\left(\mathbf{\Gamma\Gamma}^{(1)} y\right)}_{2\times 2} \cdot \underbrace{\mathbf{bb}^{(1)}}_{2\times 1} + {}_{1\times 4}\underbrace{\mathbf{KK}^{(1)}} \underbrace{\exp\left(\mathbf{\Gamma}^{(1)} y\right)}_{4\times 4} \underbrace{\mathbf{b}^{(1)}}_{4\times 1} + \\ &\quad \underbrace{\mathbf{kk}_0^{(1)}}_{1\times 1} + \underbrace{\mathbf{kk}_1^{(1)}}_{1\times 1} \exp(y) \text{ for } y \in I_1\end{aligned}$$

where $\mathbf{GG}^{(i)}$,$\mathbf{\Gamma\Gamma}^{(i)}$, $\mathbf{bb}^{(i)}$,$\mathbf{KK}^{(i)}$, $\mathbf{kk}_0^{(i)}$ and $\mathbf{kk}_1^{(i)}$ for $i \in \{1, 2\}$ are given below. In particular, the constant vector $\mathbf{bb}^{(i)}$ is determined by boundary conditions similar to those for debt.

1.4 Defaulted bonds

Consider now a defaulted bond. Its value $D_l^i(y)$ will be determined by the holding cost $hc_{def}^s(y) = \chi_s^{def}\left[N^{def} - P_s^{def}(y)\right]$. Let $\mathbf{V}(y) = \text{diag}\left(\left[v_U^G(y), v_U^G(y), v_U^B(y), v_U^B(y)\right]\right)$. Then, we can write out

$$\hat{\mathbf{R}} \cdot \mathbf{D}^{def}(y) = \theta\left[\mathbf{V}(y)\boldsymbol{\alpha} - \mathbf{D}^{def}\right] + \hat{\mathbf{Q}}^{def} \cdot \mathbf{D}^{def}(y) - \boldsymbol{\chi}^{def}\left[\mathbf{N}^{def} - \mathbf{W}^{def}\mathbf{D}^{def}(y)\right] \quad (11)$$

which is solved by

$$\mathbf{D}^{def}(y) = \left[\hat{\mathbf{R}} + \theta \mathbf{I}_n - \hat{\mathbf{Q}}^{def} - \boldsymbol{\chi}^{def}\mathbf{W}^{def}\right]^{-1}\left[\theta \mathbf{V}(y)\boldsymbol{\alpha} - \boldsymbol{\chi}^{def}\mathbf{N}^{def}\right]$$

Suppose now that $\mathbf{N}^{def} = \mathbf{V}(y)\,\mathbf{n}^{def}$. Then, we have

$$\mathbf{D}^{def}(y) = \left[\hat{\mathbf{R}} + \theta \mathbf{I}_n - \hat{\mathbf{Q}}^{def} - \boldsymbol{\chi}^{def}\mathbf{W}^{def}\right]^{-1} \mathbf{V}(y)\left[\theta\boldsymbol{\alpha} - \boldsymbol{\chi}^{def}\mathbf{n}^{def}\right] \tag{12}$$

where we used that fact that $\boldsymbol{\chi}^{def}\mathbf{V}(y) = \mathbf{V}(y)\,\boldsymbol{\chi}^{def}$ as both are diagonal matrices.

1.5 Generalization to n aggregate states and solution

We follow the Markov-modulated dynamics approach of Jobert and Rogers (2006).

We note that there are multiple possible bankruptcy boundaries, $y_{def}(s)$, for each aggregate state s one boundary. Order states s such that $s > s'$ implies that $y_{def}(s) > y_{def}(s')$ and denote the intervals $I_s = [y_{def}(s), y_{def}(s+1)]$ where $y_{def}(n+1) = \infty$, so that $I_s \cap I_{s+1} = y_{def}(s+1)$. Finally, let $\boldsymbol{y}_b = [y_{def}(1), ..., y_{def}(n)]^\top$ be the vector of bankruptcy boundaries.

It is important to have a clean notational arrangement to handle the proliferation of states. Let $D_l^{(s)}$ denote the value of debt for a creditor in individual liquidity state l and with aggregate state s. We will use the following notation: $D_l^{(s,i)} \equiv D_l^{(s)}, y \in I_i$, that is $D_l^{(s,i)}$ is the restriction of $D_l^{(s)}$to the interval I_i. It is now clear that $D_l^{(s,i)} = recovery$ for any $i < s$, as it would imply that the company immediately defaults in interval I_i for state s. Let us, for future reference, call debt in states $i < s$ *dead* and in states $i \geq s$ *alive*. Finally, let us stack the alive functions along states s but still restricted to interval i so that $\mathbf{D}^{(i)} = \left[D_H^{(1,i)}, D_L^{(1,i)}, ..., D_H^{(i,i)}, D_L^{(i,i)}\right]^\top$ where $D_l^{(s,i)}$ has s denoting the state, i denotes the interval and l denotes the individual liquidity state. The separation of s and i will clarify the pasting arguments that apply when y crosses from one interval to the next. Let

$$\underbrace{\mathbf{I}_i}_{i\times i} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \tag{13}$$

be the i-dimensional identity matrix, and let

$$\underbrace{\mathbf{1}_i}_{i\times 1} = [1, ..., 1]^\top \tag{14}$$

be a column vector of ones.

