Euler equations[*]

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Abstract

An Euler equation is a difference or differential equation that is an intertemporal first-order condition for a dynamic choice problem. It describes the evolution of economic variables along an optimal path. It is a necessary but not sufficient condition for a candidate optimal path, and so is useful for partially characterizing the theoretical implications of a range of models for dynamic behavior. In models with uncertainty, expectational Euler equations are conditions on moments, and thus directly provide a basis for testing models and estimating model parameters using observed dynamic behavior.

An Euler equation is an intertemporal version of a first-order condition characterizing an optimal choice as equating (expected) marginal costs and marginal benefits.

Many economic problems are dynamic optimization problems in which choices are linked over time, as for example a firm choosing investment over time subject to a convex cost of adjusting its capital stock, or a government deciding tax rates over time subject to an intertemporal budget constraint. Whatever solution approach one employs – the calculus of variations, optimal control theory or dynamic programming – part of the solution is typically an Euler equation stating that the optimal plan has the property that any marginal, temporary and feasible change in behavior has marginal benefits equal to marginal costs in the present and future. Assuming the original problem satisfies certain regularity conditions, the Euler equation is a necessary but not sufficient condition for an optimum. This differential or difference equation is a law of motion for the economic variables of the model, and as such is useful for (partially) characterizing the theoretical implications of the model for optimal dynamic behavior. Further, in a model with

uncertainty, the expectational Euler equation directly provides moment conditions that can be used both to test these theoretical implications using observed dynamic behavior and to estimate the parameters of the model by choosing them so that these implications quantitatively match observed behavior as closely as possible.

The term 'Euler equation' first appears in text-searchable JSTOR in Tintner (1937), but the equation to which the term refers is used earlier in economics, as for example (not by name) in the famous Ramsey (1928). The mathematics was developed by Bernoulli, Euler, Lagrange and others centuries ago jointly with the study of classical dynamics of physical objects; Euler wrote in the 1700's 'nothing at all takes place in the universe in which some rule of the maximum . . . does not appear' (Weitzman (2003), p. 18). The application of this mathematics in dynamic economics, with its central focus on optimization and equilibrium, is almost as universal. As in physics, Euler equations in economics are derived from optimization and describe dynamics, but in economics, variables of interest are controlled by forward-looking agents, so that future contingencies typically have a central role in the equations and thus in the dynamics of these variables.

For general, formal derivations of Euler equations, see texts or entries on the calculus of variations, optimal control theory or dynamic programming. This entry illustrates by means of example the derivation of a discrete-time Euler equation and its interpretation. The entry proceeds to discuss issues of existence, necessity, sufficiency, dynamics systems, binding constraints, and continuous-time. Finally, the entry discusses uncertainty and the natural estimation framework provided by the expectational Euler equation.

The Euler equation: Consider an infinitely-lived agent choosing a control variable (c) in each period (t) to maximize an intertemporal objective: $\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$ where $u(c_t)$ represents the flow payoff in t, $u' > 0$, $u'' < 0$, and β is the discount factor, $0 < \beta < 1$. The agent faces a present-value budget constraint:

$$\sum_{t=1}^{\infty} R^{1-t} c_t \leq W_1 \tag{1}$$

where R is the gross interest rate ($R = 1 + r$ where r is the interest rate) and W_1 is given.

By the theory of the optimum, if a time-path of the control is optimal, a marginal increase in the control at any t, dc_t, must have benefits equal to the cost of the decrease in $t+1$ of the same present value amount, $-Rdc_t$:

$$\beta^{t-1} u'(c_t) \, dc_t - \beta^t u'(c_{t+1}) \, Rdc_t = 0.$$

Reorganization gives the Euler equations

$$u'(c_t) = \beta R u'(c_{t+1}) \text{ for } t = 1, 2, 3... \tag{2}$$

This set of Euler equations are nonlinear difference equations that characterize the evolution of the control along any optimal path. We considered a one-period deviation; several

period deviations can be considered, but they follow from sequences of one-period deviations and so doing so does not provide additional information $(u'(c_t) = \beta^2 R^2 u'(c_{t+2}))$. These equations imply that the optimizing agent equalizes the present-value marginal flow benefit from the control across periods.

The canonical application of this problem is to a household or representative agent: call c consumption, u utility, and let $W_1 = \sum_{t=1}^{\infty} R^{1-t} y_t$, the present value of (exogenous) income, y. In this case, equations (2) imply the theoretical result that variations in income do not cause consumption to rise or fall over time. Instead, marginal utility grows or declines over time as $\beta R \gtrless 1$; for $\beta R = 1$, consumption is constant.

Existence, necessity and sufficiency: In general, to ensure that the Euler equation characterizes the optimal path, one typically requires that the objective is finite (in this example, $u' > 0$) and that some feasible path exists.

Further, since Euler equations are first-order conditions, they are necessary but not sufficient conditions for an optimal dynamic path. Thus, theoretical results based only on Euler equations are applicable to a range of models. On the other hand, the equations provide an incomplete characterization of equilibria. In the example, only by using the budget constraint also, can one solve for the time-path of consumption; its level is determined by the present value of income.

