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POSTOPTIMAL ANALYSIS OF A LINEAR PROGRAM UNDER SIMULTANEOUS CHANGES IN MATRIX COEFFICIENTS

Robert M. FREUND

Sloan School of Management, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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Dedicated to Professor George B. Dantzig on the occasion of his 70th birthday.

This paper examines the sensitivity of a linear program to simultaneous changes in matrix coefficients. Consider a linear program whose coefficient matrix depends linearly on a scalar parameter θ. Previous research has attempted to express the optimal objective value $z(\theta)$ of the problem, as well as solutions to the primal and dual, as ratios of polynomial functions of θ over a range of θ. Herein, we study properties of $z(\theta)$ and the associated optimal basic feasible solution in a neighborhood about a fixed value $\bar{\theta}$ of θ. We obtain readily computable formulas for the Taylor series' (and hence all derivatives) of $z(\theta)$ and of the primal and dual optimal basic solutions about the point $\bar{\theta}$. Furthermore, even under degeneracy, we show how to determine whether or not $\bar{\theta}$ is one of finitely many possible values of θ for which derivatives of $z(\theta)$ may not exist, by examining the lexicographic order of a certain matrix. This test also reveals whether or not the formulas given represent left-sided and/or right-sided derivatives of $z(\theta)$ at $\bar{\theta}$.

Key words: Sensitivity Analysis, Parametric Programming, Linear Program, Lexicographic Order.

1. Introduction

The subject of this paper is the sensitivity of a linear program to simultaneous changes in matrix coefficients. Consider the standard linear program:

$$P: \quad \max \quad z = c \cdot x$$
$$\text{s.t.} \quad Ax = b,$$
$$x \geq 0.$$

When the vectors c and/or b are parameterized by a scalar parameter θ, we obtain the rim parametric programming problem. This problem has been treated extensively, and the classical results in this area can be found, for example, in Dinkelbach [3] and in Gal [5, 6]. In this paper, we consider the problem:

$$P(\theta): \quad \max \quad z(\theta) = c \cdot x$$
$$\text{s.t.} \quad A^{\theta}x = b,$$
$$x \geq 0,$$

where $A^{\theta} = F + \theta G$ is an $m \times n$ matrix parameterized by θ.

The problem $P(\theta)$ arises naturally in averaging constraints of the form,

$$\frac{\sum_{i=1}^{k} x_i^t}{\sum_{i=1}^{k} x_i^t + \sum_{i=k+1}^{n} x_i^t} \geq \theta, \quad t = 1, \ldots, T,$$

which after transformation becomes

$$\sum_{i=1}^{k} (1-\theta)x_i^t + \sum_{i=k+1}^{n} (-\theta)x_i^t \geq 0, \quad t = 1, \ldots, T.$$

This constraint says that the sum of the levels of the first k activities in modelling period t must constitute at least $100\theta\%$ of the sum of all activities levels in that time period.

In addition, $P(\theta)$ arises naturally in blending constraints. For example, suppose that x_i^t, $i = 1, \ldots, n$, represent delivered tonnages of coal entering a powerplant in period t, each with a heat content h_i (in MBTU/ton) and a sulfur content s_i (in lbs. SO_2/MBTU). Then if the powerplant's coal must have an average sulfur content of at most θ lbs. SO_2/MBTU in each period t, we have

$$\sum_{i=1}^{n} h_i(\theta - s_i)x_i^t \geq 0, \quad t = 1, \ldots, T.$$

In each of these two applications, derivatives of $z(\theta)$, and derivatives of optimal primal and dual solution values, constitute valuable information concerning the sensitivity of the underlying linear program to changes in θ.

The earliest result regarding $P(\theta)$ was the formula for the derivative of $z(\theta)$ with respect to θ, at $\theta = \bar{\theta}$, given by

$$z'(\bar{\theta}) = -\bar{\pi} G \bar{x}, \tag{1}$$

where $\bar{x}$ and $\bar{\pi}$ are optimal solutions to the primal and dual of $P(\theta)$ at $\theta = \bar{\theta}$. In 1956, Mills [14] obtained this formula for linear programs by examing saddlepoints of the Lagrangian $L(x, \pi) = c \cdot x - \pi(A^{\theta}x - b)$; Golshtein [8] gave a corrected proof via saddlepoints, where it is required that the sets of optimal solutions to $P(\theta)$ at $\bar{\theta}$ be bounded. In 1959, Saaty [17] rederives (1) when $P(\bar{\theta})$ is nondegenerate, using the identity $dB^{-1}(\theta)/d\theta = -B^{-1}(\theta)[dB(\theta)/d\theta]B^{-1}(\theta)$, where $B(\theta)$ is the basis matrix for $P(\theta)$.