Fundamental parameters. Let $\hat{\mathbf{R}} = \mathrm{diag}\left([r_1, r_1, ..., r_n, r_n]\right)$ and let $\mathbf{RR} = \mathrm{diag}\left([r_1, ..., r_n]\right)$. Let $\hat{\mathbf{Q}}$ summarize the OTC-augmented transition intensities. Define the building block matrices $\mathbf{B}_s^{W_1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B}_s^{W_2} = \begin{bmatrix} 0 & 0 \\ w_H & -w_H \end{bmatrix}$ so that $\mathbf{B}_s^{W} = \begin{bmatrix} 0 & 0 \\ w_H & w_L \end{bmatrix} = \mathbf{B}_s^{W_1} + \mathbf{B}_s^{W_2}$. Then, we have

$$\mathbf{W}_1 = \mathrm{diag}\left(\left[\mathbf{B}_1^{W_1}, ..., \mathbf{B}_n^{W_1}\right]\right), \mathbf{W}_2 = \mathrm{diag}\left(\left[\mathbf{B}_1^{W_2}, ..., \mathbf{B}_n^{W_2}\right]\right), \boldsymbol{\chi} = \mathrm{diag}\left([\chi_1, \chi_1, ..., \chi_n, \chi_n]\right)$$

and we define the diagonal discount matrix $\mathbf{R}$ and the transition matrix $\mathbf{Q}$ as

$$\mathbf{R} = \hat{\mathbf{R}} - \boldsymbol{\chi}\mathbf{W}_1, \mathbf{Q} = \hat{\mathbf{Q}} + \boldsymbol{\chi}\mathbf{W}_2$$

Let $\boldsymbol{\mu} = \mathrm{diag}\left([\mu_1, \mu_1, ..., \mu_n, \mu_n]\right)$, $\boldsymbol{\Sigma} = \mathrm{diag}\left([\sigma_1^2, \sigma_1^2, ..., \sigma_n^2, \sigma_n^2]\right)$.

Define $\mathbf{X}^{(n)} = \mathbf{X}$, that is the restriction to the set $I_n = [y_{def}(n), \infty)$ does not change any of the matrices. Let $\mathbf{Q}^{(i)}$ be the transition matrix of jumping into an alive state $s' \leq i$ when $y \in I_i$ and in an alive state $s \leq i$. Let $\tilde{\mathbf{Q}}^{(i)}$ be the transition matrix of jumping into a default state $s' > i$ when $y \in I_i$ and in an alive state $s \leq i$, so that $\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}^{(n)} = 0$. The decomposition for $y \in I_i$ takes the following form

$$\begin{aligned}
\underbrace{\mathbf{Q}_{[1:2i]}}_{2i\times 2n} &= \left[\underbrace{\mathbf{Q}^{(i)} - \mathrm{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right)}_{2i\times 2i} \Big| \underbrace{\tilde{\mathbf{Q}}^{(i)}}_{2i\times 2(n-i)}\right] \\
\underbrace{\mathbf{R}_{[1:2i]}}_{2i\times 2n} &= \left[\underbrace{\mathbf{R}^{(i)} - \mathrm{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right)}_{2i\times 2i} \Big| \underbrace{\mathbf{0}_{2i\times 2(n-i)}}_{2i\times 2(n-i)}\right]
\end{aligned}$$

so that $\mathbf{Q}^{(i)}, \tilde{\mathbf{Q}}^{(i)}, \mathbf{R}^{(i)}, \boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma}^{(i)}$ are given by

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(i)} &= \mathbf{Q}_{[1:2i,2i+1:2n]} \\
\mathbf{Q}^{(i)} &= \mathbf{Q}_{[1:2i,1:2i]} + \operatorname{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right) \\
\mathbf{R}^{(i)} &= \mathbf{R}_{[1:2i,1:2i]} + \operatorname{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right) \\
\boldsymbol{\mu}^{(i)} &= \boldsymbol{\mu}_{[1:2i,1:2i]} \\
\boldsymbol{\Sigma}^{(i)} &= \boldsymbol{\Sigma}_{[1:2i,1:2i]}
\end{aligned}$$

where $\mathbf{X}_{[1:k]}$ takes the first k rows of matrix $\mathbf{X}$ and $\mathbf{X}_{[1:k,1:l]}$ takes the first k rows and l columns of matrix $\mathbf{X}$. Note that in the $n = 2$ aggregate state case, we have $\tilde{\mathbf{Q}}^{(1)} = \operatorname{diag}\left[\tilde{\mathbf{Q}}^{(1)}\mathbf{1}_2\right]$ as we have no joint jumps in liquidity state and in the aggregate state.

Let $\tilde{\mathbf{v}}^{(i)}\exp(y)$ be the recovery or salvage value of the firm when default is declared in states $s > i$ for $y \in I_i$. Let $\mathbf{v}^{(i)}$ be the vector unlevered values for states $(i+1, ..., n) \times (H, L)$ (i.e., it is of dimension $2(n-i) \times 1$), and let $\boldsymbol{\alpha}^{(i)}$ be the effective recovery ratios for those same states. Then, we have $\tilde{\mathbf{v}}^{(i)} = \boldsymbol{\alpha}^{(i)} \operatorname{diag}\left(\mathbf{v}^{(i)}\right)$.

Debt valuation within an interval I_i. Debt valuation follows the following differential equation on interval I_i:

$$\begin{aligned}
\left(\mathbf{R}^{(i)} + m\mathbf{I}_i - \mathbf{Q}^{(i)}\right)\mathbf{D}^{(i)} &= \boldsymbol{\mu}^{(i)}\left(\mathbf{D}^{(i)}\right)' + \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\left(\mathbf{D}^{(i)}\right)'' + 1_{\{i<n\}}\tilde{\mathbf{Q}}^{(i)}\tilde{\mathbf{v}}^{(i)}\exp(y) \\
&\quad + \left(c\mathbf{1}_{2i} - \boldsymbol{\chi}^{(i)}\mathbf{N}^{(i)}\right) + m \cdot p\mathbf{1}_{2i}
\end{aligned} \tag{15}$$

where $\tilde{\mathbf{Q}}^{(i)}\tilde{\mathbf{v}}^{(i)}\exp(y)$ represents the intensity of jumping into default times the recovery in the default state and $m \cdot p\mathbf{1}_{2i}$ represents the intensity of randomly maturing times the payoff in the maturity state. Next, let us conjecture a solution of the kind $\mathbf{g}\exp(\gamma y) + \mathbf{k}_0^{(i)} + \mathbf{k}_1^{(i)}\exp(y)$ where $\mathbf{g}$ is a vector and γ is a scalar. The particular part stemming from $\mathbf{c}^{(i)}$ is solved by a term $\mathbf{k}_0^{(i)}$ with

