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ON THE PRIMAL-DUAL GEOMETRY OF LEVEL SETS IN LINEAR AND CONIC OPTIMIZATION*

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Abstract. For a conic optimization problem

$$\begin{array}{lll} P: & \text{minimize}_x & c^T x \\ & \text{s.t.} & Ax = b, \\ & & x \in C \end{array}$$

and its dual

$$\begin{array}{lll} D: & \text{supremum}_{y,s} & b^T y \\ & \text{s.t.} & A^T y + s = c, \\ & & s \in C^*, \end{array}$$

we present a geometric relationship between the primal objective function level sets and the dual objective function level sets, which shows that the maximum norms of the primal objective function level sets are nearly inversely proportional to the maximum inscribed radii of the dual objective function level sets.

Key words. convex optimization, conic optimization, duality, level sets

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1. Introduction and motivation.

This paper is concerned with the interrelated geometry of the primal objective function level sets and the dual objective function level sets of the following conic convex optimization primal and dual pair:

$$\begin{array}{lll} P: & \text{minimum}_x & c^T x \\ & \text{s.t.} & Ax = b, \\ & & x \in C \end{array}$$

and

$$\begin{array}{lll} D: & \text{supremum}_{y,s} & b^T y \\ & \text{s.t.} & A^T y + s = c, \\ & & s \in C^*, \end{array}$$

where C is a closed convex cone in a finite-dimensional normed vector space X. We present a geometric relationship between the primal objective function level sets and the dual objective function level sets, namely, that the maximum norms of the primal objective function level sets are nearly inversely proportional to the maximum inscribed radii of the dual objective function level sets.

To provide motivation without yet becoming encumbered by details, consider the case when C is the nonnegative orthant, i.e., $C = \Re^n_+ := \{x \in \Re^n \mid x \geq 0\}$, in which case P and D are simply linear programming (LP) primal and dual problems. Below we list and comment on two well-known properties of LP:

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Property 1. Suppose that P and D are both feasible. Then the set of optimal solutions of P is unbounded if and only if there is no strictly feasible solution of D; that is, $A^T y + s = c, s \geq 0$ implies $s \not> 0$. This property is easily proved via LP duality, for example, and is part of the folklore of optimization. Put another way, Property 1 can be stated as follows:

> "The set of primal optimal solutions is unbounded if and only if every dual feasible s lies on the boundary of $\Re^n_+$."

Property 2. If P and D each have feasible solutions that satisfy all inequalities strictly, then the central trajectory exists, whereby for each $\mu > 0$ there exists unique feasible solutions $x(\mu)$ of P and $(y(\mu), s(\mu))$ of D for which $x_j(\mu) \cdot s_j(\mu) = \mu, j = 1, \ldots, n$. This is an elementary consequence of the optimality conditions for the logarithmic barrier functions appended to a linear program; see Wright [7], for example. Now notice here that for a given value $\mu > 0$, the norm $\|x(\mu)\|$ is large if and only if $\text{dist}(s_j(\mu), \partial\Re^n_+)$ is small. In fact, a little basic arithmetic manipulation easily shows that

$$\mu \leq \|x(\mu)\|_1 \cdot \min_j \{s_j(\mu)\} \leq n\mu,$$

which can then be used to assert the following:

> "For a given duality gap $\theta > 0$, there exists a primal feasible x and a dual feasible (y, s) with duality gap at most θ and with the property that $\theta/n \leq \|x\|_1 \cdot \text{dist}(s, \partial\Re^n_+) \leq \theta$."

This brief discussion points to an interrelationship between the norms of certain primal feasible solutions x and the distances of certain dual feasible solutions s to the boundary of the nonnegative orthant. In section 2 we make this interrelationship precise for the case of linear optimization in Theorem 2.1, which shows that the maximum norms of primal objective level sets are almost exactly inversely proportional to the maximum distances to the boundary of dual objective level sets. In fact, just as linear optimization is a special case of more general conic convex optimization, Theorem 2.1 is a special case of a more general theorem that demonstrates an inverse proportional relationship between the maximum norms of primal objective level sets and the maximum distances to the boundary of dual objective level sets in conic convex optimization. This more general result is presented in section 3 as Theorem 3.2 and is the main result of this paper. Section 4 discusses several aspects of cone geometry that arise in our development, and section 5 contains proofs.

Notation. We denote real n-dimensional space and the nonnegative n-dimensional orthant by $\Re^n$ and $\Re^n_+$, respectively. Let $e = (1, \ldots, 1)^T$ denote the vector of 1's in $\Re^n$.

