Proofs, Tables and Figures

In this electronic companion to the paper, we provide proofs of the theorems and lemmas. This companion also contains several accompanying tables and figures to the paper.

Table EC.1 Range of critical fractile values where Assumption 1 holds.

Distribution	*When is Assumption 1 satisfied?*	*Notes*
Normal(μ,σ)	$\frac{b}{b+h} \geq \frac{1}{2}$	
Exponential(λ)	$\frac{b}{b+h} \geq 0$	
Lognormal(μ,σ)	$\frac{b}{b+h} \geq \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(-\frac{\sigma}{2}\right)$	erf: error function
Pareto(x_m,α)	$\frac{b}{b+h} \geq 0$	
Uniform(A,B)	$\frac{b}{b+h} \geq 0$	
Gamma(α,β)	$\frac{b}{b+h} \geq \frac{1}{\Gamma(\alpha)}\gamma(\alpha,\alpha-1)$	Γ: gamma function, γ: incomplete gamma function
Beta(α,β)	$\frac{b}{b+h} \geq \frac{B\left(\frac{\alpha-1}{\alpha+\beta-2};\alpha,\beta\right)}{B(\alpha,\beta)}$	B: beta function
Power Law(α)	$\frac{b}{b+h} \geq 0$	
Logistic(μ,s)	$\frac{b}{b+h} \geq \frac{1}{2}$	
GEV(μ,σ,ξ)	$\frac{b}{b+h} \geq e^{-1-\xi}$	for $\xi \geq 0$
Chi(k)	$\frac{b}{b+h} \geq P\left(\frac{k}{2},\frac{k-1}{2}\right)$	for $k \geq 1$; P: regularized gamma function
Chi-squared(k)	$\frac{b}{b+h} \geq \begin{cases} \frac{1}{\Gamma\left(\frac{k}{2}\right)}\gamma\left(\frac{k}{2},\frac{k-2}{2}\right), & \text{if } k \geq 2 \\ 0, & \text{if } k < 2 \end{cases}$	
Laplace(μ,β)	$\frac{b}{b+h} \geq \frac{1}{2}$	
Weibull(λ,k)	$\frac{b}{b+h} \geq \begin{cases} 1-e^{-\frac{k-1}{k}}, & \text{if } k \geq 1 \\ 0 & \text{if } k < 1 \end{cases}$	

Table EC.2 Average errors (%) with samples from an exponential distribution.

(a) Sample average approximation

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	2.39	1.83	2.08	2.24	2.62	3.22	4.05	4.67	7.65	10.87	33.76
50	0.77	0.73	0.81	0.87	1.35	1.49	1.93	2.38	3.10	7.33	16.89
100	0.54	0.34	0.48	0.60	0.70	0.91	0.96	1.50	2.03	3.24	8.56
200	0.27	0.23	0.27	0.29	0.34	0.40	0.49	0.64	1.22	2.22	4.36

(b) Distribution fitting

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	1.88	1.54	1.54	1.69	2.03	2.60	3.37	4.26	5.81	9.64	40.06
50	0.65	0.64	0.69	0.80	0.99	1.23	1.53	1.90	2.72	4.88	22.93
100	0.36	0.34	0.39	0.46	0.57	0.73	0.96	1.33	1.91	2.62	9.03
200	0.21	0.20	0.21	0.24	0.28	0.34	0.43	0.59	0.94	1.64	7.25

Figure EC.1 Upper bound for a log-concave distribution with $\frac{b}{b+h}$ quantile q^*.