$$\underbrace{\mathbf{k}_0^{(i)}}_{2i\times 1} = \underbrace{\left(\mathbf{R}^{(i)} + m\mathbf{I}_i - \mathbf{Q}^{(i)}\right)^{-1}}_{2i\times 2i}\underbrace{(c + m \cdot p)\,\mathbf{1}_{2i} - \boldsymbol{\chi}^{(i)}\mathbf{N}^{(i)}}_{2i\times 1} \tag{16}$$

and the particular part stemming from $\tilde{\mathbf{Q}}^{(i)}\tilde{\mathbf{v}}^{(i)}$ is solved by a term $\mathbf{k}_1^{(i)} \exp(y)$ with

$$\underbrace{\mathbf{k}_1^{(i)}}_{2i\times 1} = \underbrace{\left(\mathbf{R}^{(i)} + m\mathbf{I}_i - \mathbf{Q}^{(i)} - \boldsymbol{\mu}^{(i)} - \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\right)^{-1}}_{2i\times 2i} \underbrace{\tilde{\mathbf{Q}}^{(i)}}_{2i\times 2(n-i)} \underbrace{\tilde{\mathbf{v}}^{(i)}}_{2(n-i)\times 1} \tag{17}$$

It should be clear that $\mathbf{k}_1^{(n)} = \mathbf{0}$ as on I_n there is no jump in the aggregate state that would result in immediate default. Plugging in, dropping the $\mathbf{c}^{(i)}$ and $\tilde{\mathbf{Q}}^{(i)}\tilde{\mathbf{v}}^{(i)} \exp(y)$ terms, canceling out $\exp(\gamma y) > 0$, we have

$$\mathbf{0}_{2i} = \left(\mathbf{Q}^{(i)} + m\mathbf{I}_i - \mathbf{R}^{(i)}\right)\mathbf{g} + \boldsymbol{\mu}^{(i)}\gamma\mathbf{g} + \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\gamma^2\mathbf{g} \tag{18}$$

Following JR06, we premultiply by $2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}$ and define $\mathbf{h} = \gamma\mathbf{g}$ to get

$$\gamma\mathbf{g} = \mathbf{h} \tag{19}$$

$$\gamma\mathbf{h} = -2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu}^{(i)}\mathbf{h} + 2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\left(\mathbf{R}^{(i)} + m\mathbf{I}_i - \mathbf{Q}^{(i)}\right)\mathbf{g} \tag{20}$$

Stacking the vectors $\mathbf{j} = \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix}$ we have

$$\gamma\mathbf{j} = \begin{bmatrix} \mathbf{0}_{2i} & \mathbf{I}_{2i} \\ 2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\left(\mathbf{R}^{(i)} + m\mathbf{I}_i - \mathbf{Q}^{(i)}\right) & -2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu}^{(i)} \end{bmatrix}\mathbf{j} = \underbrace{\mathbf{A}^{(i)}}_{4i\times 4i}\mathbf{j} \tag{21}$$

where $\mathbf{I}$ is of appropriate dimensions. The problem is now a simple eigenvalue-eigenvector problem and each solution j is a pair $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1}, \underbrace{\mathbf{j}_j^{(i)}}_{4i\times 1}\right)$ (or rather $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1}, \underbrace{\mathbf{g}_j^{(i)}}_{2i\times 1}\right)$, as the vector $\mathbf{j}_j^{(i)}$ contains the same information as $\mathbf{g}_j^{(i)}$ when we know $\gamma_j^{(i)}$, so we discard the lower half of $\mathbf{j}_j^{(i)}$). The number of solutions j to this eigenvector-eigenvalue problem is $4i$. Let

$$\mathbf{G}^{(i)} \equiv \left[\mathbf{g}_1^{(i)}, ..., \mathbf{g}_{2\times 2\times i}^{(i)}\right] \tag{22}$$

be the matrix of eigenvectors, and let

$$\begin{aligned}\boldsymbol{\gamma}^{(i)} &\equiv \left[\gamma_1^{(i)}, ..., \gamma_{2\times2\times i}^{(i)}\right]' &(23)\\ \boldsymbol{\Gamma}^{(i)} &\equiv \operatorname{diag}\left[\boldsymbol{\gamma}^{(i)}\right] &(24)\end{aligned}$$

be the corresponding vector and diagonal matrix, respectively, of eigenvalues.

The general solution on interval i is thus

$$\underbrace{\mathbf{D}^{(i)}}_{2i\times1} = \underbrace{\mathbf{G}^{(i)}}_{2i\times4i} \cdot \underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)}_{4i\times4i} \cdot \underbrace{\mathbf{c}^{(i)}}_{4i\times1} + \underbrace{\mathbf{k}_0^{(i)}}_{2i\times1} + \underbrace{\mathbf{k}_1^{(i)}}_{2i\times1} \exp(y) \tag{25}$$

where the constants $\mathbf{c}^{(i)} = \left[c_1^{(i)}, ..., c_{4i}^{(i)}\right]^\top$ will have to be determined via conditions at the boundaries of interval I_i (**NOTE:** $c_j^{(i)} \neq c$ where c is the coupon payment).

Boundary conditions. The different value functions $\mathbf{D}^{(i)}$ for $i \in \{1, ..., n\}$are linked at the boundaries of their domains I_i. Note that $I_i \cap I_{i+1} = \{y_{def}(i+1)\}$ for $i < n$.

For $i = n$, we can immediately rule out all positive solutions to γ as debt has to be finite and bounded as $y \to \infty$, so that the entries of $\mathbf{C}^{(n)}$ corresponding to positive eigenvalues will be zero:[2]

$$\lim_{y\to\infty} \left|\mathbf{D}^{(n)}(y)\right| < \infty \tag{26}$$

For $i < n$, we must have value matching of the value functions that are alive across the boundary, and we must have value matching of the value functions that die across the boundary:

$$\mathbf{D}^{(i+1)}(y_{def}(i+1)) = \begin{bmatrix} \mathbf{D}^{(i)}(y_{def}(i+1)) \\ \begin{bmatrix} \tilde{v}_H^{i+1} \\ \tilde{v}_L^{i+1} \end{bmatrix} \exp(y_{def}(i+1)) \end{bmatrix} \tag{27}$$

For $i < n$, we must have mechanical (i.e. non-optimal) smooth pasting of the value functions

[2] According to JR06, there are exactly $2 \times |S| = 2n$ eigenvalues of $\mathbf{A}$ in the left open half plane (i.e. negative) and $2n$ eigenvalues in the right open half plane (i.e. positive).

that are alive across the boundary:

$$\left(\mathbf{D}^{(i+1)}\right)'\left(y_{def}\left(i+1\right)\right)_{[1:2i]}=\left(\mathbf{D}^{(i)}\right)'\left(y_{def}\left(i+1\right)\right) \tag{28}$$

where $\mathbf{x}_{[1\ldots 2i]}$ selects the first $2i$ rows of vector $\mathbf{x}$.