2. Primal-dual geometry of level sets for linear optimization.

Consider the following dual pair of linear optimization problems:

$$\begin{array}{lll} LP: & \text{minimize} & c^T x \\ & \text{s.t.} & Ax = b, \\ & & x \geq 0 \end{array}$$

and

$$\begin{array}{lll} LD: & \text{maximize} & b^T y \\ & \text{s.t.} & A^T y + s = c, \\ & & s \geq 0, \end{array}$$

whose common optimal value is z^*. For $\epsilon > 0$ and $\delta > 0$, define the ϵ- and δ-level sets for the primal and dual problems as follows:

$$P_\epsilon := \left\{ x \mid Ax = b, x \geq 0, c^T x \leq z^* + \epsilon \right\}$$

and

$$D_\delta := \left\{ s \mid \exists\, y \text{ satisfying } A^T y + s = c, s \geq 0, b^T y \geq z^* - \delta \right\}.$$

Define

$$\begin{array}{rll} R_\epsilon := & \max & \|x\|_1 \\ & \text{s.t.} & Ax = b, \\ & & c^T x \leq z^* + \epsilon, \\ & & x \geq 0 \end{array} \tag{2.1}$$

and

$$\begin{array}{rll} r_\delta := & \max & \min_j \{s_j\} \\ & \text{s.t.} & A^T y + s = c, \\ & & b^T y \geq z^* - \delta, \\ & & s \geq 0. \end{array} \tag{2.2}$$

The quantity R_ϵ is simply the size of the largest vector x in the primal level set P_ϵ, measured in the L_1-norm. The quantity r_δ can be interpreted as the positivity of the most positive vector s in the dual level set D_δ or, equivalently, as the maximum distance to the boundary of the nonnegative orthant over all points s in D_δ. The following theorem presents a reciprocal relationship between R_ϵ and r_δ.

THEOREM 2.1. *Suppose that z^* is finite. If R_ϵ is positive and finite, then*

$$\min\{\epsilon, \delta\} \;\leq\; R_\epsilon \cdot r_\delta \;\leq\; \epsilon + \delta.$$

Otherwise, $R_\epsilon = 0$ if and only if $r_\delta = +\infty$, and $R_\epsilon = +\infty$ if and only if $r_\delta = 0$.

Theorem 2.1 bounds the size of the largest vector in P_ϵ and the positivity of the most positive vector in D_δ from above and below, and shows that these quantities are almost exactly inversely proportional. In fact, taking $\delta = \epsilon$, the result states that $R_\epsilon \cdot r_\epsilon$ lies in the interval $[\epsilon, 2\epsilon]$. The proof of this theorem follows as a special case of a more general result for convex conic optimization, namely Theorem 3.2 in section 3.

Remark 2.1. If $R_\epsilon < \infty$, then

$$R_{\epsilon'} \leq \left(\frac{\epsilon'}{\epsilon} \right) R_\epsilon \tag{2.3}$$

for all $\epsilon' \geq \epsilon$. $\square$

Proof. If $R_\epsilon = 0$, the result follows trivially, since then $R_{\epsilon'} = 0$ for all $\epsilon' > 0$. So suppose that $0 < R_\epsilon < +\infty$. Let x^* be an optimal solution of LP, and let $x' \in P_{\epsilon'}$ be given. Then $x := \frac{\epsilon}{\epsilon'} x' + \frac{\epsilon' - \epsilon}{\epsilon'} x^*$ satisfies $x \in P_\epsilon$, whereby $\|x\|_1 \leq R_\epsilon$. Now notice that $\|x'\|_1 = e^T x' = \frac{\epsilon'}{\epsilon} e^T x - \frac{\epsilon' - \epsilon}{\epsilon} e^T x^* \leq \frac{\epsilon'}{\epsilon} e^T x = \frac{\epsilon'}{\epsilon} \|x\|_1 \leq \frac{\epsilon'}{\epsilon} R_\epsilon$. Therefore $R_{\epsilon'} \leq \frac{\epsilon'}{\epsilon} R_\epsilon$, proving the result. $\square$

Remark 2.1 bounds the rate of growth of $R_{\epsilon'}$ as ϵ' increases and shows that $R_{\epsilon'}$ grows at most linearly in ϵ' and at a rate no greater than $\frac{R_\epsilon}{\epsilon}$. There is a version of (2.3) for r_δ and $r_{\delta'}$, namely

$$r_{\delta'} \geq \left(\frac{\delta'}{\delta}\right) r_\delta \tag{2.4}$$

for all $0 \leq \delta' \leq \delta$, which is true as an elementary consequence of the convexity of the feasible region of LD.

By exchanging the roles of the primal and dual problems, we obviously can construct analogous results for the most positive vector x in P_ϵ as well as for the size of the largest vector s in D_δ.

