

Strategic Inventory Placement in Supply Chains: Non-Stationary Demand

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The life cycle of new products is becoming shorter and shorter in all markets. For electronic products, life cycles are measured in units of months, with six to twelve-month life cycles being common. Given these short product life-cycles, product demand is increasingly difficult to forecast. Furthermore, demand is never really stationary as the demand rate evolves over the life of the product. In this paper we consider the problem of where in a supply chain to place strategic safety stocks to provide a high level of service to the final customer with minimum cost. We extend a model that we have developed for stationary demand to the case of non-stationary demand, as might occur for products with short life cycles. Key assumptions are that we can model the supply chain as a network, that each stage in the supply chain operates with a periodic-review base-stock policy, that demand is bounded and that there is a guaranteed service time between every stage and its customers. To model the non-stationary demand, we divide the product life cycle into time phases, where we assume that the demand process is stationary within each time phase. However, the parameters for the demand process will vary from phase to phase. We first establish conditions for which the placement of safety stocks is the same within each phase. As a consequence, the optimization algorithm for the case of stationary demand extends directly to determining the safety stocks within each phase. We then examine what happens to the optimal safety stock policy in the transition from one phase to the next. We provide empirical evidence, as well as a heuristic argument, for a policy that maintains the same safety stock locations through the transition between time phases. We conclude with an illustrative example that demonstrates the proposed solution and how it would be deployed.

1. Introduction

Manufacturing firms are introducing new products at a higher frequency, and these products are on the market for increasingly shorter amounts of time. For each new product introduction, a firm must determine its supply chain and the associated operating policies in order to match the product supply to the product demand to achieve the most revenue and incur the least cost. Major challenges in addressing this design task are the facts that product demand is uncertain and difficult to forecast, and that the demand process evolves over the product life cycle and is never stationary.

The primary intent of this research is to examine the problem of locating safety stocks in a supply chain, in a way that accounts for both uncertain and non-stationary demand processes. Given the inherent complexity of modeling non-stationary demand processes, we seek a pragmatic approach that requires approximations and compromises in order to get results that might apply in practice. We use the modeling framework from Graves and Willems (2000), and propose a demand model that permits consideration of non-stationary demand. Our key result is to show that the optimization from Graves and Willems (2000) for finding the safety-stock placement in a supply chain applies to this case of non-stationary demand.

In the remainder of this section we briefly discuss related literature. In section 2, we present the key assumptions for our framework for modeling a supply chain. In section 3, we review the multi-stage inventory model from Graves and Willems (2000). In section 4, we specify a non-stationary demand process as a sequence of stationary phases. We then show that for a single end-item, the location of the safety stocks within a phase does not depend on the demand parameters. When there are multiple end-items, we provide conditions on the demand process for which this result also applies. In section 5, we examine how to use this result during

the transition from one phase to another phase as the demand process changes. In section 6, we present an example to illustrate how the proposed policy might apply in practice.

Related Literature: In comparison to the inventory models for stationary demand, there has been much less work for the case of non-stationary demand. The work that has been done on non-stationary demand can be characterized by how the non-stationary demand is specified and whether the work focuses on optimization versus performance evaluation.

Morton and Pentico (1995) and Bollapragada and Morton (1999) focus on setting inventory policies for a single stage facing a general non-stationary demand process with proportional holding and backorder costs. When the order cost is zero, a time-varying base-stock policy is optimal; for a non-zero order cost, then a time-varying (s, S) policy is optimal. The optimal policies can be found by dynamic programming. This research focuses on developing computationally efficient upper and lower bounds on the optimal policy.

Ettl et al. (2000) minimize the total expected inventory capital in a multi-stage inventory system, where the key challenge is to approximate the replenishment lead-times within the supply chain. To model non-stationary demand, they assume the horizon can be broken into a set of stationary phases and adopt a rolling-horizon approach where the optimization is performed for each demand phase.

Non-stationary demand has also been modeled as a Markov-modulated Poisson demand process where the demand process is governed by a discrete time Markov chain; for example, a two-state chain corresponding to a high and low period. Chen and Song (2001) and Abhyankar and Graves (2001) are indicative of this approach. Chen and Song (2001) show that echelon base-stock policies with state-dependent order-up-to levels are optimal for serial networks. Abhyankar and Graves (2001) develop an optimization model to determine where best to place

an inventory hedge in a two-stage serial supply chain that faces Markov-modulated demand with two states.

There is a large stream of research that focuses on the bullwhip effect in supply chains. The bullwhip papers that focus on non-stationary demand generally assume each stage follows an adaptive base-stock policy. The papers then analyze the effect that different forecasting techniques and demand distributions have on the inventory requirements at each stage. Lee et al. (1997), Graves (1999) and Chen et al. (2000) are indicative of the work in this area. Lee et al. (1997) demonstrate that the adjustment of order-up-to levels at the retailer amplifies the variance of the order signal the retailer provides the manufacturer. Graves (1999) shows that the relationship between safety stock required and net replenishment lead-time becomes convex when the demand process is an integrated moving average process of order (0, 1, 1). Chen et al. (2000) demonstrate that the bullwhip effect is due, in part, to the demand stage's need to forecast its own mean and standard deviation of demand over the net replenishment lead-time.