Lastly, for $i=1$, we must have

$$\mathbf{D}^{(1)}\left(y_{def}\left(1\right)\right)=\begin{bmatrix}\tilde{v}_H^1\\ \tilde{v}_L^1\end{bmatrix}\exp\left(y_{def}\left(1\right)\right) \tag{29}$$

Full solution. We can now state the full solution to the debt valuation given cut-off strategies:

Proposition 1. *The debt value functions* $\mathbf{D}$ *for a given default vector* $\boldsymbol{y}_B$ *are*

$$\mathbf{D}\left(y\right)=\begin{cases}\underbrace{\mathbf{D}^{(n)}\left(y\right)}_{2n\times 1}=\mathbf{G}^{(n)}\cdot\exp\left(\boldsymbol{\Gamma}^{(n)}y\right)\cdot\mathbf{c}^{(n)}+\mathbf{k}_0^{(n)} & y\in I_n\\ \vdots & \vdots\\ \underbrace{\mathbf{D}^{(i)}\left(y\right)}_{2i\times 1}=\mathbf{G}^{(i)}\cdot\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\cdot\mathbf{c}^{(i)}+\mathbf{k}_0^{(i)}+\mathbf{k}_1^{(i)}\exp\left(y\right) & y\in I_i\\ \vdots & \vdots\\ \underbrace{\mathbf{D}^{(1)}\left(y\right)}_{2\times 1}=\mathbf{G}^{(1)}\cdot\exp\left(\boldsymbol{\Gamma}^{(1)}y\right)\cdot\mathbf{c}^{(1)}+\mathbf{k}_0^{(1)}+\mathbf{k}_1^{(1)}\exp\left(y\right) & y\in I_1\end{cases}$$

with the following boundary conditions to pin down vectors $\mathbf{c}^{(i)}$*:*

$$\lim_{y\to\infty}\left|\underbrace{\mathbf{D}^{(n)}(y)}_{2n\times 1}\right| < \infty \tag{30}$$

$$\underbrace{\mathbf{D}^{(i+1)}(y_{def}(i+1))}_{2(i+1)\times 1} = \underbrace{\begin{bmatrix} \mathbf{D}^{(i)}(y_{def}(i+1)) \\ \begin{bmatrix} \tilde{v}_H^{i+1} \\ \tilde{v}_L^{i+1} \end{bmatrix} \exp(y_{def}(i+1)) \end{bmatrix}}_{2(i+1)\times 1} \tag{31}$$

$$\underbrace{\left(\mathbf{D}^{(i+1)}\right)'(y_{def}(i+1))_{[1:2i]}}_{2i\times 1} = \underbrace{\left(\mathbf{D}^{(i)}\right)'(y_{def}(i+1))}_{2i\times 1} \tag{32}$$

$$\underbrace{\mathbf{D}^{(1)}(y_{def}(1))}_{2\times 1} = \underbrace{\begin{bmatrix} \tilde{v}_H^{1} \\ \tilde{v}_L^{1} \end{bmatrix} \exp(y_{def}(1))}_{2\times 1} \tag{33}$$

where $\mathbf{x}_{[1:2i]}$ *selects the first* $2i$ *rows of vector* $\mathbf{x}$*.*

Note that the derivative of the debt value vector is

$$\underbrace{\left(\mathbf{D}^{(i)}\right)'(y)}_{2i\times 1} = \mathbf{G}^{(i)}\mathbf{\Gamma}^{(i)} \cdot \exp\left(\mathbf{\Gamma}^{(i)}y\right) \cdot \mathbf{c}^{(i)} + \mathbf{k}_1^{(i)} \exp(y) \tag{34}$$

where we note that $\mathbf{\Gamma}^{(i)} \cdot \exp\left(\mathbf{\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma}^{(i)}y\right) \cdot \mathbf{\Gamma}^{(i)}$ as both are diagonal matrices (although this interchangeability only is important when $s = 1$ as it then helps collapse some equations).

The first boundary condition (30) essentially implies that we can discard any positive entries of $\boldsymbol{\gamma}^{(n)}$ by setting the appropriate coefficients of $\mathbf{C}^{(n)}$ to 0. The second boundary condition (31) implies that we have value matching at any boundary $y_{def}(i+1)$ for $i < n$, be it to a continuation state or a bankruptcy state. The third boundary condition (32) implies that we also have smooth pasting at the boundary $y_{def}(i+1)$ for those states in which the firm stays alive on both sides of the boundary. Finally, the fourth boundary condition (33) implies value matching at the boundary $y_{def}(1)$, but of course only for those states in which the firm is still alive.

1.6 Equity

The equity holders are unaffected by the individual liquidity shocks the debt holders are exposed to. The only shocks the equity holders are directly exposed to are the shifts in $\mu(s)$ and $\sigma(s)$, i.e. shifts to the cash-flow process.

However, as debt has finite maturity and has to be rolled over by assumption, equity holders are indirectly affected by liquidity shocks in the market through the effect it has on debt prices. Thus, when debt matures, it is either rolled over if the debt holders are of type H, or it is reissued to different debt holders in the case that the former debt holder is of type L. Either way, there is a cash flow (inflow or outflow) of $m\left[\mathbf{S}^{(i)}\cdot\mathbf{D}^{(i)}(y)-p\mathbf{1}_i\right]$ at each instant as a mass $m\cdot dt$ of debt holders matures on $[t,t+dt]$.