3. Conic optimization with a norm on X. We now consider the generalization of linear optimization to convex optimization in conic linear form:

$$\begin{array}{llll} P: & z^* := \text{minimum}_x & c^T x \\ & \text{s.t.} & Ax = b, \\ & & x \in C \end{array}$$

and its dual

$$\begin{array}{llll} D: & v^* := \text{supremum}_{y,s} & b^T y \\ & \text{s.t.} & A^T y + s = c, \\ & & s \in C^*, \end{array}$$

where $C \subset X$ is a closed convex cone in the (finite) n-dimensional linear vector space X, and b lies in the (finite) m-dimensional vector space Y. This format for convex optimization dates back at least to Duffin [2]. Strong duality results can be found in [2] as well as in Ben-Israel, Charnes, and Kortanek [1].

For $\epsilon > 0$ and $\delta > 0$, we define the ϵ- and δ-level sets for the primal and dual problems as follows:

$$P_\epsilon := \left\{x \mid Ax = b, x \in C, c^T x \leq z^* + \epsilon\right\}$$

and

$$D_\delta := \left\{s \mid \exists\, y \text{ satisfying } A^T y + s = c, s \in C^*, b^T y \geq v^* - \delta\right\}.$$

We make the following assumption.

Assumption A. z^* is finite. The cone C satisfies $C \neq \{0\}$, and C contains no line (whereby C^* has an interior).

Suppose that X is endowed with a norm $\|\cdot\|$, and so X^* is endowed with the dual norm $\|\cdot\|_*$. Let $B(x,r)$ and $B^*(s,r)$ denote the balls of radius r centered at $x \in X$ and $s \in X^*$, respectively, defined for the appropriate norms.

We denote the maximum norm of P_ϵ by R_ϵ, defined as

$$\begin{array}{lll} R_\epsilon := & \max_x & \|x\| \\ & \text{s.t.} & x \in P_\epsilon. \end{array} \tag{3.1}$$

We denote by r_δ the inscribed size of D_δ, defined as

$$\begin{array}{lll} r_\delta := & \max_{s,r} & r \\ & \text{s.t.} & s \in D_\delta, \\ & & B^*(s,r) \subset C^*. \end{array} \tag{3.2}$$

As in the case of linear optimization, r_δ measures the distance of the most interior point of the dual level set D_δ to the boundary of the cone C^*. Put another way, r_δ measures the "interiorness" (with respect to C^*) of the most interior point in D_δ.

Before presenting the version of Theorem 2.1 for convex conic optimization, we first review the concept of the min-width of a cone. We use the following definition of the min-width.

DEFINITION 3.1. *Let $K \subset X$ be a closed convex cone in the normed linear vector space X satisfying* (i) *K has a nonempty interior and* (ii) *$K \neq X$. The min-width of K is defined as*

$$\tau_K \;:=\; \max_{x \in \mathrm{int} K} \left\{ \frac{\mathrm{dist}\,(x, \partial K)}{\|x\|} \right\} \;= \max_{x \neq 0} \left\{ \frac{r}{\|x\|} \;\middle|\; B(x, r) \subset K \right\} \,.$$

Note that τ_K measures the maximum ratio of the radius to the norm of the center of an inscribed ball in K, and so larger values of τ_K correspond to an intuitive notion of greater minimum width of K. The quantity τ_K was called the "inner measure" of K for Euclidean norms in Goffin [5] and has been used more recently for general norms in analyzing condition measures for conic convex optimization; see [3]. Note that $\tau_K \in (0, 1]$, since K has a nonempty interior and $K \neq X$, and τ_K is attained for some $x^0 \in \mathrm{int} K$ satisfying $\|x^0\| = 1$, as well as along the ray αx^0 for all $\alpha > 0$. Let τ_{K^*} be defined similarly for the dual cone K^*.

The following is analogous to Theorem 2.1 for conic problems.

THEOREM 3.2. *Suppose that Assumption A holds. If R_ϵ is positive and finite, then $z^* = v^*$ and*

$$\tau_{C^*} \cdot \min\{\epsilon, \delta\} \;\leq\; R_\epsilon \cdot r_\delta \;\leq\; \epsilon + \delta. \tag{3.3}$$

If $R_\epsilon = 0$, then $z^* = v^*$ and $r_\delta = +\infty$; else if $R_\epsilon = +\infty$ and v^* is finite, then $r_\delta = 0$.

Here we have had to introduce the min-width τ_{C^*} into the left inequality of (3.3), somewhat weakening the result. In the next section we show that the left inequality can be tight. We also show how to define a family of cone-based norms for which $\tau_{C^*} = 1$, and we show that for norms induced by a ϑ-normal barrier function on C the min-width constant τ_{C^*} satisfies $\tau_{C^*} \geq 1/\sqrt{\vartheta}$. Theorem 3.2 is proved in section 5. Here we use Theorem 3.2 to prove Theorem 2.1.