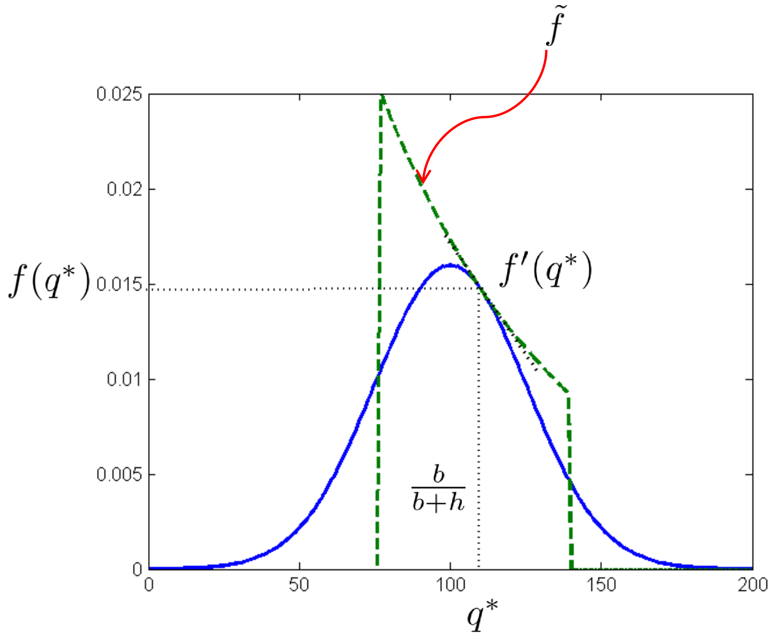

Table EC.3 Average errors (%) with samples from a normal distribution.

(a) Sample average approximation

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	6.03	3.84	3.81	3.11	2.60	2.95	3.50	4.91	6.23	8.71	42.85
50	2.31	1.69	1.62	1.58	1.41	1.60	1.59	2.06	3.26	4.57	13.76
100	1.63	1.15	0.92	0.86	0.83	0.75	0.92	1.08	1.56	2.18	5.94
200	0.81	0.45	0.38	0.36	0.30	0.29	0.38	0.47	0.81	1.41	3.65

(b) Distribution fitting

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	4.65	3.53	3.07	2.73	2.62	2.77	3.16	3.74	5.48	12.83	75.12
50	1.91	1.43	1.27	1.20	1.24	1.38	1.60	1.87	2.53	4.41	18.77
100	1.13	0.90	0.78	0.71	0.68	0.69	0.76	0.89	1.17	1.75	6.59
200	0.47	0.36	0.28	0.25	0.25	0.27	0.33	0.42	0.63	1.03	3.92

Table EC.4 Average errors (%) with samples from a Pareto distribution.

(a) Sample average approximation

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	0.88	0.69	0.83	0.93	1.14	1.59	2.18	3.12	6.70	28.33	34.35
50	0.28	0.28	0.31	0.37	0.60	0.73	1.02	1.39	2.28	6.12	33.39
100	0.19	0.13	0.19	0.24	0.29	0.39	0.45	0.83	1.54	3.28	39.74
200	0.09	0.08	0.10	0.11	0.14	0.17	0.24	0.35	0.80	1.97	6.86

(b) Distribution fitting

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	0.70	0.61	0.69	0.79	0.96	1.24	1.68	2.47	4.83	9.69	40.31
50	0.25	0.25	0.28	0.32	0.39	0.53	0.76	1.15	2.25	4.52	18.92
100	0.15	0.14	0.16	0.20	0.24	0.31	0.41	0.62	1.34	2.85	11.71
200	0.08	0.08	0.09	0.10	0.12	0.15	0.19	0.30	0.72	1.65	6.97

Table EC.5 Average errors (%) with samples from a Beta distribution.

(a) Sample average approximation

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	5.15	4.80	4.07	3.07	3.06	2.63	2.92	2.90	4.30	4.27	14.99
50	2.69	2.28	2.15	1.99	1.63	1.41	1.26	1.25	1.34	1.88	2.47
100	1.86	1.17	0.94	0.82	0.88	0.85	0.73	0.77	0.79	0.89	0.78
200	1.11	0.59	0.40	0.35	0.36	0.35	0.32	0.31	0.32	0.30	0.41

(b) Distribution fitting

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	5.62	4.42	3.41	2.70	2.38	2.32	2.34	2.39	3.40	7.13	35.94
50	2.90	2.24	1.77	1.50	1.43	1.49	1.61	1.65	1.68	2.88	9.40
100	1.43	1.06	0.83	0.69	0.62	0.61	0.62	0.63	0.64	0.99	4.45
200	0.70	0.43	0.30	0.26	0.25	0.25	0.26	0.26	0.24	0.37	2.29

Table EC.6 Average errors (%) with samples from a mixed normal distribution.

