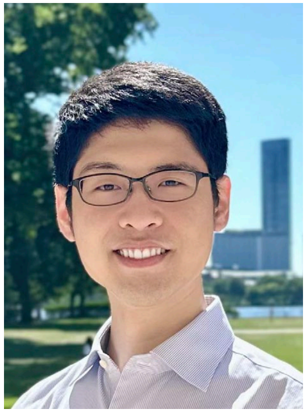
Level-Set Geometry and the Performance of Restarted-PDHG for Conic LP

Zikai Xiong (with Robert Freund)

Cargèse Workshop on Combinatorial Optimization



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Two slides on Interior-Point Methods (IPMs)

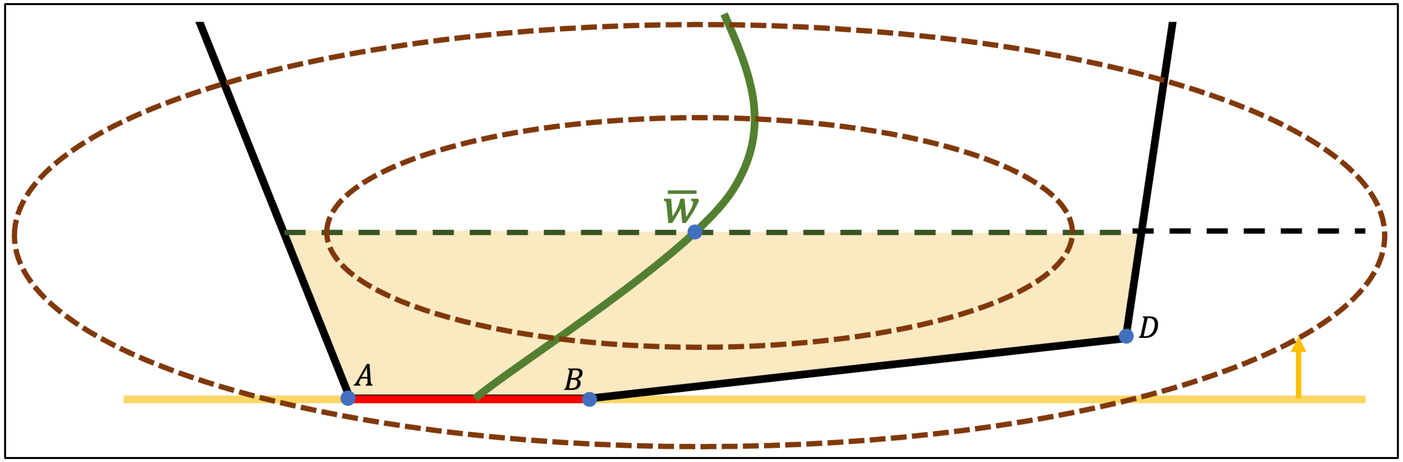
Central-Path ellipsoids have remarkable properties

Central-path solutions are:

$$x_\mu := \arg\min_x \; c^\top x + \mu \cdot F(x) \quad \text{s.t. } Ax = b,\; x \in \mathcal{K}$$

$$y_\mu, s_\mu := \arg\max_{y,s} \; b^\top y - \mu \cdot F^*(s) \quad \text{s.t. } A^\top y + s = c,\; s \in \mathcal{K}^*$$

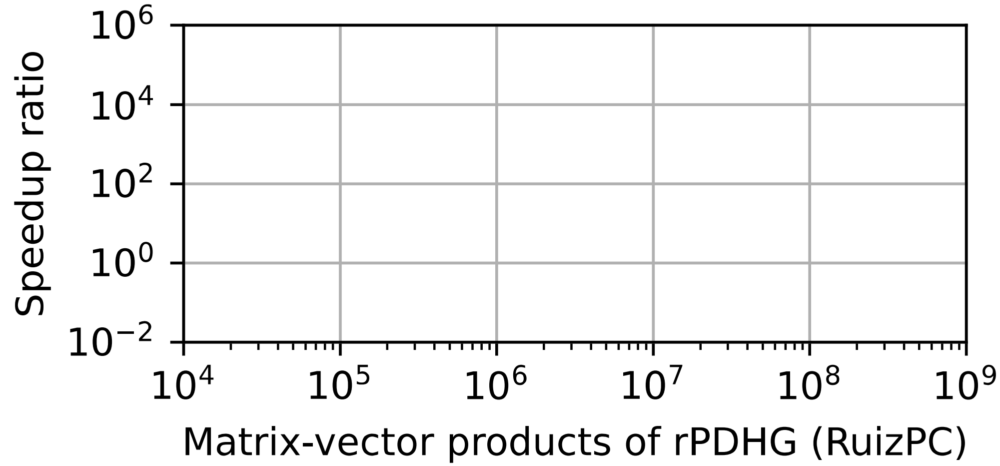
Example for LP: $F(x) := -\sum_{i=1}^{n} \ln(x_i)\,,\;\; \vartheta_F = n$