Other research on $P(\theta)$ has centered on the computation of $z(\theta)$ as θ varies over some prespecified range R. When the matrix G has only one nonzero row or only one nonzero column, the problem can be analyzed by methods from parametric analysis, see e.g. Kim [9] and Orchard-Hayes [15]. However, when more than one row or column of G is nonzero, and in particular if G is not sparse, the characterization of $z(\theta)$ for $\theta \in R$ as well as the range of optimality of a given basis becomes much more difficult. If β is a basis for $P(\theta)$, and the basis matrix $(F + \theta G)_{\beta}$ is denoted by $B(\theta)$, then

$$B^{-1}(\theta) = \text{adj}(B(\theta))/\det B(\theta),$$

each of whose coefficients is a rational function of θ, i.e. an expression of the form $p(\theta)/q(\theta)$, where $p(\theta)$ and $q(\theta)$ are polynomials in θ. The limits of optimality of a basis $B(\theta)$ will be those points where some coefficient of the primal basic solution or reduced costs changes sign, or where the determinant of $B(\theta)$ is zero. In each case, the limit is the root of the numerator or denominator of a rational function of θ, i.e. the root of a polynomial of θ. Building on the analysis of $P(\theta)$ through rational functions and roots of polynomials, Weickenmeier [19] and Finkelstein and Gumenok [4] have developed parametric programming algorithms for $P(\theta)$.

Another form of sensitivity analysis of $P(\theta)$ is the analysis of the solution set of $P(\theta)$ as a function of θ, denoted by $X(\theta)$. At issue are conditions on $P(\theta)$ which guarantee that the mapping $X(\theta)$ satisfies certain continuity properties. Pertinent references include Dantzig et al. [2], Dinkelbach [3], Robinson [16], Lorenzen [13], Klatte [10, 11], Lommatzsch [12], and Gal [6].

The main concern of this paper is postoptimal analysis of $P(\theta)$ in a neighborhood of a given value of $\theta = \bar{\theta}$, without resorting to rational functions of θ. In Section 2, we present formulas for the Taylor series of $z(\theta)$ about $\theta = \bar{\theta}$, for all derivatives of $z(\theta)$, and for the optimal primal and dual basic solutions, each of whose terms is readily computable from the problem data and the current basis inverse. These formulas are shown to be valid when $P(\bar{\theta})$ is nondegenerate and has a finite optimum. However, degeneracy *is* prevalent in most large-scale linear programs, either in fact or due to numerical round-off error. Hence, in Section 3, we show that the main results of Section 2 are valid for all but a finite number of values of θ even in the case of degeneracy. We also present a test, based on the lexicographic order of a certain matrix, that determines whether or not the current basis yields left-sided and/or right-sided directional derivatives of $z(\theta)$ at $\bar{\theta}$.

This paper's origins stem from my interest in computing $z'(\theta)$ in a particular linear programming application of the sulfur blending constraint described above. In the study of this sensitivity analysis problem, I have tried to follow the standard of George Dantzig's work—the development of theory in the solution of practical problems.

2. Postoptimal analysis at nondegenerate optimal solutions

Consider the following parameterized linear program in standard form:

$$
\begin{aligned}
P(\theta): \quad & \text{maximize} \quad z(\theta) = c \cdot x \\
& \text{subject to} \quad (F + \theta G)x = b, \\
& \qquad\qquad\quad x \geqslant 0,
\end{aligned}
$$

where F, G are $m \times n$ matrices (with $m \leqslant n$), b, c are m- and n-vectors, respectively, and $G \neq 0$. Let R denote the real numbers and $\bar{R} = R \cup \{-\infty, +\infty\}$. $z(\theta)$ is defined to be $+\infty$ if $P(\theta)$ is feasible and unbounded, and $z(\theta) = -\infty$ if $P(\theta)$ is infeasible.

If we set $A^\theta = F + \theta G$, $P(\theta)$ and its dual, $D(\theta)$, are seen to be:

$P(\theta)$: maximize $z(\theta) = c \cdot x$

subject to $A^\theta x = b$

$x \geq 0$;

$D(\theta)$: minimize $v(\theta) = \pi \cdot b$

subject to $\pi \cdot A^\theta \geq c$.

Let $\beta \subset \{1, \ldots, n\}$. Let $\mathbb{R}^{m \times n}$ and $\mathbb{R}^n$ be the space of all real $m \times n$ matrices and real n-vectors, respectively. If M and y are a matrix and a vector, M_β or y_β denotes the submatrix or subvector whose columns or components correspond to the elements of β. If A^θ_β is nonsingular at θ, then β or A^θ_β is a basis for $P(\theta)$. If β is a basis for $P(\theta)$, then the primal basic solution is given by $x_\beta(\theta) = (A^\theta_\beta)^{-1} b$, $x_\alpha(\theta) = 0$, where $\alpha = \{1, \ldots, n\} \backslash \beta$, and the dual basic solution is given by $\pi_\beta(\theta) = c_\beta (A^\theta_\beta)^{-1}$. A basis β is primal feasible at θ if $x_\beta(\theta) \geq 0$, dual feasible at θ if $c - \pi_\beta(\theta) A^\theta \leq 0$, and optimal at θ if it is both primal and dual feasible at θ. β is defined to be a nondegenerate optimal basis at θ if $\pi_\beta(\theta) A^\theta - c + x(\theta) > 0$, where $x(\theta) = (x_\beta(\theta), x_\alpha(\theta))$. This corresponds to both primal and dual nondegeneracy at the optimal solution.