(a) Sample average approximation

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	3.99	2.64	1.89	2.14	6.17	3.98	5.52	11.72	5.78	3.84	4.41
50	1.29	0.86	0.61	0.39	0.38	0.35	0.53	0.79	1.81	1.62	4.26
100	0.74	0.43	0.35	0.27	0.39	0.45	0.33	0.39	0.55	0.71	2.51
200	0.37	0.21	0.16	0.13	0.08	0.08	0.12	0.19	0.22	0.59	1.47

(b) Distribution fitting

	Critical quantile										
Sample size	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
25	2.96	1.67	1.86	3.32	3.09	2.36	10.46	16.65	6.55	13.74	32.75
50	1.56	1.14	0.52	0.47	0.33	0.54	0.49	2.18	0.24	1.84	25.18
100	0.90	0.40	0.35	1.08	0.81	0.18	0.16	1.85	0.33	1.19	7.76
200	0.69	0.38	0.15	0.59	0.42	0.48	0.57	0.37	0.94	4.61	2.97

EC.1. Proof of Theorem 2

As a preliminary for the proof, let us first state a version of Bernstein's inequality (Bernstein 1927):

THEOREM EC.1 **(Bernstein's inequality)**. *Let $X^1, X^2, \ldots, X^N$ be i.i.d. random variables such that $|X^1| \le c$ almost surely, and $Var(X^1) = \sigma^2$. Then, for any $t > 0$,*

$$\Pr\left(\frac{1}{N}\sum_{i=1}^{N} X^i - E[X^1] \ge t\right) \le \exp\left(\frac{-Nt^2}{2\sigma^2 + 2tc/3}\right).$$

For the proof of Theorem 2, we will require the following proposition.

PROPOSITION EC.1. *Suppose $\hat{Q}_N$ is the $\frac{b}{b+h}$ quantile of a random sample from D with size N. Then, for any $\gamma > 0$,*

$$\Pr\left(\partial_- C(\hat{Q}_N) \le \gamma \text{ and } \partial_+ C(\hat{Q}_N) \ge -\gamma\right) \ge 1 - 2\exp\left(\frac{-3N\gamma^2}{6bh + 8\gamma(b+h)}\right).$$

Proof. *Let $\bar{F}$ be the complementary cdf of D, i.e., $\bar{F}(q) = \Pr(D \ge q) = 1 - F(q) + \Pr(D = q)$. For a random sample $\{D^1, \ldots, D^N\}$ drawn from D, let $\hat{Q}_N$ be the $\frac{b}{b+h}$ sample quantile. Define*

$$\hat{F}_N(q) \triangleq \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{[D^i \le q]},$$
$$\hat{\bar{F}}_N(q) \triangleq \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{[D^i \ge q]}.$$

For simplicity, define $\alpha \triangleq \frac{\gamma}{b+h}$ and $\beta \triangleq \frac{b}{b+h}$. Define the events $B \triangleq [\partial_+ C(\hat{Q}_N) < -\gamma] = [F(\hat{Q}_N) < \beta - \alpha]$ and $L \triangleq [\partial_- C(\hat{Q}_N) > \gamma] = [\bar{F}(\hat{Q}_N) < 1 - \beta - \alpha]$. To prove Proposition EC.1, we need to find an upper bound for $\Pr(B)$ and for $\Pr(L)$.

Define the quantile $q_1 \triangleq \inf\{q : F(q) \ge \beta - \alpha\}$. Since F is nondecreasing, we have that $B = [\hat{Q}_N < q_1]$. Consider a monotonically decreasing, nonnegative sequence $\{\tau^k\}_{k=1}^{\infty}$, where $\tau^k \downarrow 0$. Define the sequence of events $\{B_k\}_{k=1}^{\infty}$, where

$$B_k \triangleq [\hat{Q}_N \le q_1 - \tau^k] = [\hat{F}_N(q_1 - \tau^k) \ge \beta].$$

Note that since $\hat{F}_N(q_1 - \tau^k) \le \hat{F}_N(q_1 - \tau^{k+1})$, then it follows that $B_k \subseteq B_{k+1}$. Thus, we have that $B_k \uparrow \lim_{k\to\infty} B_k \triangleq \bar{B}$, which implies $\Pr(B_k) \uparrow \Pr(\bar{B})$. Note also that $B \subseteq \bar{B}$, thus $\Pr(B) \le \Pr(\bar{B})$.