Sneak Preview:

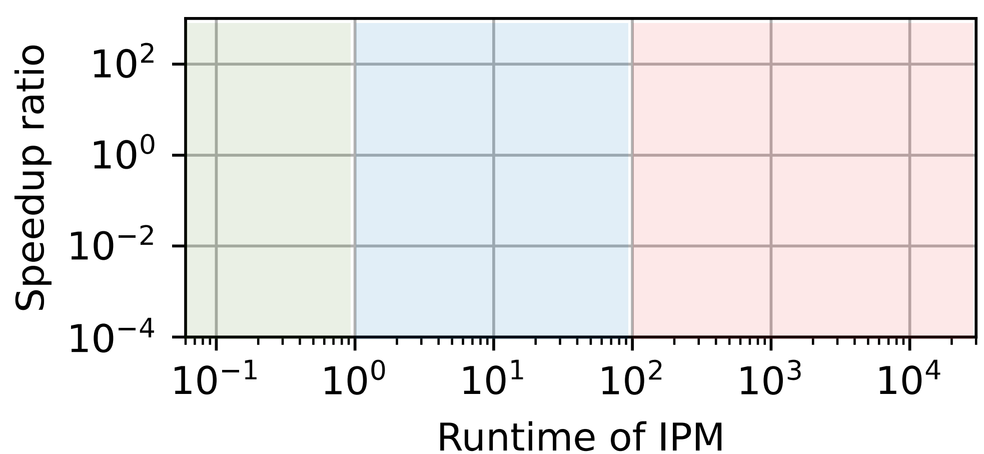
Distribution of Speedups of our method rPDHG-AHR

Speedups compared with restarted-PDHG (rPDHG) with PDLP's rescaling



Speedups compared with a "home-grown" IPM

(Predictor-corrector path-following interior-point method in Nocedal and Wright *Numerical Optimization* (2006))



Conic Linear Optimization ("CLO" or "CLP")

CLP in standard form

(primal)

$$\boxed{\begin{aligned} \min\ & c^\top x \\ \text{s.t.}\ & Ax = b \\ & x \in \mathcal{K} \end{aligned}}$$

(dual)

$$\begin{aligned} \max\ & b^\top y \\ \text{s.t.}\ & c - A^\top y \in \mathcal{K}^* \end{aligned}$$

Decision variables

- $x \in R^n$ (for primal problem)
- $y \in R^m$ (for dual problem)

CLP saddlepoint formulation

$$\boxed{\min_{x \in \mathcal{K}} \ \max_{y} \ c^\top x - y^\top A x + b^\top y}$$

Primal-Dual Hybrid Gradient for LP

PDHG

$$x^{k+1} \leftarrow \left(x^k + \tau A^\top y^k - \tau c\right)^+$$
$$y^{k+1} \leftarrow y^k - \sigma A\left(2x^{k+1} - x^k\right) + \sigma b$$

- Only requires matrix-vector multiplications
- "Simple" O($1/t$) convergence when τ and σ satisfy:

$$\tau \cdot \sigma \leq \frac{1}{\sigma_{max}(A)^2}$$

- For LP, restarts based on average iterates yield linear convergence [Applegate, Hinder, Lu, Lubin, 2023]

Challenge II: Loose/unworkable computational guarantees

Original computational guarantees:

Theorem [Applegate, Hinder, Lu, Lubin, 2023] PDHG computes an ε-optimal solution within

$$O\left((\|x^\star\| + \|y^\star\|) \cdot ||A|| \cdot \boldsymbol{H(K)} \cdot \log\left(\frac{\|x^\star\| + \|y^\star\|}{\varepsilon} \right) \right)$$