Finally, there is a growing body of work focusing on designing supply chains to handle non-stationary demand. Beyer and Ward (2000) use simulation to accurately model the inventory requirements in a two-echelon supply chain that utilizes two modes of distribution. Their model captures the impacts of non-stationary demand and the resulting competition for inventory between stages in the supply chain. Johnson and Anderson (2000) investigate the benefits of postponement in supply chains that introduce multiple products where demands face a product life-cycle pattern. For products with short life cycles, the authors show that implementing postponement can reduce costs while simultaneously increasing service levels.

2. Assumptions

In order to develop our results, we need first to introduce the modeling framework from Graves and Willems (2000), hereafter referenced as G-W, for stationary demand. For completeness, we state the key assumptions, but refer the reader to G-W for discussion and justification of these assumptions.

Multi-Stage Network: We model a supply chain as a network where nodes are stages in the supply chain and a directed arc denotes that an upstream stage supplies a downstream stage. A stage represents a major processing function such as the procurement of a raw material, or the production of a component, or the manufacture of a subassembly, or the assembly and test of a finished good, or the transportation of a finished product from a central distribution center to a regional warehouse. Each stage is a potential location for holding a safety-stock inventory of the item processed at the stage.

We associate with each arc a scalar ϕ_{ij} to indicate how many units of the upstream component i are required per downstream unit j . If a stage is connected to several upstream stages, then its production activity is an assembly requiring inputs from each of the upstream stages. A stage that is connected to multiple downstream stages is either a distribution node or a production activity that produces a common component for multiple internal customers.

Production Lead-Times: For each stage, we assume a known deterministic production lead-time, call it T_i . When a stage reorders, the production lead-time is the time from when all of the inputs are available until production is completed and available to serve demand. The production lead-time includes the waiting and processing time at the stage, plus any transportation time to put the item into inventory. For instance, suppose stage k requires inputs from stage i and j ; then for a

production request made at time t , stage k completes the production at time $t + T_k$, provided that there are adequate supplies of i and j at time t .

We assume that there are no capacity constraints that limit production at a stage. Thus, the production lead-time is invariant to the work-in-process.

Periodic-Review Base-Stock Replenishment Policy: We assume that all stages operate with a periodic-review base-stock replenishment policy with a common review period. Each period, each stage observes demand either from an external customer or from its downstream stages, and places orders on its suppliers so as to replenish the observed demand. There is no time delay in ordering, so each period each stage sees its customer demand.

Demand Process: We assume that external demand occurs only at nodes that have no successors, which we term demand nodes or stages. For each demand node j , G-W assume that the end-item demand comes from a stationary process for which the average demand per period is μ_j .

An internal stage has only internal customers or successors. Since each stage orders according to a base-stock policy, the demand at internal stage i is:

$$d_i(t) = \sum_{(i,j) \in A} \phi_{ij} d_j(t)$$

where $d_j(t)$ denotes the realized demand at stage j in period t and A is the arc set for the network representation of the supply chain. The average demand rate for stage i is:

$$\mu_i = \sum_{(i,j) \in A} \phi_{ij} \mu_j.$$

G-W assume that demand at each stage j is *bounded* by the function $D_j(\tau)$, for $\tau = 1, 2, 3, \dots, M_j$, where M_j is the maximum replenishment time for the stage¹. That is, for any period t and for $\tau = 1, 2, 3, \dots, M_j$, we have

$$D_j(\tau) \geq d_j(t-\tau+1) + d_j(t-\tau+2) + \dots + d_j(t).$$

We define $D_j(0) = 0$ and assume that $D_j(\tau)$ is increasing and concave on $\tau = 1, 2, 3, \dots, M_j$.

Guaranteed Service Times: G-W assume that each demand node j promises a *guaranteed service time* S_j by which the stage j will satisfy customer demand. That is, the customer demand at time t , $d_j(t)$, must be filled by time $t + S_j$. Furthermore, we assume that stage j provides *100% service* for the specified service time: stage j delivers exactly $d_j(t)$ to the customer at time $t + S_j$.

Similarly, an internal stage i quotes and guarantees a service time S_{ij} for each downstream stage j , $(i, j) \in A$. Given a base-stock policy, stage j places an order equal to $\phi_{ij} d_j(t)$ on stage i at time t ; then stage i delivers exactly this amount to stage j at time $t + S_{ij}$.

G-W assume that stage i quotes the same service time to all of its downstream customers, namely $S_{ij} = S_i$ for each downstream stage j , $(i, j) \in A$. We describe in Graves and Willems (1998) how to extend the model to permit customer-specific service times.

The service times for both the end items and the internal stages are *decision variables* for the optimization model, as will be seen in section 3. However, as a model input, we may impose bounds on the service times for each stage. In particular, for each demand node we are given a maximum service time, presumably set by the market place.

¹ The maximum replenishment time for node j is defined as $M_j = T_j + \max \{M_i \mid i:(i,j) \in A\}$.

3. Supply Chain Model

In this section we present the multi-stage model of a supply chain, and the optimization problem for determining safety stocks.

Inventory Model: [see Kimball 1988 or Simpson 1958] The single-stage inventory model serves as the building block for modeling a multi-stage supply chain. We assume the inventory systems starts at time 0 with initial inventory $I_j(0)$. Under the stated assumptions, we can express the inventory at stage j at the end of period t as

$$I_j(t) = B_j - d_j(t - SI_j - T_j, t - S_j) \quad (1)$$

where $B_j = I_j(0) \geq 0$ denotes the base stock, $d_j(a, b)$ denotes the demand at stage j over the time interval $(a, b]$, and SI_j is the inbound service time for stage j . For the discrete-time demand process, we understand $d_j(a, b)$ to be

$$d_j(a, b) = d_j(a+1) + d_j(a+2) + \dots + d_j(b)$$

for $a < b$, where $d_j(t) = 0$ for $t \leq 0$. When $a \geq b$, we define $d_j(a, b) = 0$.