From the definition of q_1, observe that for every $k \ge 1$, there exists $\varepsilon_k > \alpha$ such that $F(q_1 - \tau^k) = \beta - \varepsilon_k < \beta - \alpha$. Note that

$$F(q_1 - \tau^k)\left(1 - F(q_1 - \tau^k)\right) < (\beta - \alpha)(1 - \beta + \varepsilon_k). \qquad \text{(EC.1)}$$

Thus, we have that

$$\begin{aligned}\Pr(B_k) &= \Pr(\hat{F}_N(q_1-\tau^k) \geq \beta), \\ &= \Pr(\hat{F}_N(q_1-\tau^k) - F(q_1-\tau^k) \geq \varepsilon_k), \\ &\leq \exp\left(\frac{-N\varepsilon_k^2/2}{F(q_1-\tau^k)(1-F(q_1-\tau^k)) + \frac{\varepsilon_k}{3}}\right), \qquad \text{(EC.2)} \\ &\leq \exp\left(\frac{-N\varepsilon_k/2}{\frac{1}{\varepsilon_k}(\beta-\alpha)(1-\beta)+\beta-\alpha+\frac{1}{3}}\right), \qquad \text{(EC.3)}\end{aligned}$$

where (EC.2) follows from Bernstein's inequality and (EC.3) follows from inequality (EC.1). Now, since $\varepsilon_k > \alpha$, *for all* $k \geq 1$, *we have that*

$$\begin{aligned}\Pr(B_k) &\leq \exp\left(\frac{-N\alpha/2}{\frac{1}{\alpha}\beta(1-\beta)-\frac{2}{3}+2\beta-\alpha}\right), \\ &\leq \exp\left(\frac{-N\alpha/2}{\frac{1}{\alpha}\beta(1-\beta)+\frac{4}{3}-2\min(\beta,1-\beta)-\alpha}\right), \\ &\leq \exp\left(\frac{-N\alpha/2}{\frac{1}{\alpha}\beta(1-\beta)+\frac{4}{3}}\right) = \exp\left(\frac{-3N\gamma^2}{6bh+8\gamma(b+h)}\right) \triangleq \delta.\end{aligned}$$

Thus, $\Pr(B) \leq \Pr(\bar{B}) \leq \delta$. *In fact, by going through a similar argument, we can show that* $\Pr(L) \leq \delta$. *Thus, by the union bound, we have that*

$$\Pr\left(\partial_- C(\hat{Q}_N) > \gamma \ \textit{or}\ \partial_+ C(\hat{Q}_N) < -\gamma\right) = \Pr(B \cup L) \leq \Pr(B) + \Pr(L) \leq 2\delta,$$

proving Proposition EC.1. Q.E.D.

We can now proceed with the proof of Theorem 2. Note that S_ϵ^{LRS} consists of all q for which $\partial_- C(q) \leq \gamma$ and $\partial_+ C(q) \geq -\gamma$, with $\gamma = \frac{\epsilon}{3}\min(b,h)$. From Proposition EC.1, the SAA solution from a random sample with size N lies in S_ϵ^{LRS} with probability at least

$$\begin{aligned}1-2\exp\left(\frac{-N\epsilon^2\min\{b,h\}^2}{18bh+8\epsilon(b+h)\min\{b,h\}}\right) &= 1-2\exp\left(\frac{-N\epsilon^2\min\{b,h\}}{18\max\{b,h\}+8\epsilon(b+h)}\right). \\ &\geq 1-2\exp\left(\frac{-N\epsilon^2}{18+8\epsilon}\cdot\frac{\min\{b,h\}}{b+h}\right).\end{aligned}$$

EC.2. Proof of Theorem 3

Since C is convex, $S_\epsilon^f \cap [q^*, \infty)$ can be equivalently expressed as $\{q : C'(q) \leq C'(\overline{q}) \text{ and } q \geq q^*\}$. Note that,

$$\begin{aligned}C'(\overline{q}) &= (b+h)(F(\overline{q}) - F(q^*)) = (b+h)\left[(\overline{q}-q^*)f(q^*) + O(\overline{q}-q^*)^2\right] \\ &= \sqrt{2\epsilon bh\Delta(q^*)f(q^*)} + O(\epsilon), \qquad \text{(EC.4)}\end{aligned}$$

which follows from Taylor series approximation and from the definition of $\overline{q}$ in (7).

To prove Theorem 3, note that the event that $\tilde{Q}_N^\alpha \in S_\epsilon^f \cap [q^*, \infty)$, where $\alpha = C'(\overline{q})$, is equivalent to the intersection of events $[\tilde{Q}_N^\alpha \geq q^*]$ and $[C'(\tilde{Q}_N^\alpha) \leq \alpha]$. We will prove an upper bound on the probability of $[\tilde{Q}_N^\alpha < q^*]$ and on the probability of $[C'(\tilde{Q}_N^\alpha) > \alpha]$. It follows similar lines to the proof of Lemma 3.5 in Levi et al. (2007), except we will use Bernstein's inequality instead of Hoeffding's inequality.