For notational ease, we will denote by double letters (e.g. $\mathbf{xx}$) a constant for equity that takes a similar place as a single letter (i.e. $\mathbf{x}$) constant for debt. Then, the HJB for equity on interval I_i is given by

$$\begin{aligned}\left(\mathbf{RR}^{(i)}-\mathbf{QQ}^{(i)}\right)\mathbf{E}^{(i)}(y) &= \boldsymbol{\mu\mu}^{(i)}\left(\mathbf{E}^{(i)}\right)'(y)+\frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\left(\mathbf{E}^{(i)}\right)''(y)+1_{\{i<n\}}\underbrace{\widetilde{\mathbf{QQ}}^{(i)}\mathbf{0}_i}_{Default}\\ &\quad+\underbrace{\mathbf{1}_i\exp(y)}_{Cashflow}-\underbrace{(1-\pi)\,c\mathbf{1}_i}_{Coupon}+\underbrace{m\left[\mathbf{S}^{(i)}\cdot\mathbf{D}^{(i)}(y)-p\mathbf{1}_i\right]}_{Rollover}\end{aligned} \tag{35}$$

where

$$\widetilde{\mathbf{QQ}}^{(i)} = \mathbf{QQ}_{[1:i,i+1:n]} \tag{36}$$

$$\mathbf{QQ}^{(i)} = \mathbf{QQ}_{[1:i,1:i]}+\operatorname{diag}\left(\widetilde{\mathbf{QQ}}^{(i)}\mathbf{1}_{n-i}\right) \tag{37}$$

$$\mathbf{RR}^{(i)} = \mathbf{RR}_{[1:i,1:i]}+\operatorname{diag}\left(\widetilde{\mathbf{QQ}}^{(i)}\mathbf{1}_{n-i}\right) \tag{38}$$

$$\boldsymbol{\mu\mu}^{(i)} = \boldsymbol{\mu\mu}_{[1:i,1:i]} \tag{39}$$

$$\boldsymbol{\Sigma\Sigma}^{(i)} = \boldsymbol{\Sigma\Sigma}_{[1:i,1:i]} \tag{40}$$

Note that $\widetilde{\mathbf{QQ}}^{(i)}$ is the transition matrix only between aggregate states that is also an $i\times i$ square matrix, and $\mathbf{S}^{(i)}$ is a $i\times 2i$ matrix that selects which debt values the firm is able to issue (each row has to sum to 1), and m is a scalar (**NOTE**: In contrast to $\mathbf{R}$, the matrix

$\mathbf{RR}$ does not contain the maturity intensity m). For example, for $i=2$, if the company is able to place debt only to H types, then $\mathbf{S}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. It is important that for reach row i only entries $2i-1$ and $2i$ are possibly nonzero, whereas all other entries are identically zero (otherwise, one would issue bonds belonging to a different state).

Writing out $\mathbf{D}^{(i)}(y) = \mathbf{G}^{(i)} \exp\left(\mathbf{\Gamma}^{(i)} y\right) \mathbf{c}^{(i)}$and conjecturing a solution to the particular, non-constant part $\underbrace{\mathbf{KK}^{(i)}}_{i\times 4i} \underbrace{\exp\left(\mathbf{\Gamma}^{(i)} y\right)}_{4i\times 4i} \underbrace{\mathbf{c}^{(i)}}_{4i\times 1}$, we have

$$\begin{aligned} & \left(\widetilde{\mathbf{RR}}^{(i)} - \widetilde{\mathbf{QQ}}^{(i)}\right) \mathbf{KK}^{(i)} \exp\left(\mathbf{\Gamma}^{(i)} y\right) \mathbf{c}^{(i)} \\ = & \left[\boldsymbol{\mu\mu}^{(i)} \cdot \mathbf{KK}^{(i)} \cdot \mathbf{\Gamma}^{(i)} + \frac{1}{2} \mathbf{\Sigma\Sigma}^{(i)} \mathbf{KK}^{(i)} \cdot \left(\mathbf{\Gamma}^{(i)}\right)^2 + m \cdot \mathbf{S}^{(i)} \cdot \mathbf{G}^{(i)}\right] \exp\left(\mathbf{\Gamma}^{(i)} y\right) \mathbf{c}^{(i)} \end{aligned} \tag{41}$$

We can solve this by considering each $\gamma_j^{(i)}$ separately — recall that $\mathbf{c}^{(i)}$ is a vector and $\exp\left(\mathbf{\Gamma}^{(i)} y\right)$ is a *diagonal* matrix and in total there are $4i$ different roots. Consider the part of the particular part $\mathbf{S}^{(i)} \cdot \mathbf{g}_j^{(i)} \exp\left(\gamma_j^{(i)} y\right) \cdot c_j^{(i)}$ and our conjecture gives $\underbrace{\mathbf{KK}_j^{(i)}}_{i\times 1} \underbrace{\exp\left(\gamma_j^{(i)} y\right)}_{1\times 1} \cdot \underbrace{c_j^{(i)}}_{1\times 1}$ for each root $j \in [1, ..., 4i]$. Plugging in and multiplying out the scalar $\exp\left(\gamma_j^{(i)} y\right) c_j^{(i)}$, we find that

$$\left(\widetilde{\mathbf{RR}}^{(i)} - \widetilde{\mathbf{QQ}}^{(i)}\right) \mathbf{KK}_j^{(i)} = \boldsymbol{\mu\mu}^{(i)} \cdot \mathbf{KK}_j^{(i)} \cdot \gamma_j^{(i)} + \frac{1}{2} \mathbf{\Sigma\Sigma}^{(i)} \mathbf{KK}_j^{(i)} \cdot \left(\gamma_j^{(i)}\right)^2 + m \cdot \mathbf{S}^{(i)} \cdot \mathbf{g}_j^{(i)} \tag{42}$$

Solving for $\mathbf{KK}_j^{(i)}$, we have

$$\underbrace{\mathbf{KK}_j^{(i)}}_{i\times 1} = \underbrace{\left[\widetilde{\mathbf{RR}}^{(i)} - \widetilde{\mathbf{QQ}}^{(i)} - \boldsymbol{\mu\mu}^{(i)} \cdot \gamma_j^{(i)} - \frac{1}{2} \mathbf{\Sigma\Sigma}^{(i)} \cdot \left(\gamma_j^{(i)}\right)^2\right]^{-1}}_{i\times i} m \cdot \underbrace{\mathbf{S}^{(i)}}_{i\times 2i} \underbrace{\mathbf{g}_j^{(i)}}_{2i\times 1} \tag{43}$$