Proof of Theorem 2.1. Note that LP is a special case of P with $X = \Re^n$ and $C = \Re^n_+$, whereby $C^* = \Re^n_+$. Endow X with the L_1-norm $\|\cdot\| = \|\cdot\|_1$, whose dual norm on X^* is the L_∞-norm $\|\cdot\|_* = \|\cdot\|_\infty$. To prove the theorem it suffices to show that $\tau_{C^*} = 1$, which we do now. Let $s^0 = e$, and note that $\|s^0\|_\infty = 1$, and that $B^*(s^0, 1) = \{s \mid \|s - e\|_\infty \leq 1\} \subset \Re^n_+ = C^*$, whereby $\tau_{C^*} \geq 1$. However, $\tau_{C^*} \leq 1$ because C^* is a pointed cone, and so $\tau_{C^*} = 1$, completing the proof. ∎

The following remark, analogous to Remark 2.1, is proved in section 5.

Remark 3.1. If $R_\epsilon < \infty$, then

$$R_{\epsilon'} \leq \left(\frac{\epsilon'}{\epsilon} \right) \left(\frac{1}{\tau_{C^*}} \right) R_\epsilon$$

for all $\epsilon' \geq \epsilon$.

4. On the min-width constant.

4.1. The min-width constant can be tight. Here we show by example that the left inequality in (3.3) can be tight, and so the constant τ_{C^*} cannot be replaced by a larger quantity. Let $X = \Re^n$ and $C = \Re^n_+$ (whereby $C^* = \Re^n_+$), and let X be endowed with the L_p-norm $\|x\|_p := (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}}$ for $1 \le p \le +\infty$, whose dual norm is $\|s\|_* = \|s\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$ with appropriate limits for $p, q = 1$ and/or ∞. Then it is straightforward to show that $\tau_C = n^{-\frac{1}{p}}$ and $\tau_{C^*} = n^{-\frac{1}{q}}$. Consider the following LP primal and dual instance:

$$\tilde{P}: \quad \begin{array}{rl} \min_x & 0^T x \\ \text{s.t.} & Ix = e, \\ & x \in \Re^n_+, \end{array} \qquad\qquad \tilde{D}: \quad \begin{array}{rl} \max_{y,s} & e^T y \\ \text{s.t.} & Iy + s = 0, \\ & s \in \Re^n_+, \end{array}$$

whose common optimal value is $z^* = 0$. Then $R_\epsilon = \|e\|_p = n^{\frac{1}{p}}$, and $r_\delta = \frac{\delta}{n}$ for all $\epsilon, \delta > 0$. Let $\epsilon := \delta$, whereby $R_\epsilon \cdot r_\delta = n^{\frac{1}{p}} \cdot \frac{\delta}{n} = \delta \cdot n^{\left(\frac{1}{p}-1\right)} = \delta \cdot n^{-\frac{1}{q}} = \delta \cdot \tau_{C^*} = \min\{\epsilon, \delta\}\tau_{C^*}$, which shows that the left inequality of (3.3) can indeed be tight.

4.2. Min-widths for the family of norms induced by a ϑ-normal barrier. In this subsection we assume that C is a regular cone; i.e., C is pointed and has an interior. Suppose that $F(\cdot) : \text{int}C \to \Re$ is a ϑ-normal barrier for C; see [6]. Then $F^*(\cdot) : \text{int}C^* \to \Re$, the conjugate function of $F(\cdot)$, is also a ϑ-normal barrier for C^*; see [6] as well.

Let $s^0 \in \text{int}C^*$ be given. The norm induced by the ϑ-normal barrier $F(\cdot)$ at s^0 is defined as follows:

$$\|s\|_{*,s^0} := \sqrt{s^T H^*(s^0) s},$$

where $H^*(s^0)$ is the Hessian of $F^*(\cdot)$ evaluated at s^0. It then follows from Theorem 2.1.1 of [6] that $B^*(s^0, 1) \subset C^*$ and from Proposition 2.3.4 of [6] that $\|s^0\|_{*,s^0} = \sqrt{\vartheta}$. Therefore under the dual norm $\|s\|_* := \|s\|_{*,s^0}$ we have $\tau_{C^*} \ge 1/\sqrt{\vartheta}$.

4.3. A family of norms on X for which $\tau_{C^*} = 1$. In this subsection we also assume that C is a regular cone. For every $s^0 \in \text{int}C^*$, there is a norm analogous to the L_∞-norm for the nonnegative orthant for which the associated min-width is $\tau_{C^*} = 1$. To see this, consider a given interior point $s^0 \in \text{int}C^*$, and define the following norm:

$$\begin{array}{rcll} \|s\|_* & := & \min_\alpha & \alpha \\ & & \text{s.t.} & s + \alpha s^0 \in C^*, \\ & & & -s + \alpha s^0 \in C^*. \end{array}$$

It is a straightforward exercise to verify that $\|\cdot\|_*$ is indeed a norm, and its dual norm turns out to be

$$\begin{array}{rcll} \|x\| & := & \min_{x^1,x^2} & (s^0)^T(x^1 + x^2) \\ & & \text{s.t.} & x^1 - x^2 = x, \\ & & & x^1 \in C, \\ & & & x^2 \in C. \end{array}$$

Under $\|\cdot\|_*$, it is easily shown that $\|s^0\|_* = 1$ and $\tau_{C^*} = 1$.