Dynamic analysis: More generally, complete characterization of optimal behavior uses the Euler equation as one equation in a system of equations. For example, replacing the budget constraint (equation (1)) with the capital-accumulation equation

$$k_{t+1} = f(k_t) - c_t + (1 - \delta) k_t \tag{3}$$

where k is capital, $f(k)$ is output, $f' > 0$, $f'' < 0$, $f(0) = 0$, $\lim_{k \to 0} f' > \beta^{-1} - (1 - \delta)$, and $\lim_{k \to \infty} f' < \beta^{-1} - (1 - \delta)$, and adding the constraints k_1 given, $k_t \geq 0$, and $c_t \geq 0$, gives the basic Ramsey growth model. The constant real interest rate of equation (2) is replaced by the marginal product of capital in the resulting Euler equation

$$u'(c_t) = \beta (1 - \delta + f'(k_{t+1})) u'(c_{t+1}). \tag{4}$$

Equations (3) and (4) form a system of two differential equations with two steady-states that has been widely studied as a model of economic growth. Linearization shows that the interesting $(k > 0)$ steady state is locally saddle-point stable, and there is a unique feasible convergence path that pins down the dynamic path of consumption and capital.

Binding constraints: The above Euler equations are interior first-order conditions. When the economic problem includes additional constraints on choice, the resulting Euler equations have Lagrange multipliers. Consider adding a 'liquidity constraint' to our example: that the household maintain positive assets in every period s: $\sum_{t=1}^{s} R^{1-t} y_t - \sum_{t=1}^{s} R^{1-t} c_t \geq 0$ for all s. In this case, the program is more easily solved in a recursive formulation. Equation (2) holds with a single Lagrange multiplier, $\lambda_{t+1} \geq 0$, on the constraint that assets are positive in $t + 1$ since prior to $t + 1$ assets levels are

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unaffected by the choice of c_t and in period $t+1$ the present value of future consumption is unchanged by the one-period deviation considered:

$$u'(c_t) = \beta R u'(c_{t+1}) + \lambda_{t+1}.$$

The multiplier λ_{t+1} has the interpretation of a shadow price. When the constraint does not bind, $\lambda_{t+1} = 0$, the interior version of the Euler equation holds, and the marginal benefit marginal cost interpretation is straightforward. When the constraint binds, the interpretation still holds, but almost tautologically: the change in utility of an extra marginal unit of consumption in t is equal to the change in utility from the marginal decreases in consumption in $t+1$ plus the shadow price (in terms of marginal utility) of marginally relaxing the constraint on c_t. For example, if $\beta R = 1$ and $y_t = \bar{y} \ \forall t \neq 2$ and $y_2 = \underline{y} < \bar{y}$, then $\lambda_{t+1} = 0 \ \forall t \neq 2$, $\lambda_3 = u'\left(\frac{\underline{y}+R\bar{y}}{1+R}\right) - u'(\bar{y}) > 0$, and $c_1 = c_2 = \frac{\underline{y}+R\bar{y}}{1+R}$, $c_t = \bar{y} \ \forall t \geq 3$. This example illustrates that, relative to the unconstrained equilibrium $\left(c_t = \bar{y} - r\left(\bar{y} - \underline{y}\right)\right)$, the constraint can postpone consumption ($t = 1, 2$ relative to $t \geq 3$), create a causal link from an increase in income to consumption ($t = 2$ to 3), and can lower consumption in unconstrained periods ($t = 1$).

Continuous time: In general, continuous-time models have differential Euler equations that are equivalent to the difference-equation versions of their discrete-time counterparts. In the example, replacing $t+1$ with $t + \Delta t$, $c_{t+\Delta t} = c_t + \Delta c_t$, $\beta = 1 - \rho \Delta t$, $R = 1 + r \Delta t$, expanding $u'(c_t + \Delta c_t)$ around c_t, and letting $\Delta t \to 0$ gives:

$$\frac{\dot{c}_t}{c_t} = \sigma_t (r - \rho)$$

where $\sigma_t = -\frac{u'(c_t)}{c_t u''(c_t)}$. While the marginal-costs-marginal-benefit interpretation of the

equation is less obvious in continuous time, it is still clear that consumption rises over time according to the difference between the interest rate (r) and the discount rate (ρ), and more obvious that the strength of this response is governed by σ_t, which for this reason is called the elasticity of intertemporal substitution.