The inbound service time SI_j is the time for stage j to get supplies from its immediate suppliers. In period t , stage j places an order equal to $\phi_{ij} d_j(t)$ on each upstream stage i for which $\phi_{ij} > 0$. The inbound service time is the time for all of these orders to be delivered to stage j , so that stage j can commence production to replenish $d_j(t)$. Thus we have $SI_j = \max_{(i,j) \in A} \{S_i\}$.

Determination of Base Stock: In order for stage j to provide 100% service to its customers, we require that $I_j(t) \geq 0$; we see from (1) that this requirement equates to

$$B_j \geq d_j(t - SI_j - T_j, t - S_j).$$

Since demand is bounded, we can satisfy the above requirement with the least inventory by setting the base stock as:

$$B_j = D_j(\tau) \quad \text{where } \tau = \max [0, SI_j + T_j - S_j] = [SI_j + T_j - S_j]^+. \quad (2)$$

Thus, the base stock is the maximum possible demand over the *net* replenishment time for the stage, equal to its replenishment time ($SI_j + T_j$) minus its quoted service time (S_j).

Safety Stock Model: We use (1) and (2) to find the expected inventory level $E[I_j]$:

$$E[I_j] = B_j - E[d_j(t - SI_j - T_j, t - S_j)] = D_j(SI_j + T_j - S_j) - [SI_j + T_j - S_j]^+ \mu_j. \quad (3)$$

The expected inventory represents the safety stock held at stage j , and depends on the net replenishment time and the demand bound.

The expected work-in-process depends only on the lead-time at stage j and is not a function of the service times. Hence, in posing an optimization problem, we ignore the pipeline inventory and only model the safety stock.

Multi-Stage Model: To model the multi-stage system, we use (3) for every stage, but where the inbound service time is a function of the outbound service times for the upstream stages; to wit, the model for stage j is

$$E[I_j] = D_j(SI_j + T_j - S_j) - (SI_j + T_j - S_j) \mu_j \quad (4)$$

$$SI_j + T_j - S_j \geq 0 \quad (5)$$

$$SI_j - S_i \geq 0 \quad \text{for all } (i, j) \in A \quad (6)$$

where (5) assures, without loss of generality, that the net replenishment time is nonnegative, and (6) constrains the inbound service time to equal or exceed the service times for the upstream stages.

This suggests the following optimization problem **P** for finding the optimal service times:

$$\begin{aligned}
 \mathbf{P} \quad & \min \sum_{j=1}^N h_j \left\{ D_j (SI_j + T_j - S_j) - (SI_j + T_j - S_j) \mu_j \right\} \\
 \text{s. t.} \quad & S_j - SI_j \leq T_j \quad \text{for } j= 1, 2 \dots N \\
 & SI_j - S_i \geq 0 \quad \text{for all } (i, j) \in A \\
 & S_j \leq s_j \quad \text{for all demand nodes } j \\
 & S_j, SI_j \geq 0 \text{ and integer} \quad \text{for } j = 1, 2 \dots N
 \end{aligned}$$

where h_j denotes the per-unit holding cost for inventory at stage j and s_j is the maximum service time for demand node j . The objective of problem **P** is to minimize the holding cost for the safety stock in the supply chain. The constraints assure that the net replenishment times are nonnegative, the inbound service times are the maximum supplier service time, and the end-item stages satisfy their service guarantee. The decision variables are the service times. Graves and Willems (2000) present a dynamic programming algorithm for solving **P** for supply chains that are modeled as spanning trees.

4. Non-Stationary Demand

In this section we provide a specification for non-stationary demand. We then show conditions on the demand process for which the solution to \mathbf{P} is invariant. We then discuss the reasonableness of the demand process assumptions.

Specification of Non-Stationary Demand: We assume that we can divide the planning horizon for a product into K phases. We assume that the demand within each phase is from a stationary demand process. For instance, if the product is subject to seasonal demand, there might be three or four phases, corresponding to the different seasons. For each phase, we have a relatively stationary demand process. But the demand process changes from phase to phase.

As another example, consider a new product for which the product life cycle might consist of four phases:

- a start-up or product-launch phase in which the product is introduced to the market;
- a growth or demand-ramp phase over which the demand rate grows rapidly;
- a peak-demand phase during which the product sells at its maximum rate;
- and an end-of-life phase during which the product demand declines as it is removed from the market, replaced by the next generation of products.

Then we assume that we can approximate the demand as being stationary within each phase.

Coughlin (1998) describes an application in which he used this modeling approach to partition the demand for high-tech products into several stationary intervals for the purposes of determining the placement of strategic inventory across a supply chain.

For a new product, the assumption of stationarity within a phase is a strong assumption, as by definition the demand is growing (declining) through the growth (end-of-life) phase and thus not stationary. Nevertheless, for the purpose of setting safety stocks and service times, we

propose this assumption as an approximation. We will see soon why this assumption may be helpful in developing safety stock polices for non-stationary demand.