In the case when $X = \Re^n$, $C = C^* = \Re^n_+$, and $s^0 = e$, we recover the L_∞-norm as $\|s\|_*$ for $s \in X^* = \Re^n$ and the L_1-norm as $\|x\|$ for $x \in X = \Re^n$.

5. Proofs of main results. We start by pointing out a fact about strong duality in general conic convex optimization that we will use in our proof of Theorem 3.2. Suppose we have a primal and dual pair of conic convex optimization problems

$$\begin{array}{llll} \hat{P}: & \hat{z}^* := & \inf_x & f^T x \\ & & \text{s.t.} & Mx = g, \\ & & & x \in K, \end{array} \qquad \begin{array}{llll} \hat{D}: & \hat{v}^* := & \sup_{y,s} & g^T y \\ & & \text{s.t.} & M^T y + s = f, \\ & & & s \in K^*, \end{array}$$

where $K \subset X$ is a closed convex cone in the (finite) n-dimensional linear vector space X, and g lies in the (finite) m-dimensional vector space Y. The following lemma presents a sufficient condition for this pair to exhibit strong duality.

LEMMA 5.1. *Assume that $\hat{z}^*$ is finite and for some $\epsilon > 0$ the level set $\hat{P}_\epsilon := \{x \mid Mx = g, x \in K, f^T x \leq \hat{z}^* + \epsilon\}$ is bounded. Then $\hat{P}$ attains its optimum and $\hat{z}^* = \hat{v}^*$.*

Proof. Note that $\hat{P}$ attains its optimum, since $\hat{P}_\epsilon$ is bounded. The boundedness of $\hat{P}_\epsilon$ also implies that

$$\{0\} = \{x \in X \mid Mx = 0, x \in K, f^T x \leq 0\}. \tag{5.1}$$

It is elementary to show that $\hat{z}^* \geq \hat{v}^*$. Suppose that $\hat{z}^* > \hat{v}^*$, let $\bar{\epsilon}$ be such that $0 < \bar{\epsilon} < \hat{z}^* - \hat{v}^*$, and let

$$S = \{(w, \alpha) \mid \exists y \in Y^*, s \in K^* \text{ satisfying } w = M^T y + s - f, g^T y \geq \hat{v}^* + \bar{\epsilon} - \alpha\}.$$

Then S is a nonempty convex set in $X^* \times \Re$, and $(0,0) \notin S$, whereby there exists $(x, \theta) \neq 0$ satisfying $x^T w + \theta\alpha \geq 0$ for all $(w, \alpha) \in S$. Therefore

$$x^T \left(M^T y + s - f\right) + \theta\left(-g^T y + \hat{v}^* + \bar{\epsilon} + \eta\right) \geq 0 \quad \forall y \in Y^*, \forall s \in K^*, \forall \eta \geq 0. \tag{5.2}$$

This implies that $Mx = g\theta$, $\theta \geq 0$, and $x \in K$. We now have two cases.

Case 1. $\theta > 0$. Without loss of generality we can assume that $\theta = 1$. Therefore x is feasible for $\hat{P}$, and (5.2) also implies that $\hat{z}^* \leq f^T x \leq \hat{v}^* + \bar{\epsilon} < \hat{z}^*$, which is a contradiction.

Case 2. $\theta = 0$. In this case $x \neq 0$, $x \in K$, $Mx = 0$, and (5.2) implies that $f^T x \leq 0$, contradicting (5.1).

In both cases we have a contradiction, and so $\hat{z}^* = \hat{v}^*$. ∎

We next state some properties of norms and the min-width. The following is a special case of the Hahn–Banach theorem; for a short proof of this proposition based on the subdifferential operator, see Proposition 2 of [4].

PROPOSITION 5.2. *For every $x \in X$, there exists $\bar{x} \in X^*$ with the property that $\|\bar{x}\|_* = 1$ and $\|x\| = \bar{x}^T x$.* ∎

The following exhibits some useful properties of the min-width of a cone.