Define $\beta \triangleq \frac{b}{b+h}$ and $\gamma \triangleq \frac{1}{2}\frac{\alpha}{b+h}$. First, let us bound the probability of $B \triangleq [\tilde{Q}_N^\alpha < q^*]$. For a real-valued sequence $\{\tau^k\}_{k=1}^\infty$ where $\tau^k \downarrow 0$, define

$$B_k \triangleq [\tilde{Q}_N^\alpha \leq q^* - \tau^k] = \left[-b + (b+h)\hat{F}_N(q^* - \tau^k) \geq \frac{\alpha}{2}\right] = [\hat{F}_N(q^* - \tau^k) \geq \beta + \gamma].$$

Note that since $\hat{F}_N$ is monotonically increasing, it follows that $B_k \subseteq B_{k+1}$. Thus, if $\bar{B}$ is the limiting event of the sequence of events $\{B_k\}_{k=1}^\infty$, then $B_k \uparrow \bar{B}$, implying that $\Pr(B_k) \uparrow \Pr(\bar{B})$. Note also that $B \subseteq \bar{B}$, thus $\Pr(B) \leq \Pr(\bar{B})$. Therefore, to bound $\Pr(B)$, we only need to find a uniform upper bound for $\Pr(B_k)$.

Note that for any $k \geq 1$, there exists $\varepsilon^k > 0$ such that $F(q^* - \tau^k) = \beta - \varepsilon^k$. Thus,

$$F(q^* - \tau^k)\left(1 - F(q^* - \tau^k)\right) = (\beta - \varepsilon^k)(1 - \beta + \varepsilon^k) < \beta(1 - \beta + \varepsilon^k).$$

From Bernstein's inequality, we have that

$$\begin{aligned}
\Pr(B_k) &= \Pr\left(\hat{F}_N(q^* - \tau^k) \geq \beta + \gamma\right) = \Pr\left(\hat{F}_N(q^* - \tau^k) - F(q^* - \tau^k) \geq \gamma + \varepsilon^k\right) \\
&\leq \exp\left(\frac{-N(\gamma + \varepsilon^k)^2}{2F(q^* - \tau^k)(1 - F(q^* - \tau^k)) + \frac{2}{3}(\gamma + \varepsilon^k)}\right) \\
&= \exp\left(\frac{-N(\gamma + \varepsilon^k)}{\frac{2}{(\gamma+\varepsilon^k)}(\beta - \varepsilon^k)(1 - \beta - \gamma) + 2(\beta - \varepsilon^k) + \frac{2}{3}}\right) \\
&\leq \exp\left(\frac{-N(\gamma + \varepsilon^k)}{\frac{2}{(\gamma+\varepsilon^k)}\beta(1 - \beta - \gamma) + 2\beta + \frac{2}{3}}\right) \leq \exp\left(\frac{-N\gamma}{\frac{2}{\gamma}\beta(1 - \beta - \gamma) + 2\beta + \frac{2}{3}}\right)
\end{aligned}$$

where the inequality follows when $1 - \beta - \gamma \geq 0$. Hence, for all $k \geq 1$,

$$\Pr(B_k) \leq \exp\left(\frac{-3N\alpha^2}{24bh + 4\alpha(b+h)}\right).$$

Since $\alpha = C'(\overline{q})$, from (EC.4) we have that

$$\Pr(B_k) \leq \exp\left(-\frac{6N\epsilon bh\Delta(q^*)f(q^*) + O(\epsilon^{3/2})}{24bh + O(\epsilon^{1/2})}\right) \triangleq U(\epsilon). \tag{EC.5}$$

Now, let us bound the probability of $L \triangleq [C'(\tilde{Q}_N^\alpha) > \alpha] = \left[\bar{F}(\tilde{Q}_N^\alpha) < \frac{h}{b+h} - \frac{\alpha}{b+h}\right]$. Define $q_0 \triangleq \sup\left\{q : \bar{F}(q) \geq \frac{h}{b+h} - \frac{\alpha}{b+h}\right\}$. Thus, $L = [\tilde{Q}_N^\alpha > q_0]$. Note that $\tilde{Q}_N^\alpha = \sup\{q : h - (b+h)\hat{\bar{F}}_N(q) \leq \frac{\alpha}{2}\}$. For a real-valued sequence $\{\tau^k\}_{k=1}^\infty$ where $\tau^k \downarrow 0$, define

$$\begin{aligned}
L_k &\triangleq [\tilde{Q}_N^\alpha \geq q_0 + \tau^k] = \left[h - (b+h)\hat{\bar{F}}_N(q_0 + \tau^k) \leq \frac{\alpha}{2}\right] \\
&= \left[\hat{\bar{F}}_N(q_0 + \tau^k) \geq \frac{h}{b+h} - \frac{1}{2}\frac{\alpha}{b+h}\right] = \left[\hat{\bar{F}}_N(q_0 + \tau^k) \geq 1 - \beta - \gamma\right].
\end{aligned}$$