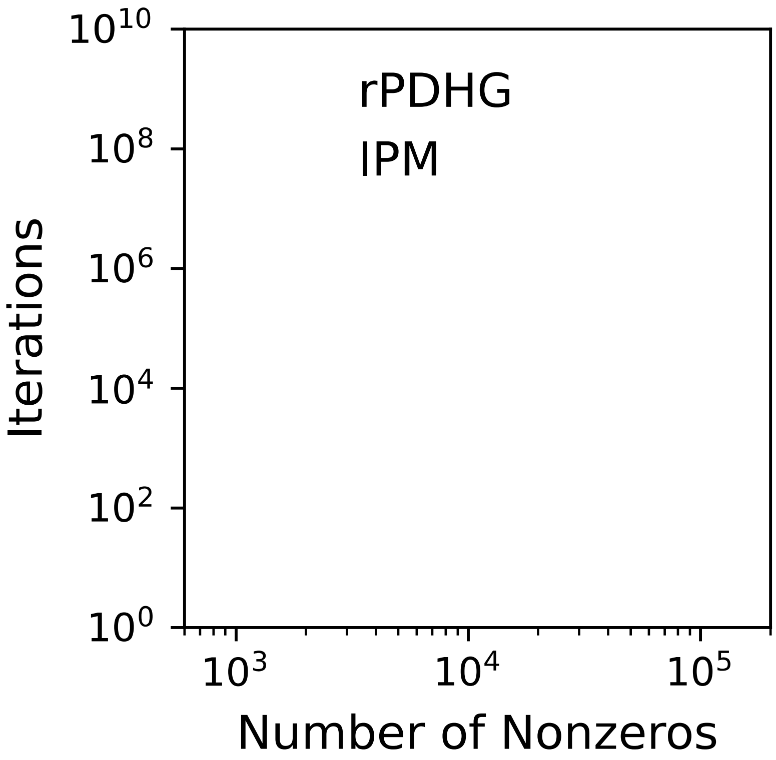
iterations.

$\boldsymbol{H(K)}$ is the global Hoffman constant of the matrix $\boldsymbol{K}$ of the KKT system

$$\boldsymbol{K} = \begin{pmatrix} A & \\ -A & \\ & A^\top \\ c^\top & -b^\top \end{pmatrix} \qquad \boldsymbol{H(K)} \approx \frac{1}{\min_{J \subseteq \{1,\dots,2m+n+1\}} \sigma^{+}_{\min}(K_J)}$$

Performance of rPDHG

Iterations Required for
LP relaxations from MIPLIB 2017



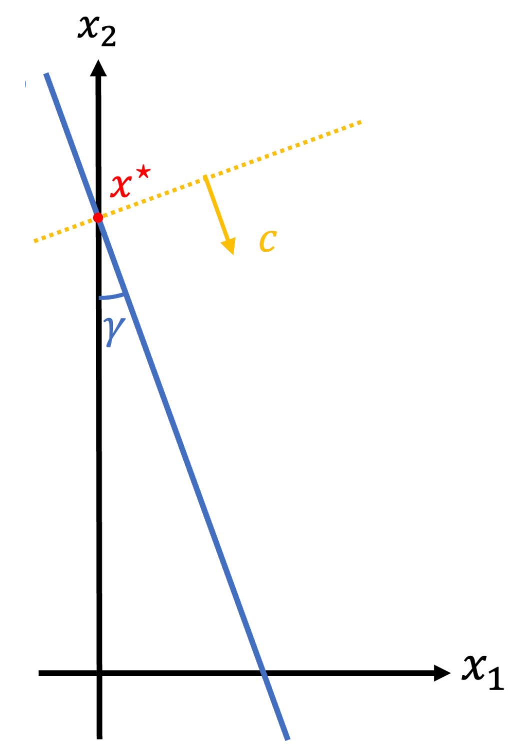
- rPDHG uses more iterations than IPM
  makes sense…

- Some small instances require a very large number of iterations
  a real challenge for rPDHG

A seemingly easy LP instance

For $\gamma \in \left(0, \frac{\pi}{2}\right)$ define:

$$P(\gamma): \quad \min_{x_1, x_2} \ \sin(\gamma)\, x_1 - \cos(\gamma)\, x_2 \quad \text{s.t.} \ \sin(\gamma)\, x_1 + \cos(\gamma)\, x_2 = 1, \quad x_1 \geq 0, x_2 \geq 0$$



$P(\gamma)$ is always easy for the simplex method and interior-point methods

However, when γ is small, PDHG requires at least 1,000,000 iterations. What **conditions** of $P(\gamma)$ make it so hard for PDHG?

Goals for improving theory for rPDHG for LP

Challenges:

- Current theoretical complexity is loose and can be hard to compute/validate

We seek iteration bounds that are more natural, tighter, and easier to analyze

- Some seemingly "easy" problems are hard for rPDHG

We seek to understand what properties of these "easy" problems make them hard

Computational guarantees

Original computational guarantees:

Theorem [Applegate, Hinder, Lu, Lubin, 2023] PDHG computes an ε-optimal solution within

$$O\left((\|x^\star\| + \|y^\star\|) \cdot ||A|| \cdot \boldsymbol{H(K)} \cdot \log\left(\frac{\|x^\star\| + \|y^\star\|}{\varepsilon}\right)\right)$$

iterations.