PROPOSITION 5.3. *Suppose K^* is a convex cone whose min-width τ_{K^*} is attained at some point $s^0 \in \text{int}K^*$ satisfying $\|s^0\|_* = 1$. Then*

(i) *$\tau_{K^*}\|x\| \leq (s^0)^T x \leq \|x\|$ for all $x \in K$, and*

(ii) *if $s - \lambda s^0 \in K^*$, then $B^*(s, \lambda\tau_{K^*}) \subset K^*$.*

Proof. For a given $x \in K \subset X$, there exists $\bar{x} \in X^*$ for which $\|\bar{x}\|_* = 1$ and $\|x\| = \bar{x}^T x$ from Proposition 5.2. By construction of s^0 we have $B^*(s^0, \tau_{K^*}) \subset K^*$, and so $s^0 - \tau_{K^*}\bar{x} \in K^*$. Therefore $\|x\| = \|x\|\|s^0\|_* \geq (s^0)^T x = (s^0 - \tau_{K^*}\bar{x} + \tau_{K^*}\bar{x})^T x \geq \tau_{K^*}\bar{x}^T x = \tau_{K^*}\|x\|$, proving (i). To prove (ii), let $u := s - \lambda s^0$. Then $s = u + \lambda s^0$, where $u \in K^*$ and $B^*(s^0, \tau_{K^*}) \subset K^*$, whereby it follows that $B^*(s, \lambda\tau_{K^*}) \subset K^*$. ∎

We are now ready to prove Theorem 3.2, which we do by proving the following four statements:

(i) If R_ϵ is positive and finite, then $z^* = v^*$ and $R_\epsilon \cdot r_\delta \le \epsilon + \delta$.
(ii) If R_ϵ is positive and finite, then $R_\epsilon \cdot r_\delta \ge \tau_{C^*} \min\{\epsilon, \delta\}$.
(iii) If $R_\epsilon = 0$, then $z^* = v^*$ and $r_\delta = +\infty$.
(iv) If $R_\epsilon = +\infty$ and v^* is finite, then $r_\delta = 0$.

Proof of (i). Since R_ϵ is finite, it follows that P_ϵ is bounded, and so $z^* = v^*$ from Lemma 5.1. Let $x \in P_\epsilon$ be given, and let $\bar{x}$ satisfy $\|\bar{x}\|_* = 1$ and $\|x\| = \bar{x}^T x$; see Proposition 5.2. Now suppose that $s \in D_\delta$ satisfies $B^*(s, r) \subset C^*$ for some $r \ge 0$. It follows that $\epsilon + \delta \ge c^T x - z^* - b^T y + v^* = c^T x - b^T y = x^T s = x^T(s - r\bar{x} + r\bar{x}) \ge r x^T \bar{x} = r\|x\|$. As this is true for all $x \in P_\epsilon$ and all $s \in D_\delta$ satisfying $B^*(s, r) \subset C^*$, it follows that $\epsilon + \delta \ge R_\epsilon \cdot r_\delta$. ☐

Proof of (ii). Let s^0 satisfy $\|s^0\|_* = 1$ and $B^*(s^0, \tau_{C^*}) \subset C^*$, and consider the following conic convex dual programs:

$$\begin{array}{llcc} \bar{P}: & \bar{R}_\epsilon := & \max_x & (s^0)^T x \\ & & \text{s.t.} & Ax = b, \\ & & & c^T x \le z^* + \epsilon, \\ & & & x \in C, \end{array} \qquad \begin{array}{llcc} \bar{D}: & \bar{Q} := & \inf_{y,s,\theta} & -b^T y + (z^* + \epsilon)\theta \\ & & \text{s.t.} & A^T y + s = \theta c, \\ & & & s - s^0 \in C^*, \\ & & & \theta \ge 0. \end{array}$$

From Proposition 5.3 it follows that $\tau_{C^*}\|x\| \le (s^0)^T x \le \|x\|$ for any $x \in C$, whereby $\tau_{C^*} R_\epsilon \le \bar{R}_\epsilon \le R_\epsilon$, and, in particular, the level sets of $\bar{P}$ are bounded. Then we can invoke Lemma 5.1 on the pair $\bar{P}, \bar{D}$ and assert that $\bar{P}$ attains its optimum and $\bar{R}_\epsilon = \bar{Q}$.

For $\alpha \in (0, \min\{\epsilon, \delta\})$ we show below that

$$r_\delta \ge \frac{\tau_{C^*}}{\bar{R}_\epsilon + \alpha} \left(\min\{\epsilon, \delta - \alpha\}\right) \tag{5.3}$$

and letting $\alpha \to 0$ will complete the proof since (5.3) and $\alpha \to 0$ imply that $R_\epsilon \cdot r_\delta \ge \bar{R}_\epsilon \cdot r_\delta \ge \tau_{C^*} \min\{\epsilon, \delta\}$. For $\alpha \in (0, \min\{\epsilon, \delta\})$ let (y, s, θ) be a feasible solution of $\bar{D}$ satisfying

$$-b^T y + (z^* + \epsilon)\theta \le \bar{Q} + \alpha = \bar{R}_\epsilon + \alpha, \tag{5.4}$$

and define $w := s - s^0 \in C^*$. We prove (5.3) by considering three cases.