For a given vector y or matrix M, we define $\|y\| = \max |y_j|$ and $\|M\| = \max |m_{ij}|$, the standard supremum norm. An interval I in R is defined to be any set of the form (a, b), $[a, b]$, $[a, b)$, or $(a, b]$, where $a, b \in \bar{R}$. The ith row or jth column of a matrix M is denoted by $M_{i.}$ or $M_{.j}$, respectively. A property $\mathbb{P}$ is said to be true near $\bar{\theta}$ if there exists $\varepsilon > 0$ such that $\mathbb{P}$ is true for all $\theta \in (\bar{\theta} - \varepsilon, \bar{\theta} + \varepsilon)$. $\mathbb{P}$ is said to be true near $\bar{\theta}^-$ or near $\bar{\theta}^+$ if there exists $\varepsilon > 0$ such that $\mathbb{P}$ is true for all $\theta \in (\bar{\theta} - \varepsilon, \bar{\theta}]$, or $\theta \in [\bar{\theta}, \bar{\theta} + \varepsilon)$, respectively.

If β is a basis for $P(\bar{\theta})$, $(A^{\bar{\theta}}_\beta)^{-1}_{ij} = (\det(A^{\bar{\theta}}_\beta))^{-1}$ adj $(A^{\bar{\theta}}_\beta)_{ij}$, from which we see that each component of $(A^{\bar{\theta}}_\beta)^{-1}$ is given by $p(\bar{\theta})/q(\bar{\theta})$, where $p(\theta)$ and $q(\theta)$ are polynomials in θ of degree less than or equal to $m-1$ and m, respectively, and $q(\bar{\theta}) \neq 0$. For notational convenience, let $B = A^{\bar{\theta}}_\beta$, where $\bar{\theta}$ is a fixed value of θ and β is a given basis for $P(\bar{\theta})$; thus, B is a basis matrix at $\theta = \bar{\theta}$.

The main result for the nondegenerate case is:

Theorem 1. *Let β be a (unique) nondegenerate optimal basis for $P(\bar{\theta})$. Let $\bar{x}$ and $\bar{\pi}$ be the (unique) primal and dual optimal solutions to $P(\bar{\theta})$. Then for all θ near $\bar{\theta}$, β is a nondegenerate optimal basis for $P(\theta)$, and*

(i) $$z(\theta) = \sum_{i=0}^{\infty} c_\beta (\theta - \bar{\theta})^i (-B^{-1} G_\beta)^i \bar{x}_\beta,$$

(ii) $$z^k(\theta) = \sum_{i=k}^{\infty} \frac{(i)!}{(i-k)!} c_\beta (\theta - \bar{\theta})^{(i-k)} (-B^{-1} G_\beta)^i \bar{x}_\beta \quad \text{for } k = 1, \ldots,$$

where $z^k(\theta)$ is the kth *derivative of $z(\theta)$,*

(iii) $$x(\theta) = (x_\beta(\theta), x_\alpha(\theta)) = \left(\sum_{i=0}^{\infty} (\theta - \bar{\theta})^i (-B^{-1}G_\beta)^i \bar{x}_\beta, 0 \right)$$

is the unique optimal solution to $P(\theta)$,

(iv) $$\pi_\beta(\theta) = \sum_{i=0}^{\infty} (\theta - \bar{\theta})^i \bar{\pi} (-G_\beta B^{-1})^i$$

is the unique optimal solution to $D(\theta)$, and

(v) $$z^k(\bar{\theta}) = (k!) c_\beta (-B^{-1}G_\beta)^k \bar{x}_\beta,$$

where $B = A_\beta^{\bar{\theta}}$.

Note that when $k = 1$, (v) states that $z'(\theta) = -c_\beta B^{-1} G_\beta \bar{x} = -\bar{\pi} G \bar{x}$, which is a restatement of (1). Formula (1) can also be justified on more intuitive grounds. At $\theta = \bar{\theta}$, $\bar{x}$ and $\bar{\pi}$ are primal and dual optimal solutions to $P(\bar{\theta})$ and $D(\bar{\theta})$, and $\partial z / \partial b_i = \bar{\pi}_i$. As $\bar{x}$ is kept fixed, and θ changes to $\theta = \bar{\theta} + \Delta$, the new primal system satisfies:

$$(F + (\bar{\theta} + \Delta)G)\bar{x} = b + \Delta G \bar{x}.$$

In order to regain the original system of equations, with a right-hand side of b, we must change b to $b(\Delta) = b - \Delta G \bar{x}$. Using the chain rule for differentiation yields

$$\frac{\partial z}{\partial \Delta} = \sum_{i=1}^{m} \frac{\partial z}{\partial b_i} \frac{\partial b_i}{\partial \Delta} = \sum_{i=1}^{m} \bar{\pi}_i (-G\bar{x})_i = -\bar{\pi} G \bar{x}.$$

This is not a rigorous proof, inasmuch as $\bar{x}$ and $\bar{\pi}$ are assumed to remain fixed as θ varies.