Generalized Euler equations: Dynamic games can also lead to 'generalized' Euler equations. For example, Harris and Laibson (2001) considers a modification of the example as a game among agents at different times who disagree because their preferences are not time consistent due to hyperbolic discounting. At any s, an agent has objective: $u(c_t) + \beta \sum_{\tau=1}^{\infty} \delta^\tau u(c_{s+\tau})$, where $0 < \delta < 1$. Defining recursively $W_{t+1} = R(W_t - c_t)$, the generalized Euler equation is

$$u'(c_t) = R\underbrace{\left[\beta\delta\left(\frac{\partial c_{t+1}(W_{t+1})}{\partial W_{t+1}}\right) + \delta\left(1 - \frac{\partial c_{t+1}(W_{t+1})}{\partial W_{t+1}}\right)\right]}_{\text{``}Effective\text{'' discount factor}} u'(c_{t+1}).$$

where $c_{t+1}(W_{t+1})$ is the optimal consumption choice made in $t+1$ as a function of W_{t+1}. The effective discount rate is a function of the (endogenous) marginal propensity to consume wealth in $t+1$.

Uncertainty: Models that contain uncertainty lead to expectational Euler equations. Add to the discrete-time example that the agent believes income y_s for $s > t$ to be stochastic from the perspective of period t. The Euler equation becomes

$$u'(c_t) = \beta R \hat{E}\left[u'(c_{t+1})|I_t\right] \tag{5}$$

where $\hat{E}[.|I_t]$ represents the agent's expectation given information set I_t. The stochastic version of the consumption Euler equation has an analogous interpretation to that under certainty: the household equates *expected* (discounted) marginal utility over time.

Taking a second-order approximation to marginal utility in $t+1$ around c_t and reorganizing gives

$$\hat{E}\left[\frac{c_{t+1} - c_t}{c_t}|I_t\right] = \sigma_t\left(1 - (\beta R)^{-1}\right) + \tfrac{1}{2}\phi_t\hat{E}\left[(c_{t+1} - c_t)^2|I_t\right]$$

where $\phi_t = -\frac{c_t u'''(c_t)}{u''(c_t)}$ is the coefficient of relative prudence (see for example Dynan (1991)). It is now *expected* consumption growth that rises with the real interest rate and falls with impatience. Additionally, for $\phi_t > 0$, risk leads to precautionary saving: higher expected consumption growth (much like liquidity constraints). Finally, actual consumption growth is also driven by the realization of uncertainty about current and future income.

Testing and estimation: An expectational Euler equation is a powerful tool for testing and estimating economic models in large samples, because, along with a model of expectations, it provides orthogonality conditions on which estimation can be based. Only randomization, as under experimental settings, delivers such a clean basis for estimation without near-complete specification of an economic model, including the sources of uncertainty.

Considering our main example, define $\varepsilon_{t+1} = u'(c_{t+1}) - (\beta R)^{-1}u'(c_t)$. Hall (1978) pointed out that equation (5) implies that $\hat{E}[\varepsilon_{t+1}z_t|I_t] = z_t\hat{E}[\varepsilon_{t+1}|I_t] = 0$ for *any* z_t in the agent's information set, I_t. Under the assumption of rational expectations, mathematical expectations can be used in place of the agent's expectations. Thus, this equation predicts that observed changes in discounted marginal utility are unpredictable using I_t, or that marginal utility is a Martingale, a strong theoretical prediction that Hall (1978) tests. Hansen and Singleton (1983) use a version of the stochastic Euler equation with a portfolio choice as the basis for estimation (and testing) of the parameters of the representative agent's parameterized utility function.

Since these papers (and others), large-sample testing and estimation of Euler equations under the assumption of rational expectations has played a central role in the evaluation of dynamic economic models. Most research applies the Generalized Method of Moments (GMM) of Hansen (1982) using the restrictions on the moments of time series implied

by the expectational Euler equation. Considering a $J \times 1$ vector of z_t's, $\mathbf{z}_t$, and, based on our example, define the column vector $\mathbf{g}(c_{t+1}, c_t, \mathbf{z}_t) = (\beta R u'(c_{t+1}) - u'(c_t)) \mathbf{z}_t$, so that we have the J moment restrictions $E[\mathbf{g}(c_{t+1}, c_t, \mathbf{z}_t)] = \mathbf{0}_{J \times 1}$. For example, letting $u'(c_t) = c_t^{-(1/\sigma)}$ and assuming that second moments exist and the model is covariance stationary, the time-series average of $\mathbf{g}(c_{t+1}, c_t, \mathbf{z}_t)$ should converge to $E[\mathbf{g}(c_{t+1}, c_t, \mathbf{z}_t)]$ for the true σ, β, and R The GMM estimates of σ, β, and R are those that minimize the difference (according to a given metric) between the observed empirical moments and their theoretical counterparts, $\mathbf{0}_{J \times 1}$.

This general approach has the advantage that complete specification of the model is not necessary. In our example, the stochastic process for income need not be specified nor the stochastic process for consumption determined (which can be quite demanding in terms of computer programming and run-time). That said, more complete specification can give more theoretical restrictions and thus more power in asymptotic estimation. Gourinchas and Parker (2002) for example uses numerical methods to bring more theoretical structure to bear in estimation. Further, more complete specification can allow one to use small-sample distribution theory and thus avoid the approximations inherent in using asymptotic distribution theory for inference in finite samples. A recent cautionary example is provided by the literature showing that standard asymptotic inference can be highly misleading in large samples with 'weak instruments.'

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