Key questions:

- What conditions of the problem actually drive the performance of PDHG? Sublevel-set Geometry
- Can we improve these condition numbers and so improve computational performance in theory/practice?

  Yes, we will improve the geometry using Hessian rescaling

Condition numbers related to Level-set Geometry

Reformulation of Standard Form CLP

CLP in standard form

(Primal)

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t.}\ & Ax = b \\ & x \in \mathcal{K} \end{aligned}$$

(Dual)

$$\begin{aligned} \max\ & b^\top y \\ \text{s.t.}\ & A^\top y + s = c \\ & s \in \mathcal{K}^\star \end{aligned}$$

Assumptions (similar as in IPMs)

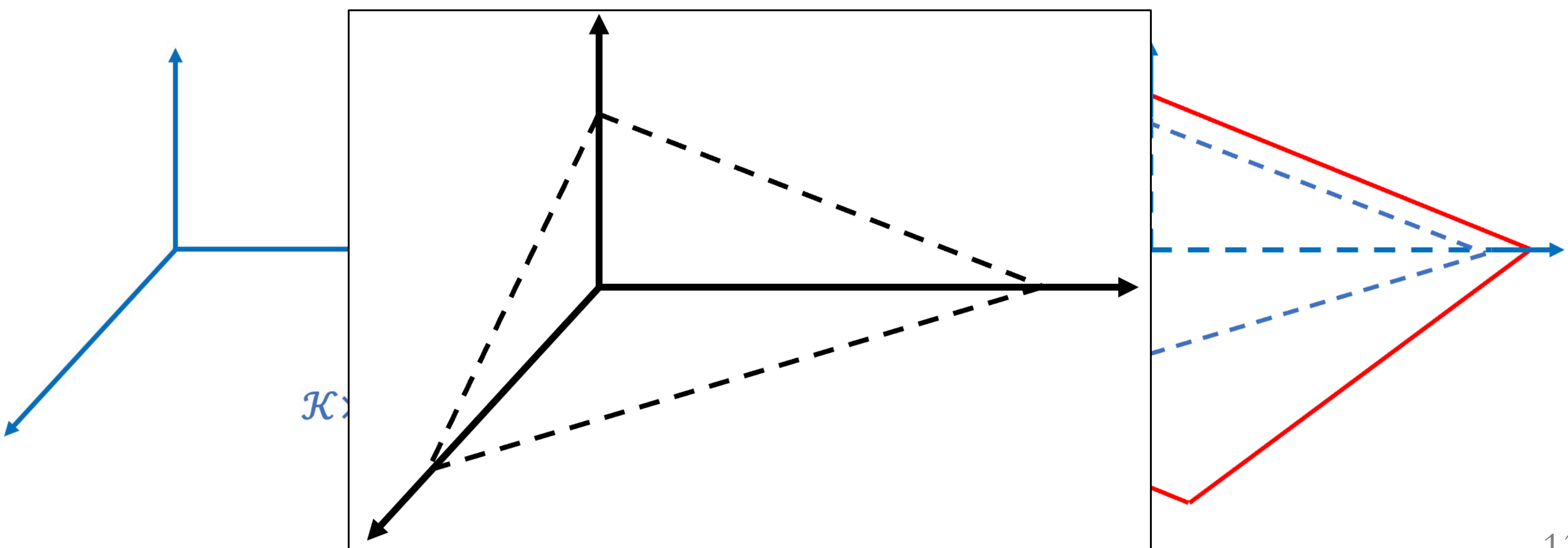
1. Strict feasibility of P and D (though not necessary in context of computation)
2. Rows of A are linearly independent
   - y and s correspond one-to-one
3. Objective vector $c \in \mathrm{Null}(A)$
   - $Ac = 0$
   - not essential, but keeps things simpler

The feasible region in the space of (x, s)

$\min\ c^\top x$	$\max\ b^\top y$
$\text{s.t.}\ Ax = b$	$\text{s.t.}\ A^\top y + s = c$
$x \in \mathcal{K}$	$s \in \mathcal{K}^\star$

(x, s) lies in the cone $\mathcal{K} \times \mathcal{K}^*$
For example, the nonnegative orthant for LP