Since $\hat{\bar{F}}_N$ is nonincreasing, then it follows that $L_k \subseteq L_{k+1}$. Thus, if $\bar{L}$ is the limiting event of the sequence $\{L_k\}_{k=1}^{\infty}$, then $L_k \uparrow \bar{L}$, implying that $\Pr(L_k) \uparrow \Pr(\bar{L})$. Note also that $L \subseteq \bar{L}$, implying that $\Pr(L) \le \Pr(\bar{L})$. Therefore, to prove a bound on $\Pr(L)$, it is sufficient to prove a uniform upper bound on $\Pr(L_k)$.

Note that for some $\epsilon^k > 0$, we have that $\bar{F}(q_0 + \tau^k) = 1 - \beta - 2\gamma - \epsilon^k$. Thus, $L_k = [\hat{\bar{F}}_N(q_0 + \tau^k) - \bar{F}(q_0 + \tau^k) \ge \gamma + \epsilon^k]$. Finally, from Bernstein's inequality, we have that

$$\begin{aligned}
\Pr(L_k) &\le \exp\left(\frac{-N(\gamma+\epsilon^k)^2}{2\bar{F}(q_0+\tau^k)(1-\bar{F}(q_0+\tau^k)) + \frac{2}{3}(\gamma+\epsilon^k)}\right) \\
&= \exp\left(\frac{-N(\gamma+\epsilon^k)}{\frac{2}{\gamma+\epsilon^k}(1-\beta-2\gamma-\epsilon^k)(\beta+2\gamma+\epsilon^k) + \frac{2}{3}}\right) \\
&= \exp\left(\frac{-N(\gamma+\epsilon^k)}{\frac{2}{\gamma+\epsilon^k}(1-\beta-2\gamma-\epsilon^k)(\beta+\gamma) + 2(1-\beta-2\gamma-\epsilon^k) + \frac{2}{3}}\right) \\
&\le \exp\left(\frac{-N(\gamma+\epsilon^k)}{\frac{2}{\gamma+\epsilon^k}(1-\beta-2\gamma)(\beta+\gamma) + 2(1-\beta-2\gamma) + \frac{2}{3}}\right) \\
&\le \exp\left(\frac{-N\gamma}{\frac{2}{\gamma}(1-\beta-2\gamma)(\beta+\gamma) + 2(1-\beta-2\gamma) + \frac{2}{3}}\right) \\
&= \exp\left(\frac{-N\gamma}{\frac{2}{\gamma}\beta(1-\beta-2\gamma) + 4(1-\beta-2\gamma) + \frac{2}{3}}\right).
\end{aligned}$$

Therefore, we have that for all $k \ge 1$,

$$\Pr(L_k) \le \exp\left(\frac{-3N\alpha^2}{24bh + 4\alpha(7h - 5b - 6\alpha)}\right).$$

Since $\alpha = C'(\overline{q})$, we have from (EC.4) that

$$\Pr(L_k) \le \exp\left(-\frac{6N\epsilon bh\Delta(q^*)f(q^*) + O(\epsilon^{3/2})}{24bh + O(\epsilon^{1/2})}\right) \triangleq U(\epsilon). \tag{EC.6}$$

Summarizing from (EC.5) and (EC.6), we have that $\Pr(B) \le \Pr(\bar{B}) \le U(\epsilon)$ and that $\Pr(L) \le \Pr(\bar{L}) \le U(\epsilon)$. Thus,

$$\begin{aligned}
\Pr\left\{\tilde{Q}_N^{\alpha} < q^* \text{ or } C'(\tilde{Q}_N^{\alpha}) > C'(\overline{q})\right\} &= \Pr(B \cup L) \le \Pr(B) + \Pr(L) \\
&\le 2U(\epsilon) \sim 2\exp\left(-\frac{1}{4}N\epsilon\Delta(q^*)f(q^*)\right), \text{ as } \epsilon \to 0.
\end{aligned}$$