Proof of Theorem 1. For all θ near $\bar{\theta}$, $(A_\beta^\theta)^{-1}$ exists and so

$$A_\beta^\theta (A_\beta^\theta)^{-1} = (B + (\theta - \bar{\theta})G_\beta)(A_\beta^\theta)^{-1} = I.$$

Premultiplying this system by B^{-1} and rearranging, we obtain:

$$(A_\beta^\theta)^{-1} = B^{-1} - (\theta - \bar{\theta})(B^{-1}G_\beta)(A_\beta^\theta)^{-1}. \tag{2}$$

By recursively substituting for $(A_\beta^\theta)^{-1}$ in this last expression, we obtain:

$$(A_\beta^\theta)^{-1} = \sum_{i=0}^{\infty} (\theta - \bar{\theta})^i (-B^{-1}G_\beta)^i B^{-1}.$$

This series converges for all $|\theta - \bar{\theta}| < \varepsilon = (m \| -B^{-1}G_\beta \|)^{-1}$. The series in (iii) follows from the fact that $x_\beta(\theta) = (A_\beta^\theta)^{-1} b$. (i) follows from the equation $z(\theta) = c_\beta x_\beta(\theta)$,

and (ii) and (v) derive from (i). Furthermore,

$$\pi_\beta(\theta) = c_\beta(A_\beta^\theta)^{-1} = \sum_{i=0}^{\infty} (\theta-\bar{\theta})^i c_\beta(-B^{-1}G_\beta)^i B^{-1}$$

$$= \sum_{i=0}^{\infty} (\theta-\bar{\theta})^i c_\beta B^{-1}(-G_\beta B^{-1})^i$$

$$= \sum_{i=0}^{\infty} (\theta-\bar{\theta})^i \bar{\pi}(-G_\beta B^{-1})^i,$$

which is the series in (iv). Because β is a nondegenerate basis for $P(\bar{\theta})$, $\pi_\beta(\bar{\theta})A^{\bar{\theta}} - c + x(\bar{\theta}) > 0$, and so for all θ near $\bar{\theta}$, $\pi_\beta(\theta)A^\theta - c + x(\theta) > 0$ as well, thereby showing that $x(\theta)$ and $\pi(\theta)$ are feasible and nondegenerate solutions to $P(\theta)$ and $D(\theta)$. □

(The series (2) can also be derived by appeal to the well known fact that $dM^{-1}/dt = -M^{-1}(t)(dM/dt)M^{-1}(t)$, where $M(t)$ is a nonsingular matrix whose coefficients are functions of t. This formula can be used to inductively prove that $d^kM^{-1}(t)/d^kt = (k!)(-M^{-1}(t)D)^kM^{-1}(t)$, in the case when $M(t) = C + Dt$, thereby obtaining the Taylor series $M^{-1}(t) = \sum_{k=0}^{\infty}(t-\bar{t})^k(-M^{-1}(\bar{t})D)^kM^{-1}(\bar{t})$. Substituting $\theta = t$, $\bar{\theta} = \bar{t}$, $A_\beta^\theta = M(\theta)$, $M(\bar{t}) = B$, and $G_\beta = D$, we obtain (2).)

Because most large-scale linear programming computer codes compute and record the primal and dual solutions and the basis inverse or the $L-U$ decomposition of the basis, each of the terms of the series in (i)-(v) can readily be computed. The computational burden of computing the higher order terms of these series is probably excessive, unless $B^{-1}G$ is very sparse. (Even when G is sparse, $B^{-1}G$ may not be sparse.)

From a theoretical point of view, the nondegeneracy hypothesis of Theorem 1 is satisfied for $P(\bar{\theta})$ except for a certain collection of problem data (b, c) which has measure zero. However, as a matter of experience, most large-scale linear programs exhibit substantial degeneracy in the optimal basis, either in fact (primarily the result of special structures) or due to numerical roundoff. It thus is necessary to examine the general (degenerate or nondegenerate) case of $P(\theta)$ if the formulas of Theorem 1 are to have practical significance.

3. Postoptional analysis at degenerate or nondegenerate optimal solutions

We begin this section with a few defintions. Let $K = \{\theta \mid P(\theta)$ is feasible and has a finite solution$\}$, i.e. K is the set of θ for which $-\infty < z(\theta) < +\infty$. For each $\beta \subset \{1, \ldots, n\}$, with $|\beta| = m$, define $R_\beta = \{\theta \mid \beta$ is an optimal basis for $P(\theta)\}$. Each R_β is called the 'critical region' for β, see e.g. Gal [7] or Dinkelbach [3]. Finally, we define $U = \{\theta \mid z(\theta) = +\infty\}$ and $V = \{\theta \mid z(\theta) = -\infty\}$. The following lemma, which has been obtained in a different formulation by Dinkelbach [3], will serve as a basis for the theorems of this section. Its proof is included here for completeness.