Case 1. $\theta = 0$. In this case $A^T y + s = 0$ and $-b^T y \le \bar{R}_\epsilon + \alpha$. Let $(\bar{y}, \bar{s})$ be any feasible solution of D satisfying $b^T \bar{y} \ge z^* - \alpha$, and define

$$(\hat{y}, \hat{s}) := (\bar{y}, \bar{s}) + \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha}(y, s).$$

Then $(\hat{y}, \hat{s})$ is feasible for D, and

$$b^T \hat{y} = b^T \bar{y} + \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha} b^T y \ge z^* - \alpha - \delta + \alpha = z^* - \delta.$$

Also, $\hat{s} - \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha} s^0 = \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha} w + \bar{s} \in C^*$, whereby $\hat{s} \in D_\delta$ and $B^*(\hat{s}, \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha}\tau_{C^*}) \subset C^*$ from Proposition 5.3. This then implies that $r_\delta \ge \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha}\tau_{C^*}$, which implies (5.3).

Case 2. $\theta > 0$ and $\frac{\bar{R}_\epsilon + \alpha}{\theta} - \epsilon \le \delta$. Define

$$(\hat{y}, \hat{s}) = \frac{1}{\theta}(y, s),$$

whereby $(\hat{y}, \hat{s})$ satisfies $\hat{s} \in C^*$, $A^T\hat{y} + \hat{s} = c$, and

$$b^T\hat{y} = \frac{1}{\theta}b^T y \geq -\frac{\bar{R}_\epsilon + \alpha}{\theta} + z^* + \epsilon \geq z^* - \delta,$$

which shows that $\hat{s} \in D_\delta$. Furthermore, $\hat{s} = \frac{s^0}{\theta} + \frac{w}{\theta}$, $w \in C^*$, and so $\hat{s} - \frac{1}{\theta}s^0 \in C^*$. Now it follows from Proposition 5.3 that $B^*\left(\hat{s}, \frac{\tau_{C^*}}{\theta}\right) \subset C^*$, and so $r_\delta \geq \frac{\tau_{C^*}}{\theta}$. However,

$$z^* \geq b^T\hat{y} \geq -\frac{\bar{R}_\epsilon + \alpha}{\theta} + z^* + \epsilon,$$

and so $\frac{1}{\theta} \geq \frac{\epsilon}{\bar{R}_\epsilon + \alpha}$, whereby $r_\delta \geq \frac{\tau_{C^*}}{\theta} \geq \frac{\epsilon}{\bar{R}_\epsilon + \alpha}\tau_{C^*}$, which then implies (5.3).

Case 3. $\theta > 0$ and $\frac{\bar{R}_\epsilon + \alpha}{\theta} - \epsilon \geq \delta$. Let $(\bar{y}, \bar{s})$ be any feasible solution of D satisfying

$$b^T\bar{y} \geq z^* - \alpha, \tag{5.5}$$

and define

$$(\hat{y}, \hat{s}) = \lambda\left(\frac{(y, s)}{\theta}\right) + (1 - \lambda)(\bar{y}, \bar{s}),$$

where

$$\lambda = \frac{\delta - \alpha}{\frac{\bar{R}_\epsilon + \alpha}{\theta} - \epsilon - \alpha}.$$

Then $\lambda \in [0, 1]$ for $\alpha \in (0, \delta)$, and so $(\hat{y}, \hat{s})$ is a convex combination of $\frac{(y,s)}{\theta}$ and $(\bar{y}, \bar{s})$ and so satisfies $A^T\hat{y} + \hat{s} = c$, $\hat{s} \in C^*$. It also follows from (5.4) and (5.5) that $b^T\hat{y} \geq z^* - \delta$, whereby $\hat{s} \in D_\delta$. Finally, $\hat{s} - \frac{\lambda}{\theta}s^0 \in C^*$, and so from Proposition 5.3 we have $B^*(\hat{s}, \frac{\lambda\tau_{C^*}}{\theta}) \subset C^*$. Therefore

$$r_\delta \geq \frac{\lambda\tau_{C^*}}{\theta} = \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha - \alpha\theta - \epsilon\theta}\tau_{C^*} \geq \frac{\delta - \alpha}{\bar{R}_\epsilon + \alpha}\tau_{C^*},$$

from which (5.3) follows.