For each phase, we assume that we can specify the demand bound as follows:

$$D_{jk}(\tau) = \mu_{jk}\tau + v_k \sigma_j \tau^\beta \quad (7)$$

where $D_{jk}(\tau)$ is the maximum demand for stage j over a time interval of length τ that is totally within phase k . Thus, we interpret μ_{jk} as the mean demand rate, per time unit, for stage j in phase k . We can view σ_j as the nominal standard deviation of demand per time unit for stage j . Then v_k is a multiplier that depends on the phase of the planning horizon or life cycle for the product, analogous to a multiplicative seasonal index. The final term τ^β captures how demand variability grows with the length of the interval, where $0 < \beta < 1$; for instance, $\beta=0.5$ would reflect the standard assumption of adding variances when demand is independent across time periods.

We will first examine the significance of (7) in solving **P**. We will then discuss the reasonableness of the form assumed in (7).

Invariance Property

Consider problem **P** for a single phase with demand bounds given by (7); we term this to be the intra-phase problem. We equate $\mu_j = \mu_{jk}$ in Problem **P**, and then use (7) to restate the objective function of **P** for phase k as:

$$\begin{aligned} & \sum_{j=1}^N h_j \left\{ D_{jk} (SI_j + T_j - S_j) - (SI_j + T_j - S_j) \mu_{jk} \right\} \\ &= \sum_{j=1}^N h_j \left\{ v_k \sigma_j (SI_j + T_j - S_j)^\beta \right\} \\ &= v_k \sum_{j=1}^N h_j \left\{ \sigma_j (SI_j + T_j - S_j)^\beta \right\} \end{aligned}$$

We make the following observation:

Proposition 1: For the demand bound given by (7), the optimal solution to the intra-phase problem \mathbf{P} does not depend on the phase. That is, the optimal service times do not depend on the phase multiplier v_k and are the same for all phases.

Discussion: The significance of this observation is that *the optimal policy within each phase is invariant*. We choose the same stages to place safety stock, independent of the phase. The actual size of the safety stock, though, does depend on the phase. From (2), we set the base stock within each phase as

$$B_{jk} = D_{jk}(\tau) \quad \text{where } \tau = \max [0, SI_j + T_j - S_j].$$

where SI_j, S_j are the optimal service times from \mathbf{P} . Thus, the base stock and the resulting expected safety stock (from (4)) do change from phase to phase.

We will investigate later what happens when we transition from phase to phase. But first we explore the reasonableness of the demand bound (7).

Justification for Demand Bound: Single End-Item

Consider a supply chain with a single end-item, such as a serial network or assembly network. We assume that stage N is the end-item and suppose that the end-item demand is normally distributed in each phase k with mean m_{Nk} and standard deviation s_{Nk} per time unit. Then, for the purposes of setting safety stocks, we might define the demand bound in phase k as

$$D_{Nk}(\tau) = m_{Nk}\tau + z_{Nk} s_{Nk} \tau^\beta \tag{8}$$

where z_{Nk} is set to assure that the safety stock covers the demand variation some percentage of time. The choice of z_{Nk} indicates how frequently management is willing to resort to other tactics to cover demand variability in phase k . For instance during a growth or ramp phase of a product, we might set this factor very high so as to assure that the safety stock provides very good service

during this critical phase. Conversely, during the end-of-life phase, we expect this factor would be smaller, due to the high cost of obsolescence relative to the stock-out cost.

For notational convenience let the scalar f_{jN} denote the number of units of item j per unit of end-item N , where $f_{NN} = 1$. Then it is reasonable to assume that every stage j has a proportional demand bound, namely

$$D_{jk}(\tau) = f_{jN} D_{Nk}(\tau) \quad \text{for all } k. \quad (9)$$

Then we see that these assumptions are consistent with the demand-bound model (7).

Namely we can set the parameters for (7) as follows:

$$\mu_{jk} = f_{jN} m_{Nk} \quad \text{for all } j, k$$

$$v_k = z_{Nk} s_{Nk} \quad \text{for all } k,$$

$$\sigma_j = f_{jN} \quad \text{for all } j.$$

We can now make the following observation:

Proposition 2: For a supply chain with a single end-item, if the demand bounds in each phase are given by (8) and (9), then the optimal solution of \mathbf{P} for each phase does not depend on the demand parameters m_{Nk} , s_{Nk} and z_{Nk} .

Discussion: This observation establishes the robustness of the optimal solution of \mathbf{P} to the specification of the demand-bound function. *If the demand bound is of the functional form given in (8), then we do not need to know the values for the parameters m_{Nk} , s_{Nk} or z_{Nk} in order to determine the optimal service times.* We do need to know the value of β , which signifies how variability accrues over time. In addition to β , the solution to \mathbf{P} depends on the holding costs h_j , the production lead times T_j , and the network topology. Furthermore, Proposition 1 assures that the optimal service times for each phase are the same.

Justification for Demand Bound: Multiple End-Items

Consider a supply chain with multiple end-items. Let J denote the index set for the end-item nodes. We assume that for each $j \in J$, the end-item demand is normally distributed in each phase k with mean m_{jk} and standard deviation s_{jk} per time unit. Then, for the purposes of setting safety stocks, we might again define the demand bound in phase k for end-item j as

$$D_{jk}(\tau) = m_{jk}\tau + z_{jk} s_{jk} \tau^\beta \quad (10)$$

where z_{jk} is a safety factor.