Lemma 1 (see Dinkelbach [3])

(i) *U consists of a finite union of intervals,*

(ii) *R_β consists of a finite union of intervals, for each potential basis β,*

(iii) *K consists of a finite union of intervals, and*

(iv) *V consists of a finite union of intervals.*

Proof. For any given value of θ, we can assume that A has full rank, by appropriate addition of artificial variables, or deletion of rows, if necessary. Thus, if $z(\theta) = +\infty$ for some θ, there exists a basis β such that $x_\beta(\theta) \geq 0$, and a column $j \notin \beta$ such that $(A^\theta_\beta)^{-1}A^\theta_j \leq 0$ and $c_j - \pi_\beta(\theta)A^\theta_{\cdot j} > 0$. Therefore,

$$\{\theta \mid z(\theta) = +\infty\} = \bigcup_\beta \bigcup_{j \notin \beta} \{\theta \mid \det(A^\theta_\beta) \neq 0\}$$
$$\cap \{\theta \mid (A^\theta_\beta)^{-1}b \geq 0\} \cap \{\theta \mid (A^\theta_\beta)^{-1}A^\theta_{\cdot j} \leq 0\}$$
$$\cap \{\theta \mid c_j - c_\beta(A^\theta_\beta)^{-1}A^\theta_{\cdot j} > 0\}.$$

Because $\det(A^\theta_\beta)$ is a polynomial in θ (of degree of at most m), $\{\theta \mid \det(A^\theta_\beta) \neq 0\}$ is a finite union of intervals. We can assume that if $\det(A^\theta_\beta) \neq 0$, then $\det(A^\theta_\beta) > 0$, by rearranging columns, if necessary, whereby $\{\theta \mid (A^\theta_\beta)^{-1}b \geq 0\} = \{\theta \mid \mathrm{adj}(A^\theta_\beta)b \geq 0$, and each constraint of this latter formulation is a polynomial. Hence this set is the intersection of a finite union of intervals, which is a finite union of intervals. Similarly, $\{\theta \mid (A^\theta_\beta)^{-1}A^\theta_{\cdot j} \leq 0\} = \{\theta \mid \mathrm{adj}(A^\theta_\beta)A^\theta_{\cdot j} \leq 0\}$, each constraint of which is a polynomial in θ, so this set also is a finite union of intervals. Finally, $\{\theta \mid c_j - c_\beta(A^\theta_\beta)^{-1}A^\theta_{\cdot j} > 0\} = \{\theta \mid (\det A^\theta_\beta))c_j > c_\beta(\mathrm{adj}(A^\theta_\beta)A^\theta_{\cdot j})\}$ which is also a finite union of intervals. Thus U is the union over all β of the intersection of a finite union of intervals, which is itself a finite union of intervals. This proves (i). To prove (ii), note that

$$R_\beta = \{\theta \mid \det(A^\theta_\beta) \neq 0, (A^\theta_\beta)^{-1}b \geq 0, \text{ and } c_\beta(A^\theta_\beta)^{-1}A^\theta \geq c\}$$
$$= \{\theta \mid \det(A^\theta_\beta) \neq 0\} \cap \{\theta \mid \mathrm{adj}(A^\theta_\beta)b \geq 0\} \cap \{\theta \mid c_\beta(\mathrm{adj}(A^\theta_\beta))\theta A^\theta \geq c(\det(A^\theta_\beta)\}.$$

Using the logic employed above, we see that the latter formulation is the intersection of three sets, each of which is a finite union of intervals. (iii) follows from (ii) and the fact there are a finite number of bases, and (iv) follows from (i) and (iii). □

Let E be the union over all $\beta \subset \{1, \ldots, n\}$ of the set of endpoints of the intervals of R_β. E then is the set of 'breakpoints' of the function $z(\theta)$, i.e., E is the set of points at which a basis changes from primal or dual feasible to infeasible, or the basis matrix becomes singular.

In view of Lemma 1, we have:

Theorem 2. *Let β be an optimal basis for $P(\bar\theta)$. Let $\bar x$ and $\bar\pi$ be the primal and dual basic optimal solutions to $P(\bar\theta)$ corresponding to β. Then, except for a finite number of values of $\bar\theta \in K$, equations* (i)-(v) *of Theorem* 1 *are true for all θ near $\bar\theta$.*

Proof of Theorem 2. For any $\bar{\theta} \in K \setminus E$, and any optimal basis β for $P(\bar{\theta})$, there is an open interval $(\bar{\theta}-\varepsilon, \bar{\theta}+\varepsilon)$ such that β is an optimal basis for $P(\theta)$ for all $\theta \in (\bar{\theta}-\varepsilon, \bar{\theta}+\varepsilon)$. This being the case, the power series' of (i)-(v) of Theorem 1 converge. Since E is a finite union (over all $\beta \subset \{1, \ldots, n\}$) of a finite number of endpoints, E is finite, proving the theorem. □

We now turn our attention to the task of determining for a given problem $P(\bar{\theta})$ if $\bar{\theta}$ is a breakpoint, i.e., an element of E. If $P(\bar{\theta})$ has a non-degenerate solution, then $\bar{\theta}$ is not an element of E, and so the conclusions of Theorem 1 are valid. However, even if $P(\bar{\theta})$ has a degenerate optimal basic solution, $\bar{\theta}$ need not be an element of E. This possibility is illustrated in the following example, where Initial Tableau A is shown, followed by Tableaus 1-3, which illustrate four different bases and basic solutions by pivoting on the initial tableau. In these tableaus, the bottom row represents the objective function $z(\theta)$. Note that this example is a transformation of a rim parametric program, as Tableau 1 shows.