Therefore (5.3) is true in all cases, and the proof is complete. □

Proof of (iii). Since $R_\epsilon = 0$ it follows that $P_\epsilon = \{0\}$ is bounded, and so $z^* = v^*$ from Lemma 5.1. It then follows that $b = 0$, and so $z^* = v^* = 0$. To prove that $r_\delta = +\infty$ it suffices to prove that there exists $(\tilde{y}, \tilde{s})$ satisfying

$$A^T\tilde{y} + \tilde{s} = 0 \quad \text{and} \quad \tilde{s} \in \text{int}C^*. \tag{5.6}$$

Let $s^0, \bar{P}$, and $\bar{D}$ be exactly as in the proof of (ii), and the same logic as in the proof of (ii) yields $\bar{R}_\epsilon = \bar{Q} = 0$; notice that because $b = 0$ and $z^* = 0$ it follows that the objective function of $\bar{D}$ is simply $\epsilon\theta$. If $\bar{D}$ attains its optimal value $\bar{Q} = 0$, then any optimal solution (y^*, s^*, θ^*) of $\bar{D}$ will satisfy $\theta^* = 0$, and so (5.6) will be satisfied by setting $(\tilde{y}, \tilde{s}) = (y^*, s^*)$. Alternatively, if $c = 0$, then the (y, s) variables of any feasible solution (y, s, θ) of $\bar{D}$ will satisfy (5.6). It remains to consider the case when $\bar{D}$ does not attain its optimum and $c \neq 0$. Let $\alpha := \frac{\epsilon \cdot \tau_{C^*}}{2\|c\|_*}$, and let (y, s, θ) be a feasible solution of $\bar{D}$ satisfying $\epsilon\theta = -b^T y + (z^* + \epsilon)\theta \leq \bar{R}_\epsilon + \alpha = \frac{\epsilon \cdot \tau_{C^*}}{2\|c\|_*}$; then, in particular, $\theta \leq \frac{\tau_{C^*}}{2\|c\|_*}$. Define $w := s - s^0 \in C^*$. Let $(\tilde{y}, \tilde{s}) = (y, s^0 + w - \theta c)$. Then $A^T\tilde{y} + \tilde{s} = A^T y + s - \theta c = 0$, and $\tilde{s} = s^0 + w - \theta c = w + \frac{1}{2}s^0 + \frac{1}{2}(s^0 - 2\theta c)$. Notice that $\|2\theta c\|_* \le \tau_{C^*}$, and so $s^0 - 2\theta c \in C^*$, and also $w \in C^*$ and $s^0 \in \mathrm{int} C^*$, whereby it follows that $\tilde{s} \in \mathrm{int} C^*$, validating (5.6). □

Proof of (iv). Because $R_\epsilon = +\infty$ it follows that there exists $x \neq 0$ satisfying $x \in C, Ax = 0$, and $c^T x = 0$. From Proposition 5.2 there also exists $\bar{x} \in X^*$ for which $\|\bar{x}\|_* = 1$ and $\|x\| = \bar{x}^T x$. Now suppose that v^* is finite, and let $\hat{s} \in D_\delta$ satisfy $B^*(\hat{s}, r) \subset C^*$ for some $r \ge 0$. Then there exists $\hat{y}$ for which $A^T \hat{y} + \hat{s} = c$, and so $x^T \hat{s} = x^T(c - A^T \hat{y}) = 0 - 0 = 0$. Also, $\hat{s} - r\bar{x} \in C^*$, and $x \in C$ implies that $0 \le x^T(\hat{s} - r\bar{x}) = -r x^T \bar{x} = -r\|x\|$, whereby $r = 0$. This then implies that r_δ $= 0$. □

Proof of Remark 3.1. If $R_\epsilon = 0$, the result follows trivially, since then $R_{\epsilon'} = 0$ for all $\epsilon' > 0$. So suppose that $0 < R_\epsilon < +\infty$. Let x^* be an optimal solution of P (P attains its optimum; see Lemma 5.1), and let $x' \in P_{\epsilon'}$ be given. Then $x := \frac{\epsilon}{\epsilon'} x' + \frac{\epsilon' - \epsilon}{\epsilon'} x^*$ satisfies $x \in P_\epsilon$, whereby $\|x\| \le R_\epsilon$. Let s^0 satisfy $\|s^0\|_* = 1$ and $B^*(s^0, \tau_{C^*}) \subset C^*$. Then from Proposition 5.3 we have $\tau_{C^*}\|x'\| \le (s^0)^T x' = \frac{\epsilon'}{\epsilon}(s^0)^T x - \frac{\epsilon' - \epsilon}{\epsilon}(s^0)^T x^* \le \frac{\epsilon'}{\epsilon}(s^0)^T x \le \frac{\epsilon'}{\epsilon}\|x\| \le \frac{\epsilon'}{\epsilon} R_\epsilon$. Therefore $\|x'\| \le \frac{\epsilon'}{\epsilon}\frac{1}{\tau_{C^*}} R_\epsilon$ for all $x' \in P'_\epsilon$, and so $R_{\epsilon'} \le \frac{\epsilon'}{\epsilon}\frac{1}{\tau_{C^*}} R_\epsilon$, proving the result. □

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