Now we suppose that for all $j \in J$ and all k :

$$z_{jk}s_{jk} = v_k\sigma_j. \quad (11)$$

That is, for each end-item j , we have a nominal standard deviation of demand, σ_j , for all phases.

For each phase k , we have a multiplier that applies to all end-items. For instance, if the phases correspond to seasons, then we assume that the seasonal impact on demand variability is the same for all end-items. If the phases correspond to the life cycle of a product, then we assume that the end-items have concurrent life cycles. For instance, the end-items might be from a product family, and are complementary products that use the same generation of technology.

Another context in which (10) and (11) might hold is when the end-items represent different distribution channels for the same product. Then one might expect the relative variability to be the same across the different phases for the different distribution channels.

Thus, by assumption, the demand bound for the end-items, as given by (10) and (11), satisfies (7).

Furthermore, for each component stage i , let f_{ij} denote the number of units required from stage i per unit of end-item j for $j \in J$. Then we might construct the demand bound for stage i as follows:

$$D_{ik}(\tau) = m_{ik}\tau + z_{ik} s_{ik} \tau^\beta \quad (12)$$

where $m_{ik} = \sum_{j \in J} f_{ij} m_{jk}$ and $(z_{ik} s_{ik})^\alpha = \sum_{j \in J} (f_{ij} z_{jk} s_{jk})^\alpha$ for some positive α . For instance, we might

assume that the demands for the end items are independent of each other, so that the demand variance at stage i is the sum of the induced variances from the demand for each end-item; then we would set $\alpha = 2$. For the general case we substitute (11) to find

$$z_{ik} s_{ik} = \sqrt[\alpha]{\sum_{j \in J} (f_{ij} v_k \sigma_j)^\alpha} = v_k \sqrt[\alpha]{\sum_{j \in J} (f_{ij} \sigma_j)^\alpha} = v_k \sigma_i, \quad (13)$$

where $\sigma_i^\alpha = \sum_{j \in J} (f_{ij} \sigma_j)^\alpha$. Thus, from (12) and (13), we see that all component stages satisfy (7).

Hence, for supply chains with multiple end items, we contend that (7) is reasonable if the demand bounds for each end-item satisfy (7), and if we can construct the demand bound for the internal stages as prescribed above. In particular, if the end-item demands were independent, then we might develop demand bounds for the component stages by adding their variances.

5. Inter-Phase Example

The purpose of this section is to examine what happens in our supply chain model when we transition from one phase to another phase. We define an intra-phase policy as an inventory policy that applies to a specific phase and an inter-phase policy as a policy that applies to the time window for transitioning from one phase to the next phase. In the prior section we have shown that the optimal intra-phase safety stock policy is invariant to the phase, whereby we use the same set of service times within each phase.

Having discovered this property, we ask whether the intra-phase service times are also optimal when we transition from one phase to another phase. If this were true, then we would have a very strong and pragmatic result: namely, if demand is non-stationary as specified, then the optimal service times are stationary for the entire planning horizon, and are given by the solution to **P**. The optimal safety stocks will vary, but can be found from the service times and the demand bound.

Unfortunately, we can show by example that the intra-phase service times *need not be optimal* for the inter-phase transition window. In spite of this negative result, we will argue that the intra-phase policy is near optimal and is also much easier to implement. We develop these arguments by means of an example.

Consider a two-stage system, with stage 1 being the upstream stage serving stage 2 and $f_{12} = 1$. The cost and time parameters for the example are given in Table 1:

	Stage 1	Stage 2
Holding cost	0.5	1.0
Production lead time	10	5

Table 1: Cost and Time Parameters for Example

There are two phases and the demand bound for the end item is given by (7), with $\sigma = 2$, $\beta = 0.5$, and phase parameters given in Table 2:

	Mean demand rate (μ)	Phase multiplier (v)
Phase 1	100	30
Phase 2	150	50

Table 2: Phase Parameters for Example

Thus, the demand bounds for the two phases are:

$$D_{21}(\tau) = 100 \tau + 60 \sqrt{\tau} \qquad D_{22}(\tau) = 150 \tau + 100 \sqrt{\tau}$$

We first determine the optimal service times for the intra-phase problem, given by \mathbf{P} , when the maximum service time for the end-item (stage 2) is zero ($s_2 = 0$). For this problem, there is a single decision variable, namely the outbound service time at stage 1 (S_1) equal to the inbound service time for stage 2 (SI_2). For either phase, the solution to \mathbf{P} sets $S_1 = SI_2 = 0$. Thus, we hold a safety stock of intermediate product between stage 1 and stage 2, as well as an end-item safety stock at stage 2 to serve external demand. The resulting objective function value, equal to the holding cost for the safety stock, is \$229.03 in phase 1 and \$381.72 in phase 2. The base stocks are $B_{11} = 1189$ and $B_{21} = 634$ for phase 1, and $B_{12} = 1816$ and $B_{22} = 974$ for phase 2.

Now we examine what happens during the transition from phase 1 to phase 2. Suppose we specify time $t = 0$ to be the end of phase 1. *We define the inter-phase time window to be the time interval over which the inventory at one or more stages of the supply chain depends on the demand from both phases.* We note from the base stock model (1) that the inventory at stage j at time t depends on demand from both phases when

$$t - SI_j - T_j \leq 0 \qquad \text{and} \qquad t - S_j > 0.$$

For this example the inter-phase time window can range from 0 to $T_1 + T_2 = 15$, depending on the service times.