Finally, for the homogenous part we use the same approach as above, but now we have less states as the individual liquidity state drops out. Thus, we conjecture $\mathbf{gg} \exp(\gamma\gamma \cdot y)$ to get

$$\mathbf{0}_i = \left(\widetilde{\mathbf{QQ}}^{(i)} - \widetilde{\mathbf{RR}}^{(i)}\right) \mathbf{gg} + \boldsymbol{\mu\mu}^{(i)} \gamma\gamma \mathbf{gg} + \frac{1}{2} \mathbf{\Sigma\Sigma}^{(i)} \gamma\gamma \mathbf{gg} \tag{44}$$

so that, again, we have the following eigenvector eigenvalue problem

$$\gamma\gamma \mathbf{jj} = \left[\begin{array}{cc} \mathbf{0}_i & \mathbf{I}_i \\ 2\left(\boldsymbol{\Sigma\Sigma}^{(i)}\right)^{-1}\left(\widetilde{\mathbf{RR}}^{(i)} - \widetilde{\mathbf{QQ}}^{(i)}\right) & -2\left(\boldsymbol{\Sigma\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu\mu}^{(i)} \end{array} \right] \mathbf{jj} = \underbrace{\mathbf{AA}^{(i)}}_{2i\times 2i}\mathbf{jj} \tag{45}$$

which gives $\left(\gamma\gamma_j^{(i)}, \mathbf{gg}_j^{(i)}\right)$ for $j \in [1, ..., 2i]$ solutions. We stack these into a matrix of eigenvectors $\mathbf{GG}^{(i)}$ and a vector of eigenvalues $\boldsymbol{\gamma\gamma}^{(i)}$, from which we define the diagonal matrix of eigenvalues $\boldsymbol{\Gamma\Gamma}^{(i)} \equiv \text{diag}\left(\boldsymbol{\gamma\gamma}^{(i)}\right)$. What remains is to solve for $\mathbf{kk}_0^{(i)}$ and $\mathbf{kk}_1^{(i)}$. We have

$$\mathbf{kk}_0^{(i)} = \left[\widetilde{\mathbf{RR}}^{(i)} - \widetilde{\mathbf{QQ}}^{(i)}\right]^{-1}\left[-\left(1-\pi\right)c\mathbf{1}_i + m\left(\mathbf{S}^{(i)}\mathbf{k}_0^{(i)} - p\mathbf{1}_i\right)\right] \tag{46}$$

and

$$\mathbf{kk}_1^{(i)} = \left[\widetilde{\mathbf{RR}}^{(i)} - \widetilde{\mathbf{QQ}}^{(i)} - \boldsymbol{\mu\mu}^{(i)} - \frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\right]^{-1}\left(\mathbf{1}_i + m\cdot\mathbf{S}^{(i)}\mathbf{k}_1^{(i)}\right) \tag{47}$$

with $\mathbf{k}_1^{(n)} = \mathbf{0}$.

We are left with the following proposition.

Proposition 2. *The equity value functions* $\mathbf{E}$ *for a given default vector* $\boldsymbol{y}_B$ *are*

$$\mathbf{E}(y) = \begin{cases} \underbrace{\mathbf{E}^{(n)}(y)}_{n\times 1} = \mathbf{GG}^{(n)}\cdot\exp\left(\boldsymbol{\Gamma\Gamma}^{(n)}y\right)\cdot\mathbf{cc}^{(n)} + \mathbf{KK}^{(n)}\exp\left(\boldsymbol{\Gamma}^{(n)}y\right)\mathbf{c}^{(n)} + \mathbf{kk}_0^{(n)} + \mathbf{kk}_1^{(n)}\exp(y) & y \in I_n \\ \vdots & \vdots \\ \underbrace{\mathbf{E}^{(i)}(y)}_{i\times 1} = \mathbf{GG}^{(i)}\cdot\exp\left(\boldsymbol{\Gamma\Gamma}^{(i)}y\right)\cdot\mathbf{cc}^{(i)} + \mathbf{KK}^{(i)}\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)} + \mathbf{kk}_0^{(i)} + \mathbf{kk}_1^{(i)}\exp(y) & y \in I_i \\ \vdots & \vdots \\ \underbrace{\mathbf{E}^{(1)}(y)}_{1\times 1} = \mathbf{GG}^{(1)}\cdot\exp\left(\boldsymbol{\Gamma\Gamma}^{(1)}y\right)\cdot\mathbf{cc}^{(1)} + \mathbf{KK}^{(1)}\exp\left(\boldsymbol{\Gamma}^{(1)}y\right)\mathbf{c}^{(1)} + \mathbf{kk}_0^{(1)} + \mathbf{kk}_1^{(1)}\exp(y) & y \in I_1 \end{cases}$$

with the following boundary conditions to pin down the vector $\mathbf{cc}^{(i)}$*:*

$$\lim_{y\to\infty}\left|\underbrace{\mathbf{E}^{(n)}(y)\exp(-y)}_{n\times 1}\right| < \infty \tag{48}$$

$$\underbrace{\mathbf{E}^{(i+1)}(y_{def}(i+1))}_{(i+1)\times 1} = \underbrace{\begin{bmatrix}\mathbf{E}^{(i)}(y_{def}(i+1)) \\ 0\end{bmatrix}}_{(i+1)\times 1} \tag{49}$$

$$\underbrace{\left(\mathbf{E}^{(i+1)}\right)'(y_{def}(i+1))_{[1:i]}}_{i\times 1} = \underbrace{\left(\mathbf{E}^{(i)}\right)'(y_{def}(i+1))}_{i\times 1} \tag{50}$$

$$\underbrace{\mathbf{E}^{(i)}(y_{def}(1))}_{i\times 1} = 0 \tag{51}$$

where $\mathbf{x}_{[1:i]}$ *selects the first* i *rows of vector* $\mathbf{x}$*.*