Initial Tableau A

RHS	x_1	x_2	x_3	x_4	x_5
1	1	0	0	0	0
10	θ	1	0	1	2
0	$-\theta$	0	1	1	3
0	1	0	0	0	−5

Tableau 1

RHS	x_1	x_2	x_3	x_4	x_5	Basis	Range of Optimality
1	1	0	0	0	0	$\beta_1 = \{1, 2, 3\}$,	$0 \leq \theta \leq 10$
$10-\theta$	0	1	0	1	2		
θ	0	0	1	1	3		
−1	0	0	0	0	−5		

Tableau 2

RHS	x_1	x_2	x_3	x_4	x_5		
1	1	0	0	0	0		
$10-\theta$	0	1	0	1	2	$\beta_2 = \{1, 3, 4\}$,	$5 \leq \theta \leq 10$
$-10+2\theta$	0	−1	1	0	1		
−1	0	0	0	0	−5		

Tableau 3

RHS	x_1	x_2	x_3	x_4	x_5	
1	1	0	0	0	0	
θ	0	0	1	1	3	$\beta_3 = \{1, 2, 4\}, \quad 0 \leq \theta \leq 5$
$10-2\theta$	0	1	−1	0	−1	
−1	0	0	0	0	−5	

This example has degenerate optimal solutions for $0 \leq \theta \leq 10$, yet the only breakpoints are $E = \{0, 5, 10\}$. For $0 < \theta < 10$, there are multiple optimal solutions to the primal. β_1 is optimal over the range $0 \leq \theta \leq 10$, yet the ranges of optimality for β_2 and β_3 are [5, 10] and [0, 5], respectively. As θ decreases below 0, β_1 and β_3 become infeasible.

We now show how to determine if $\bar{\theta}$ is a breakpoint or not (without pivoting beyond the final tableau) given the problem data (F, G, b, c) and an optimal basis β for $P(\bar{\theta})$. In order to demonstrate how this can be done, some more notation and a result from linear algebra are first presented.

Let $f'_+(\theta)$ and $f'_-(\theta)$ denote the directional derivative of $f(\theta)$ in the plus and minus direction, i.e.,

$$f'_+(\theta) = \lim_{h \downarrow 0} \left(\frac{f(\theta+h) - f(\theta)}{h} \right), \qquad f'_-(\theta) = \lim_{h \uparrow 0} \left(\frac{f(\theta) - f(\theta+h)}{h} \right).$$

Let $\geqslant$ denote the lexicographic ordering for vectors and extended to matrices M, see e.g. Dantzig [1], where $M \geqslant 0$ if $M_{i\cdot} \geqslant 0$ for each row i of M. Given a vector y, define $M \geqslant 0 \bmod y$ if $M_{i\cdot} \geqslant 0$ whenever $y_i = 0$, and $M \approx 0 \bmod y$ if $M_{i\cdot} = (0, \ldots, 0)$ whenever $y_i = 0$. The ordering $\geqslant \bmod y$ corresponds to the lexicographic ordering when $y = 0$, otherwise it corresponds to the lexicographic ordering restricted only to those rows of M for which the corresponding components of y are zero. The following intermediate results will be used in the analysis:

Lemma 2 (see Veinott [18]). *If $D \in \mathbb{R}^{m \times m}$ is a matrix of rank r, $M \in \mathbb{R}^{l \times m}$, and $v \in \mathbb{R}^m$, then there exists a $j \leq r$ such that $MD^1v, MD^2v, MD^3v, \ldots$, are all linear combinations of $MD^1v, \ldots, MD^jv$.*

One version of this lemma is presented in [18]. It is proved here for completeness.

Proof. The vectors D^iv, $i = 1, \ldots$, all lie in the subspace L spanned by the columns of D. Thus, the vectors $D^1v, \ldots, D^{r+1}v$ cannot all be linearly independent, whereby there exists $j \leq r$ such that $D^{j+1}v$ is a linear combination of $D^1v, \ldots, D^jv$. We now show by induction that D^iv is a linear combination of $D^1v, \ldots, D^jv$ for all $i = 1, \ldots$. Clearly, this is true for $i = 1, \ldots, j+1$. Suppose it is true for $k \geqslant j+1$. Then $D^kv = \sum_{l=1}^{j} \lambda_l D^l v$, and $D^{k+1}v = \sum_{l=2}^{j+1} \lambda_l D^l v$. But since D^{j+1} is a linear combination of $D^1v, \ldots, D^jv$, the result follows after premultiplication by M. □

Lemma 3. *For given matrices* $M \in \mathbb{R}^{l \times m}$, $D \in \mathbb{R}^{m \times m}$ *and a vector* $v \in \mathbb{R}^m$, *let* $y(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i MD^i v$ *for all* ε *near* 0, *and suppose* $y(0) = Mv \geq 0$. *Define* Q^k *and* P^k *to be the k-column matrices* $Q^k = [MD^1 v, \ldots, MD^k v]$ *and*

$$P^k = [M(-D)^1 v, \ldots, M(-D)^k v] \quad \text{for } k = 1, \ldots.$$

Let r be the rank of D.