EC.3. Proof of Lemma 1

Denote by $\partial_- g(x)$ (or $\partial_+ g(x)$) the left-side (or right-side) derivative of a function g at x. The failure rate and reverse hazard rate is given by $\bar{r}(x) = \frac{f(x)}{1-F(x)}$ and $r(x) = \frac{f(x)}{F(x)}$. Since f is a log-concave distribution, it has an increasing failure rate. This implies that $\log \bar{r}(x) = \log f(x) - \log(1 - F(x))$ is increasing, and $\partial_- \log \bar{r}(x) \geq 0$ for all x. Thus,

$$\gamma_1 + \gamma_0 \frac{b+h}{h} = \gamma_1 + \frac{f(q^*)}{1 - F(q^*)} \geq \partial_- \log f(q^*) + \frac{f(q^*)}{1 - F(q^*)} = \partial_- \log \bar{r}(q^*) \geq 0. \tag{EC.7}$$

A log-concave distribution also has a decreasing reversed hazard rate. This implies that $\log r(x) = \log f(x) - \log F(x)$ is decreasing and $\partial_+ \log r(x) \leq 0$ for all x. Thus,

$$\gamma_1 - \gamma_0 \frac{b+h}{b} = \gamma_1 - \frac{f(q^*)}{F(q^*)} \leq \partial_+ \log f(q^*) - \frac{f(q^*)}{F(q^*)} = \partial_+ \log \bar{r}(q^*) \leq 0. \tag{EC.8}$$

Combining (EC.7) and (EC.8), we have that $-\frac{b+h}{h} \leq \frac{\gamma_1}{\gamma_0} \leq \frac{b+h}{b}$.

EC.4. Proof of Lemma 2

Note that since $\log f$ is concave, then $\log f(x) \leq \log \gamma_0 + \gamma_1 (x - t)$, for all x such that $f(x) > 0$. Taking the exponent on both sides proves our result.

EC.5. Proof of Lemma 3

Note that $\frac{d}{dx} F_1(x) \leq \frac{d}{dx} F_2(x)$ by our assumption that $f_1(x) \leq f_2(x)$. Moreover, since $F_1(t) = F_2(t)$, then $F_1(x) \geq F_2(x)$ for all $x \leq t$ and $F_1(x) \leq F_2(x)$ for all $x \geq t$. Note that

$$\begin{aligned} E(D_1 - t | D_1 > t) &= \int_0^\infty \Pr(D_1 > t + s | D_1 > t) ds, \\ &= \tfrac{1}{1 - F_1(t)} \int_0^\infty (1 - F_1(t+s)) ds, \\ &\geq \tfrac{1}{1 - F_2(t)} \int_0^\infty (1 - F_2(t+s)) ds, \\ &= E(D_2 - t | D_2 > t) \end{aligned}$$

With the same technique, we can also prove that $E(t - D_1 | D_1 \leq t) \geq E(t - D_2 | D_2 \leq t)$. Combining these results proves the lemma.

EC.6. Proof of Lemma 4

We first introduce the following notation:

$$\begin{aligned} G(\alpha) &\triangleq \left(\frac{1}{1-\beta} + \alpha \right) \log\left(1 + \alpha(1-\beta)\right) + \left(\frac{1}{\beta} - \alpha \right) \log\left(1 - \alpha\beta\right) - \min\{\beta, 1-\beta\} \alpha^2, \\ U(\beta) &\triangleq \frac{\beta}{1-\beta} \log\left(\frac{1}{\beta} \right) - \beta, \\ L(\beta) &\triangleq \frac{1-\beta}{\beta} \log\left(\frac{1}{1-\beta} \right) - (1-\beta). \end{aligned}$$

We need to prove that each of the three functions are nonnegative.