Note first the dimensionalities: $\underbrace{\mathbf{\Gamma\Gamma}^{(i)}}_{2i\times 2i}, \underbrace{\mathbf{GG}^{(i)}}_{i\times 2i}$ and $\underbrace{\mathbf{\Gamma}^{(i)}}_{4i\times 4i}, \underbrace{\boldsymbol{G}^{(i)}}_{2i\times 4i}$. Note second the derivative of the equity value vector is

$$\underbrace{\left(\mathbf{E}^{(i)}\right)'(y)}_{i\times 1} = \mathbf{GG}^{(i)}\mathbf{\Gamma\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right)\cdot\mathbf{cc}^{(i)} + \mathbf{KK}^{(i)}\mathbf{\Gamma}^{(i)}\exp\left(\mathbf{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)} + \mathbf{kk}_1^{(i)}\exp(y) \tag{52}$$

where we note that $\mathbf{\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{\Gamma}^{(i)}$ and $\mathbf{\Gamma\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right)\cdot \mathbf{\Gamma\Gamma}^{(i)}$as both are diagonal matrices (although this interchangeability only is important when $s = 1$ as it then helps collapse some equations).

The optimality conditions for bankruptcy boundaries $\{y_{def}(i)\}_i$ are given by

$$\left(\mathbf{E}^{(i)}\right)'(y_{def}(i))_{[i]} = 0 \tag{53}$$

i.e., a smooth pasting condition at the boundaries at which default is declared.

Finally, we summarize all the different matrices and vectors involved in the derivation of the propositions in Table 1.

Table 1: Matrix & Vector Dimensions.

Debt Parameters			Equity Parameters		
Symbol	Interpretation	Dimension	Symbol	Interpretation	Dimension
$\mathbf{D}^{(i)}(y)$	Debt Value Function	$2i \times 1$	$\mathbf{E}^{(i)}(y)$	Equity Value Function	$i \times 1$
$\boldsymbol{\mu}^{(i)}$	(Log)-Drifts	$2i \times 2i$	$\boldsymbol{\mu\mu}^{(i)}$	(Log-)Drifts	$i \times i$
$\boldsymbol{\Sigma}^{(i)}$	Volatilities	$2i \times 2i$	$\boldsymbol{\Sigma\Sigma}^{(i)}$	Volatilities	$i \times i$
$\mathbf{R}^{(i)}$	Discount rates and maturity	$2i \times 2i$	$\mathbf{RR}^{(i)}$	Discount rates	$i \times i$
$\boldsymbol{\chi}^{(i)}$	Holding cost slopes	$2i \times 2i$	c	Coupon	1×1
$\mathbf{N}^{(i)}$	Holding cost intercepts	$2i \times 1$	π	Tax rate	1×1
$\mathbf{Q}^{(i)}$	Transition to cont. states	$2i \times 2i$	$\mathbf{QQ}^{(i)}$	Transition to cont. states	$i \times i$
$\tilde{\mathbf{Q}}^{(i)}$	Transition to default states	$2i \times 2(n-i)$	$\mathbf{AA}^{(i)}$	Matrix to be decomposed	$2i \times 2i$
$\tilde{\mathbf{v}}^{(i)}$	Vector of recovery values	$2(n-i) \times 1$	$\boldsymbol{\Gamma\Gamma}^{(i)}$	Diag matrix of eigenvalues	$2i \times 2i$
$\mathbf{W}^{(i)}$	Mid-point weighting matrix	$2i \times 2i$	$\mathbf{GG}^{(i)}$	Matrix of eigenvectors	$i \times 2i$
$\boldsymbol{\Gamma}^{(i)}$	Diag matrix of eigenvalues	$4i \times 4i$	$\mathbf{kk}_0^{(i)}, \mathbf{kk}_1^{(i)}$	Coeff. of particular sol.	$i \times 1$
$\mathbf{G}^{(i)}$	Matrix of eigenvectors	$2i \times 4i$	$\mathbf{S}^{(i)}$	Issuance matrix	$i \times 2i$
$\mathbf{k}_0^{(i)}, \mathbf{k}_1^{(i)}$	Coeff. of particular sol.	$2i \times 1$	$\mathbf{KK}^{(i)}$	Coeff. of particular sol.	$i \times 4i$
$\mathbf{c}^{(i)}$	Vector of constants	$4i \times 1$	$\mathbf{cc}^{(i)}$	Vector of constants	$2i \times 1$

2 Computing the Model-implied Moments by Ratings

We follow the steps below in computing the model-implied aggregate moments on default probability, credit spreads, and others. These are also described, in more concise form, at the beginning of Section 3.3 in the paper. Due to a proliferation of material in the appendix in this revision, we have so far opted to leave the detailed description given below out of the appendix.

Step 1: Construct the model-implied mapping between market leverage and credit spread. We use Monte-Carlo simulation method to compute the total credit spread for a given maturity and aggregate state on a grid of log cash flow level. We then translate the cash flow level to its model-implied market leverage level through the analytical solution to the model. This gives us a scattered plot of credit spread over market leverage. We then fit a piece-wise hyperbolic curve to the scatters to derive the model-predicted credit spread over the continuous range of market leverage. The result is presented in Figure 3 as the solid line. Consistent with the findings in the previous literature as David (2008), Bhamra, Kuehn, and Strebulaev (2010); credit spread is a convex function of the market leverage in the model.

This means, due to the dispersion in market leverage for a given rating class (See figure 2), average credit spread of the firms for a given rating class will be bigger than the credit spread of the average firm in that group; and the former is empirically observed and therefore shall be the target of our calibration.

Step 2: Matching with the empirical distribution of market leverage in Compustat data. We compute the market leverage (i.e. book debt over the sum of market equity and book debt) of each Compustat firm-quarter observation where we have the rating information. We report the leverage distribution by rating class in Figure 2. We then compute the model-implied credit spread for each empirical firm-quarter observation using the fitted curve we constructed in Step 1. This gives a credit spread value the model predicts for each firm-quarter observation in the data.