Then

(i) $y(\varepsilon) \geq 0$ *for all* ε *near* 0^+ *if and only if* $Q^r \geq 0 \bmod y(0)$,

(ii) $y(\varepsilon) \geq 0$ *for all* ε *near* 0^- *if and only if* $P^r \geq 0 \bmod y(0)$,

(iii) $y(\varepsilon) \geq 0$ *for all* ε *near* 0 *if and only if* $Q^r \geq 0 \bmod y(0)$ *and* $P^r \geq 0 \bmod y(0)$.

Proof. Because $y(0) \geq 0$, $y(\varepsilon) \geq 0$ for all ε near 0^+ if and only if the infinite matrix Q^∞ has a lexicographically nonnegative row i whenever $y(0)_i = 0$, i.e., whenever $Q^\infty \geq 0 \bmod y(0)$. However, in view of Lemma 2, $Q^\infty \geq 0 \bmod y(0)$ if and only if $Q^r \geq 0 \bmod y(0)$. This shows (i). (ii) follows from a parallel argument and (iii) follows from (i) and (ii). □

We are now ready to examine the question of whether or not $\bar{\theta}$ is a breakpoint. For a given problem $P(\bar{\theta})$ with an optimal basis β, let $\bar{x}$ and $\bar{\pi}$ denote the optimal primal and dual basic solutions for β. Define the following $m \times k$ and $n \times k$ matrices, for $k = 1, \ldots$:

$\bar{X}^k = [(-B^{-1}G_\beta)^1 \bar{x}_\beta, (-B^{-1}G_\beta)^2 \bar{x}_\beta, \ldots, (-B^{-1}G_\beta)^k \bar{x}_\beta]$,

$\bar{Y}^k = [(B^{-1}G_\beta)^1 \bar{x}_\beta, (B^{-1}G_\beta)^2 \bar{x}_\beta, \ldots, (B^{-1}G_\beta)^k \bar{x}_\beta]$,

$\bar{C}^k = [\bar{\pi}(-G_\beta B^{-1})A, \bar{\pi}(-G_\beta^{-1}B)^2 A, \ldots, \bar{\pi}(-G_\beta B^{-1})^k A]$, and

$\bar{D}^k = [\bar{\pi}(G_\beta B^{-1})A, \bar{\pi}(G_\beta B^{-1})^2 A, \ldots, \bar{\pi}(G_\beta B^{-1})^k A]$.

Theorem 3. *Let* β *be an optimal basis for* $P(\bar{\theta})$. *Let* $\bar{x}$, $\bar{\pi}$, $\bar{X}^m$, $\bar{Y}^m$, $\bar{C}^m$, *and* $\bar{D}^m$ *be defined as above. Then*

(i) $\bar{\theta}$ is not a breakpoint if and only if $\bar{X}^m \geq 0 \bmod \bar{x}_\beta$ and $\bar{C}^m \geq 0 \bmod(\bar{\pi}A - c)$, $\bar{Y}^m \geq 0 \bmod \bar{x}_\beta$, and $\bar{D}^m \geq 0 \bmod(\bar{\pi}A - c)$.

(ii) If $\bar{X}^m \geq 0 \bmod \bar{x}_\beta$ and $\bar{C}^m \geq 0 \bmod(\bar{\pi}A - c)$, then β is an optimal basis for $P(\theta)$ for all θ near $\bar{\theta}^+$, and equations (i)-(v) of Theorem 1 are valid for all θ near $\bar{\theta}^+$, with $z^k(\cdot)$ replaced by $z_+^k(\cdot)$.

(iii) If $\bar{Y}^m \geq 0 \bmod \bar{x}_\beta$ and $\bar{D}^m \geq 0 \bmod(\bar{\pi}A - c)$, then β is an optimal basis for $P(\theta)$ for all θ near $\bar{\theta}^-$, and equations (i)-(v) of Theorem 1 are valid for all θ near $\bar{\theta}^-$, with $z^k(\cdot)$ replaced by $z_-^k(\cdot)$.

Proof. We first prove (ii). Suppose $\bar{X}^m \geq 0 \bmod \bar{x}_\beta$ and $\bar{C}^m \geq 0 \bmod(\bar{\pi}A - c)$. Let $M = I$, $D = (-B^{-1}G_\beta)$, and $v = \bar{x}_\beta$. Note that $x_\beta(\theta) = \sum_{i=0}^{\infty} (\theta - \bar{\theta})^i MD^i v$, and so by Lemma 3, $x_\beta(\theta) \geq 0$ for all θ near $\bar{\theta}^+$ if and only if $Q^r \geq 0 \bmod \bar{x}_\beta$. However, $Q^r = \bar{X}^r$ and $\bar{X}^r \geq 0 \bmod \bar{x}_\beta$ if and only if $\bar{X}^m \geq 0 \bmod \bar{x}_\beta$, since $r \leq m$. Thus β is primal feasible for all θ near $\bar{\theta}^+$ if and only if $\bar{X}^m \geq 0 \bmod \bar{x}_\beta$. As regards the dual, let $M = A$, $D = (-G_\beta B^{-1})$, and $v = \bar{\pi}$. Note that $\pi_\beta(\theta)A = \sum_{i=0}^{\infty} (\theta - \bar{\theta})^i (MD^i v)^t$, and so by Lemma 3, $\pi_\beta(\theta)A - c \geq 0$ for all θ near $\bar{\theta}^+$ if and only if $Q^r \geq 0 \bmod(\bar{\pi}A - c)$.