To model the inventory during this inter-phase time window, we need to characterize the maximum demand function. To construct the maximum demand for a time interval that spans the two phases, we assume that we can combine the maximum demands over the two phases as if

they were independent random variables. Then we set the maximum demand over the interval (a, b] to be $D(a, b)$, given as follows:

$$D(a, b) = (b-a)\mu_1 + 2v_1\sqrt{b-a} \quad \text{for } b \leq 0 \quad (14a)$$

$$D(a, b) = (-a\mu_1 + b\mu_2) + 2\sqrt{-av_1^2 + bv_2^2} \quad \text{for } a < 0 < b \quad (14b)$$

$$D(a, b) = (b-a)\mu_2 + 2v_2\sqrt{b-a} \quad \text{for } a \geq 0 \quad (14c)$$

where $\mu_1=100$, $\mu_2=150$, $v_1=30$ and $v_2=50$.

We assume that during the inter-phase time window we adapt the base stocks for each stage so as to continue to assure 100% service. That is, for a given specification of the service times, we set the base stock at stage j at time t as:

$$B_j(t) = D(t - SI_j - T_j, t - S_j), \quad (15)$$

for $D(a, b)$ given by (14). We can then substitute (14) and (15) into (3) to obtain the expected inventory at stage j at time t , as a function of the service times.

To find the best service times, we solve \mathbf{P} , modified to account for (14) and (15), for every value of $t \in (0, 15)$. We find that the optimal service times are $S_1 = S_2 = 10$ for every $t \in (0, 15)$. That is, during the inter-phase time window, the optimal safety-stock policy is to eliminate the intermediate inventory between stages 1 and 2, and enlarge the base stock of the end-item to cover the total lead-time of 15 time periods. *The intra-phase optimal policy is not optimal during the inter-phase time window.*

Time t	Intra-phase policy			Inter-phase policy		
	Holding cost for $S_1 = 0$	Stage 1 base stock for $S_1 = 0$	Stage 2 base stock for $S_1 = 0$	Holding cost for $S_1 = 10$	Stage 1 base stock for $S_1 = 10$	Stage 2 base stock for $S_1 = 10$
0	\$229	1189	634	\$232	0	1732
1	\$259	1256	706	\$246	0	1796
2	\$286	1321	775	\$258	0	1858
3	\$310	1385	843	\$271	0	1921
4	\$333	1448	909	\$282	0	1982
5	\$354	1511	974	\$293	0	2043
6	\$360	1573	974	\$304	0	2104
7	\$366	1634	974	\$314	0	2164
8	\$371	1695	974	\$324	0	2224
9	\$377	1756	974	\$334	0	2284
10	\$382	1816	974	\$344	0	2344
11	\$382	1816	974	\$353	0	2403
12	\$382	1816	974	\$362	0	2462
13	\$382	1816	974	\$370	0	2520
14	\$382	1816	974	\$379	0	2579
15	\$382	1826	974	\$387	0	2637

Table 3: Comparison of Policies for Two-Stage Example, with optimal policy in **Bold**

We report in Table 3 the holding cost for the safety stock for the two policies $S_1 = 0$ and $S_1 = 10$ for $t \in [0, 15]$, as well as the base stocks. We see that the cost difference can be as much as 20% in a single period. The cost penalty for deviating from the optimal policy in periods $t = 1, 2, \dots, 14$ is 11%.

To explain why the optimal policy switches, we must consider the structure of the two policies. In Figure 1 we plot the stage and system expected safety stock costs for both policies for $t = 0, 1, \dots, 15$.