Step 3: Aggregation by macro state, maturity and rating class. For a given rating class and maturity, we aggregate all model-implied spreads first by quarter then by aggregate state (Note: We define a quarter to be in the B state if any of its three months is classified as a NBER recession month) to arrive at the model implied moments reported in Table 2.

We used the same method in computing the model-implied moments on default probability and structural liquidity-default decomposition.

3 Calibration With Lower Liquidity Shock Intensities

While we think the broader interpretation of liquidity shocks helps justify the approach of matching liquidity shock intensity with corporate bond market turnover to some extent, we also consider an alternative calibration to demonstrate the sensitivity of our results to this concern. In the new calibration, both ξ_G and ξ_B are reduced by half. Then, we recalibrate the three free parameters N, χ_G, χ_B by targeting the 6 bid-ask spread moments (as discussed in Point 1).

The results from this calibration are reported in Table 2. The re-calibrated model still does a good matching the moments of bid-ask spreads, but the total credit spreads and bond-CDS spreads become significantly lower. The reason is that lowering liquidity shock intensities, all else equal, raises bid-ask spreads (see also Table 4 in the main text on this point). To reduce the bid-ask spreads, the calibration then reduces the holding costs by lowering N, χ_G, χ_B, which, together with lower liquidity shock intensities, reduce the bond-CDS spreads and total credit spreads. We also show the decomposition of these re-calibrated total credit spreads in Table 3. The pure-default components of the spreads under the calibration is similar to the original calibration, while the pure liquidity component, default-driven liquidity component, and the liquidity-driven default components all roughly drop by half.

These findings show that our model's performance is indeed sensitive to the calibration of ξ_G, ξ_B. Put differently, they show that in order to jointly explain the bid-ask spreads and bond-CDS spreads, our model calls for relatively high levels of liquidity shock intensities.

4 Dollar Decomposition

We decompose corporate bond prices into 4 components in a similar way as we decompose the spreads. Specifically, consider the following 5 different bond prices:

1. B^{NL}: The price of a default-free bond in a setting without any liquidity frictions, with maturity T, face value F, and annualized coupon rate c equal to the risk-free rate r. This bond will be priced at par, $B^{NL} = F$.

2. B: The price of the same default-free bond in the presence of liquidity frictions.

3. P: The price of the T-year defaultable bond in the presence of liquidity frictions, using the default policy computed from the benchmark model.

4. P^{NL}: The price of the T-year defaultable bond in a setting without any liquidity frictions, but keeping the default policy the same as the benchmark model.

5. P^{DEF}: The price of the T-year bond in a setting without any liquidity frictions, with an updated default policy that is optimal for the firm in the absence of liquidity frictions.

Table 2: **Comparison of benchmark to re-calibrated $\xi = 0.35, 0.5$ case (10 year bonds).** The "benchmark" case is our benchmark calibration. The " $\xi = 0.35, 0.5$" is when we halve the liquidity shock intensities in both states, and recalibrate the remaining parameters. Thus, this is not a comparative statics exercise.

	Panel A. Credit spreads (bps)							
	State G				State B			
	Aaa/Aa	A	Baa	Ba	Aaa/Aa	A	Baa	Ba
benchmark	86	122	182	301	136	185	261	404
$\xi = 0.35, 0.5$	58	89	143	252	96	136	201	327

	Panel B. Bid-Ask Spreads (bps)					
	State G			State B		
	Superior	Investment	Junk	Superior	Investment	Junk
benchmark	40	47	61	114	137	186
$\xi = 0.35, 0.5$	40	48	62	109	135	186

	Panel C. Bond-CDS Spreads (bps)							
	State G				State B			
	Aaa/Aa	A	Baa	Ba	Aaa/Aa	A	Baa	Ba
benchmark	48	53	61	60	69	79	92	107
$\xi = 0.35, 0.5$	22	24	26	61	31	35	40	42

Table 3: **Structural Decomposition for 10-Year Bonds Across Ratings for re-calibrated $\xi = 0.35, 0.5$ case.** We perform the structural liquidity-default decomposition for a 10-year bond, given rating and aggregate state, and then aggregate over the empirical leverage distribution in Compustat. The reported credit spreads are relative to the risk-free rate.

Rating	State	Credit Spread	Structural Decomposition			
			Pure Def	$Liq \to Def$	*Pure Liq*	$Def \to Liq$
Aaa/Aa	G	43	20	0.5	21	3
	(%)		(45)	(1)	(47)	(6)
	B	56	22	1	29	4
	(%)		(39)	(2)	(52)	(7)
A	G	74	47	1	21	6
	(%)		(63)	(2)	(28)	(8)
	B	96	56	3	29	9
	(%)		(58)	(3)	(30)	(10)
Baa	G	128	94	3	21	10
	(%)		(74)	(2)	(16)	(8)
	B	161	110	5	29	18
	(%)		(68)	(3)	(18)	(11)
Ba	G	237	194	6	21	17
	(%)		(82)	(3)	(9)	(7)
	B	287	219	8	29	31
	(%)		(76)	(3)	(10)	(11)

The price gap between the default-free bond and defaultable bond $B^{NL} - P$ can be viewed as the "value lost" due to default and liquidity. With the set of 5 prices $(B^{NL}, B, P^{DEF}, P^{NL}, P)$, we can now decompose the total gap $B^{NL} - P$ into 4 components:

$$B^{NL} - P = \Delta P^{pureDEF} + \Delta P^{LIQ \to DEF} + \Delta P^{pureLIQ} + \Delta P^{DEF \to LIQ} \tag{54}$$

where

$$\Delta P^{pureDEF} = B^{NL} - P^{DEF} \tag{55}$$

$$\Delta P^{LIQ \to DEF} = (B^{NL} - P^{NL}) - \Delta P^{pureDEF} \tag{56}$$

$$\Delta P^{pureLIQ} = B^{NL} - B \tag{57}$$

$$\Delta P^{DEF \to LIQ} = (P^{NL} - P) - \Delta P^{pureLIQ} \tag{58}$$

After conducting the decomposition at firm level (separately for the good and bad aggregate states), we average the results across firms within each rating class to get the average decompositions by ratings.

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