The latter is true if and only if $\bar{C}^m \geq 0 \bmod(\bar{\pi}A - c)$. Thus β is dual feasible for all θ near $\bar{\theta}^+$, and so is optimal for $P(\theta)$ for all θ near $\bar{\theta}^+$, whereby equations (i)-(v) of Theorem 1 are valid, with $z^k(\cdot)$ replaced by $z_+^k(\cdot)$.

The proof of (iii) parallels that of (ii). (i) follows from (ii) and (iii). □

Note that if the rank of G is known, it can be used instead of m in the above theorem.

We close with an example of 'worst case' behavior of $z(\theta)$. Theorem 3 states that the formulas of Theorem 1 are valid when certain submatrices of $\bar{X}^m$, $\bar{Y}^m$, $\bar{C}^m$, and $\bar{D}^m$ are either ≥ 0, ≤ 0, or $= 0$. However, if it is not true that '$\bar{X}^m \geq 0 \bmod \bar{x}_\beta$ and $\bar{C}^m \geq 0 \bmod(\bar{\pi}A - c)$' nor true that '$\bar{Y}^m \geq 0 \bmod \bar{x}_\beta$ and $\bar{D}^m \geq 0 \bmod(\bar{\pi}A - c)$', then $\bar{\theta}$ is an isolated point at which β is an optimal basis; and for all θ near $\bar{\theta}$ and not equal to $\bar{\theta}$, β is *not* an optimal basis, whereby equations (i)-(v) may be completely false. This phenomenon is illustrated in the following example, in which initial tableau B is shown first, followed by Tableaus I-III, which illustrate different bases and basic solutions, obtained by pivoting from the initial tableau.

Initial Tableau B

RHS	x_1	x_2	x_3	x_4	x_5	x_6	x_7
1	1	0	0	0	0	0	0
1	0	1	0	1	0	0	0
1	$1-\theta$	0	1	0	−1	0	0
0	0	0	0	θ	−1	−1	1
0	0	0	0	0	0	1	−1

Tableau I

RHS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Basis	Range of Optimality
1	1	0	0	0	0	0	0	$\beta_1 = \{1, 2, 3, 6\}$,	$\theta = 0$
1	0	1	0	1	0	0	0		
θ	0	0	1	0	−1	0	0		
0	0	0	0	$-\theta$	1	1	−1		
0	0	0	0	θ	−1	0	0		

Tableau II

RHS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
1	1	0	0	0	0	0	0	$\beta_2 = \{1, 3, 4, 6\}$, $\theta \geq 0$
1	0	1	0	1	0	0	0	
θ	0	0	1	0	−1	0	0	
θ	0	θ	0	0	1	1	−1	
$-\theta$	0	$-\theta$	0	0	−1	0	0	

Tableau III

RHS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
1	1	0	0	0	0	0	0	$\beta_3 = \{1, 2, 5, 7\}$, $\theta \leqslant 0$
1	0	1	0	1	0	0	0	
$-\theta$	0	0	−1	0	1	0	0	
$-\theta$	0	0	−1	θ	0	−1	1	
$-\theta$	0	0	−1	θ	0	0	0	

For this particular problem, β_2 is an optimal basis for $\theta \geqslant 0$, β_3 is an optimal basis for $\theta \leqslant 0$, and β_1 is an optimal basis only for $\theta = 0$. For all θ, $z(\theta) = \theta$, whereby $z'(\theta) = 1$. However, at $\bar{\theta} = 0$ with optimal basis β_1, $\bar{\pi} = (0, 0, 0, -1)$ and $-\bar{\pi}G\bar{x} = 0$, thus yielding a wrong determination of $z'(\theta)$. This occurs because $\bar{\theta} = 0$ is a breakpoint of $z(\theta)$, and because $\bar{X}^m \geqslant 0$ but $\bar{C}^m \leqslant 0$, whereby $\bar{\theta}$ is an isolated point for which β_1 is an optimal basis, i.e., β_1 is not an optimal basis for any θ near $\bar{\theta}$, except $\theta = \bar{\theta}$.

Note that the above example is just a transformation of the parametric programming problem:

$$\begin{aligned} \text{maximize} \quad & z(\theta) = \theta x_4 - x_5 \\ \text{subject to} \quad & x_2 + x_4 = 1, \\ & x_3 - x_5 = \theta, \\ & x_2, \ x_3, \ x_4, \ x_5 \geqslant 0, \end{aligned}$$

which shows that even this seemingly well-behaved rim parametric programming problem can have a badly behaved breakpoint.

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