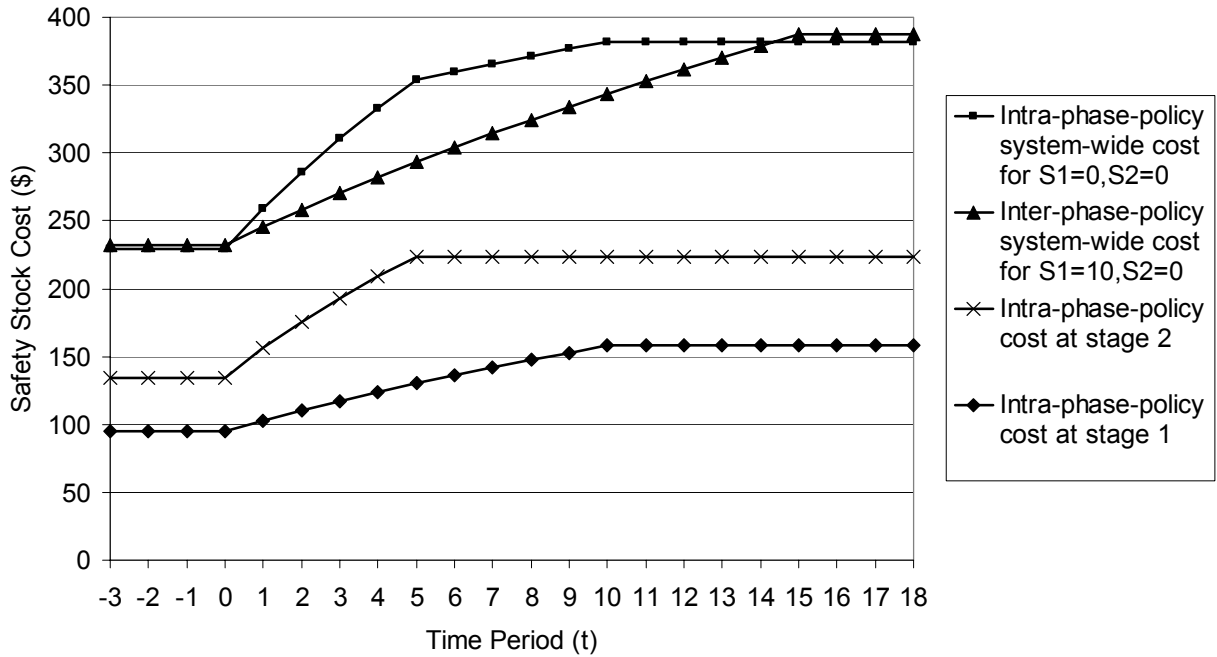


Figure 1: Expected Safety Stock as a Function of Time

During the transition window, $t \in [1, 14]$, the intra-phase policy increases the safety stock at both stages to protect against the increased variability in phase 2. In contrast, the inter-phase policy with $S_1 = 10$ maintains only an end-item safety stock; when we transition to phase 2, the end-item safety stock increases but less dramatically than that for the intra-phase policy. The inter-phase policy benefits from being better able to pool the increased demand variability from phase 2 with the lower demand variability of phase 1, due to having only a single safety stock.

Although not optimal, an open question is how well does the intra-phase optimal policy perform in the transient inter-phase time window, relative to the inter-phase optimal policy.

In Table 4 we report the percentage cost penalty for using the intra-phase policy during the time periods $t = 1, 2, \dots, 14$ relative to the optimal policy for various values of the component holding cost (or equivalently the ratio h_1/h_2). We see that there is only a cost penalty when 0.26

$< h_1 < 0.52$, as otherwise the intra-phase policy is also optimal in the transient time window. We also see that the maximum cost penalty is 12.0 % and occurs when $h_1 = 0.51$.

Component holding cost h_1	≤ 0.26	0.3	0.4	0.5	0.51	≥ 0.52
Cost penalty	0 %	0.3 %	4.0 %	11.1 %	12.0 %	0 %

Table 4: Cost penalty for intra-phase optimal policy during transient time window

In Table 5 we report the percentage cost penalty for using the intra-phase policy during the time periods $t = 1, 2, \dots, 14$ relative to the optimal policy for various values of the phase multiplier v_2 with $h_1 = 0.5$. The cost penalty grows with the demand variability in phase 2, albeit at a declining rate. And we see that if the phase multipliers are the same in both phases, then the intra-phase policy is optimal for the inter-phase time window; indeed, from (14) we can show this invariance property when $v_1 = v_2$.

Phase multiplier v_2	30	40	50	60	70
Cost penalty	0 %	6.1%	11.1 %	14.4%	16.8%

Table 5: Cost penalty for intra-phase optimal policy during transient time window

To assess the significance of the cost penalty in this example, we make three comments. First, from Table 4, we see that there is a non-negligible cost penalty for a fairly limited range of choices for the cost parameter h_1 . Second, from Table 5, we see that doubling the phase multiplier, as we transition from phase 1 to 2, results in only a 14.4% cost penalty. Third, we need to consider the cost penalty relative to the total length of the supply-chain planning horizon. If the intra-phase time window $(0, 15)$ is a fraction f of the overall planning horizon, then the cost penalty for following the intra-phase optimal policy is about $(0.11)f$ for the base case in Table 3.

We also comment on the practicality of switching policies when transitioning from one phase to another. Indeed, the optimal policy described in Table 3 would be quite difficult to implement. It requires the supply chain to consolidate the component inventory into the end-item inventory at the start of the transition time window, and then to reverse this consolidation when

the transition ends; and it requires the supply chain to change the internal service times during the transition time window. It would be much easier to follow the intra-phase policy by which the base stocks are gradually increased over the course of the transition time window, and there is no change to the internal service times.

6. Example – Consumer Packaged Goods Manufacturer

In this section, we illustrate the impact of non-stationary demand and validate the modeling approach presented in this paper by presenting a real-world example from a consumer packaged goods company². Figure 2 graphically represents the supply chain for a product family:

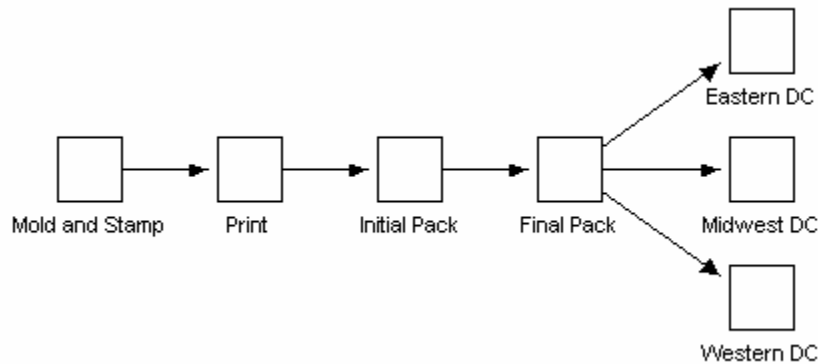


Figure 2: Supply chain map for consumer packaged goods company

The product is created through an injection molding and stamping process. Logos are then printed onto the plastic part. After the product goes through two packing operations, it is sent to one of three regional distribution centers where it serves the region’s demand. The (disguised) costs and production lead-times, in days, are:

Stage Name	Production Lead-Time	Cost Added
Mold and Stamp	15	\$0.85
Print	3	\$0.60
Initial Pack	3	\$0.15
Final Pack	3	\$0.10
Eastern DC	25	\$0.05
Midwest DC	20	\$0.05
Western DC	15	\$0.05

Table 6: Supply Chain Information

The actual demand data for the three DCs are:

² The data in this section has been disguised to protect company-confidential data. The insights drawn from the disguised data are identical to the conclusions based on the actual data.

i	Phase One		Phase Two		Phase Three	
	m _{i1}	s _{i1}	m _{i2}	s _{i2}	m _{i3}	s _{i3}
Eastern DC	1068.5	161.2	1402.0	195.9	2275.5	318.9
Midwest DC	670.5	87.7	1035.0	105.4	1532.0	186.9
Western DC	322.0	54.8	577.5	80.9	938.0	133.2

Table 7: Demand Parameters (in thousands)

Each phase covers four months of the product's life cycle. The data in Table 7 represents the monthly averages for each phase. The company seeks to maintain a 95% service level in all phases for all products.

We first want to verify whether or not the actual demand data corresponds to the demand bounds presented in (10). If so, there will be three scalars corresponding to the phases and three nominal standard deviations of demand corresponding to each of the FGI stages. We solve a nonlinear program to estimate the nominal standard deviations and the phase multipliers that provide the best fit to (11). Namely, we find estimates for v_k and σ_j that minimize the squared difference of the estimate errors:

$$\sum_{k,j} (z_{jk} s_{jk} - v_k \sigma_j)^2.$$

For the demand parameters in Table 7, the estimated phase-scale multipliers are $v_1 = 2.58$, $v_2 = 3.27$, $v_3 = 5.44$ and the estimated standard deviations are $\sigma_1 = 59.48$, $\sigma_2 = 33.83$, $\sigma_3 = 23.20$.

This results in the following estimated parameters for the three DCs:

i	Phase One		Phase Two		Phase Three	
	est. s _{i1}	% error	est. s _{i2}	% error	est. s _{i3}	% error
Eastern DC	153.3	-4.89%	194.7	-0.62%	323.4	1.40%
Midwest DC	87.2	-0.58%	110.7	5.05%	183.9	-1.59%
Western DC	59.8	9.12%	75.9	-6.18%	126.1	-5.36%

Table 8: Estimated Demand Parameters

The phase-scale multipliers increase over time because the magnitude of demand uncertainty is increasing as volumes increase over the product life cycle. The estimated standard deviations of demand reflect the relative volumes of the respective regions.

The optimal intra-phase policy using the demand bound estimates is:

Stage Name	Phase One		Phase Two		Phase Three	
	Safety stock (units)	Safety stock (cost)	Safety stock (units)	Safety stock (cost)	Safety stock (units)	Safety stock (cost)
Mold and Stamp	1186	\$353	1507	\$448	2503	\$745
Print	0	\$0	0	\$0	0	\$0
Initial Pack	0	\$0	0	\$0	0	\$0
Final Pack	0	\$0	0	\$0	0	\$0
Eastern DC	1470	\$901	1867	\$1,144	3102	\$1,900
Midwest DC	772	\$473	981	\$601	1629	\$998
Western DC	482	\$295	612	\$375	1016	\$622

Table 10: Optimal Inventory Results for Estimated Demand Parameters (in thousands)

The optimal intra-phase policy is identical for both the actual and estimated demand parameters.

The total annual safety stock cost for actual demand is \$2,972 versus \$2,951 for the estimated demand. This is a difference of less than 0.7%.

The optimal inter-phase inventory policy is different from the optimal intra-phase policy. Between phases 1 and 2 and between phases 2 and 3, the optimal inventory policy eliminates the safety stock at Mold and Stamp and increases the safety stock at the three distribution centers. For a given period (day), the cost difference can be as much as 12% but over the product life-time of 365 days, maintaining the intra-phase inventory policy increases safety stock costs by less than 1.5% when compared to implementing the optimal inter-phase inventory policy.

7. Conclusion

In this paper we introduce and develop a model for positioning safety stock in a supply chain subject to non-stationary demand. We assume that time can be broken into phases where demand is stationary within each phase, and show how to extend the model from Graves Willems (2000) to find the optimal placement of safety stocks. In particular we show that if the demand bound can be factored into two components, a phase-specific multiplier and a stage-specific scalar, then the optimal inventory locations remain constant within each phase. We also show that the inter-phase optimal policy need not be the same as the intra-phase optimal policy. Nevertheless, we argue and provide limited evidence that applying the intra-phase policy during the transition between phases is near optimal and has obvious implementation advantages.

We test the model with an application at a consumer-packaged goods company. We validate the reasonableness of the assumed form for the demand bound by estimating its parameters with data from the application, and show that the intra-phase policy is near optimal when applied over the entire planning horizon for the supply chain.

There are several interesting areas to extend this research. First, one might explore more the effectiveness of using the intra-phase policy during the transition time windows so as to understand better when it works well and when not. Second, we would hope to do more work to assess the validity and utility of the assumed form for the demand bound, as given by (7). In particular, we would like to understand how well it models real demand processes. Finally, we want to investigate what happens when the phase intervals are so short that the transition time window spans multiple phases.

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