1. Let us prove the result for G. First, we prove the result for the case when $\beta \geq \frac{1}{2}$. Note that

$$G'(\alpha) = \log\left(\frac{1 + \alpha(1-\beta)}{1 - \alpha\beta} \right) - 2(1-\beta)\alpha.$$

The derivative is nonnegative if and only if $G_1(\alpha) \triangleq (1+\alpha(1-\beta))e^{-(1-\beta)\alpha} - (1-\alpha\beta)e^{(1-\beta)\alpha} \geq 0$. Note that for $\alpha \geq 0$,

$$\begin{aligned} G_1'(\alpha) &= -\alpha(1-\beta)^2 e^{-(1-\beta)\alpha} + \beta e^{(1-\beta)\alpha} - (1-\beta)(1-\alpha\beta)e^{(1-\beta)\alpha}, \\ &\geq -\alpha(1-\beta)^2 e^{-(1-\beta)\alpha} + \beta(1-\beta)\alpha e^{(1-\beta)\alpha}, \\ &\geq \alpha(1-\beta)^2\left(e^{(1-\beta)\alpha} - e^{-(1-\beta)\alpha}\right) \geq 0 \end{aligned}$$

Note that $G_1(0)=0$, thus, $G_1(\alpha)\geq 0$ for all $\alpha \geq 0$. Now define $G_2(\alpha) \triangleq (1+\alpha(1-\beta))e^{-(1-\beta)\alpha} - (1-\alpha(1-\beta))e^{(1-\beta)\alpha}$. Note that $G_2(\alpha) \geq G_1(\alpha)$ if $\alpha \leq 0$. We have

$$G_2'(\alpha) = \alpha(1-\beta)^2\left(e^{(1-\beta)\alpha} - e^{-(1-\beta)\alpha}\right) \geq 0, \quad \text{for } \alpha \leq 0.$$

Note that $G_2(0)=0$, thus, $G_1(\alpha) \leq G_2(\alpha) \leq 0$ for all $\alpha \leq 0$. Thus, $G(\alpha)$ is nondecreasing in $\alpha \geq 0$, and non-increasing in $\alpha \leq 0$. Since at $\alpha = 0$, this function is zero, then $G(\alpha)\geq 0$ for all α. Now we can also prove the result for $\beta \leq \frac{1}{2}$, if we define the function $\tilde{\beta} = 1-\beta \geq \frac{1}{2}$ and $\tilde{G}(\alpha) = G(-\alpha)$. Q.E.D.

2. Let us prove the result for U. The result is true if and only if $-\log\beta \geq 1-\beta$. Note that $-\log\beta$ is a convex function of β, thus the linear approximation at $\beta = 1$ (i.e., the function $1-\beta$) bounds it from below. Q.E.D.

3. Let us prove the result for L. Defining $\tilde{\beta} = 1-\beta$, note that $L(\beta) = U(\tilde{\beta}) \geq 0$, which follows from (2).

EC.7. Proof of Theorem 4

Recall that if $q \in S_\epsilon^f \cap [q^*, \infty)$, then $C(q) \leq (1+\epsilon)C(q^*)$. Also, $S_\epsilon^f \cap [q^*,\infty)$ can be equivalently expressed as $\{q : C'(q) \leq C'(\overline{q}) \text{ and } q \geq q^*\}$. Let $\tilde{Q}_N^\alpha$ be defined in (8), but with $\alpha = \sqrt{2\epsilon bh\frac{\min\{b,h\}}{b+h}} + O(\epsilon)$. Since,

$$\begin{aligned} C'(\overline{q}) &= (b+h)(F(\overline{q}) - F(q^*)) = (b+h)\left[(\overline{q}-q^*)f(q^*) + O(\overline{q}-q^*)^2\right] \\ &= \sqrt{2\epsilon bh\Delta(q^*)f(q^*)} + O(\epsilon), \end{aligned}$$

then it follows from Proposition 2 that $\alpha \leq C'(\overline{q})$ when the demand distribution is log-concave. This implies that

$$\left[\tilde{Q}_N^\alpha \geq q^*\right] \cap \left[C'(\tilde{Q}_N^\alpha) \leq \alpha\right] \subseteq \left[\tilde{Q}_N^\alpha \geq q^*\right] \cap \left[C'(\tilde{Q}_N^\alpha) \leq C'(\overline{q})\right].$$

Thus, we only need to derive a lower bound on the probability of the left-hand side event to prove Theorem 4. Modifying the proof of Theorem 3 by letting $\alpha = \sqrt{2\epsilon bh\frac{\min\{b,h\}}{b+h}} + O(\epsilon)$, we can prove that

$$\Pr\left(\tilde{Q}_N^\alpha < q^* \text{ or } C'(\tilde{Q}_N^\alpha) > \alpha\right) \leq 2U^*(\epsilon) \sim 2\exp\left(-\frac{1}{4}N\epsilon\frac{\min\{b,h\}}{b+h}\right), \text{ as } \epsilon \to 0.$$