$Ax = b, \exists\ y$ s.t. $A^\top y + s = c$
(x, s) lies in an affine subspace



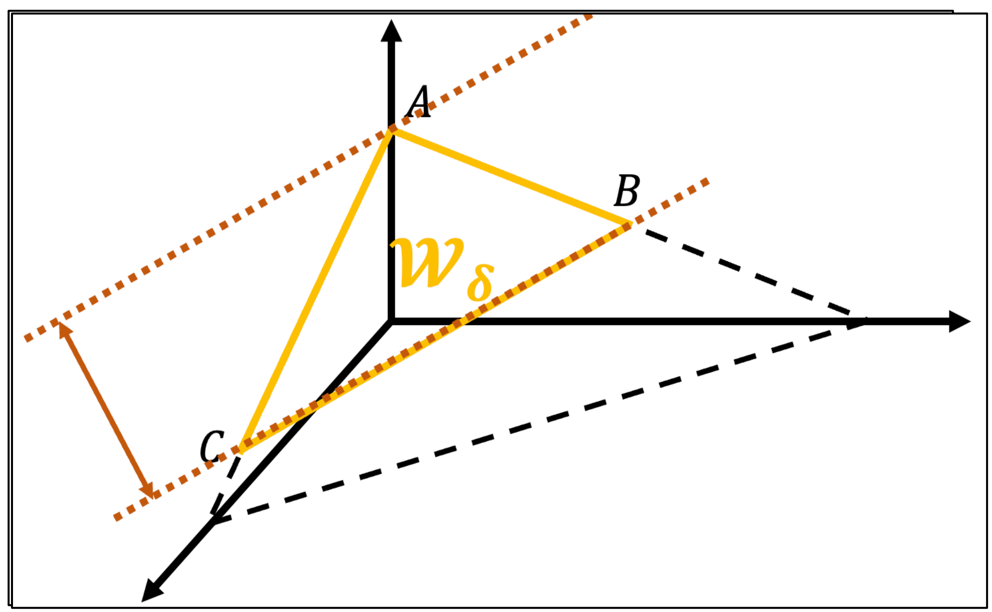
Primal-Dual Sublevel Sets

(x, s) is feasible

- The duality gap is a **linear function** of (x, s)
- The optimal solution set is $\mathcal{W}^\star$ = set of optimal (x, s)
- The optimal solution is the point A in the figure

$$\mathcal{W}_\delta := \left\{ w := (x,s) \;\middle|\; \begin{array}{c} Ax = b,\ \exists\ y \text{ s.t. } A^\top y + s = c \\ x \in \mathcal{K},\ s \in \mathcal{K}^* \\ c^\top x - b^\top (AA^\top)^{-1} A(c - s) \le \delta \end{array} \right\}$$

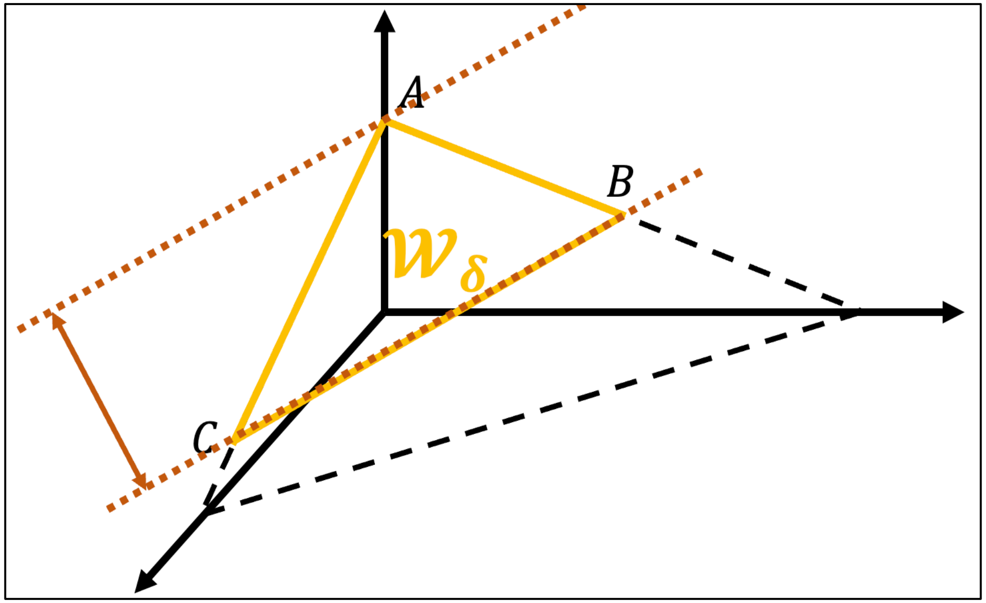
duality gap is at most δ



Note: $\mathcal{W}_0 = \mathcal{W}^\star$

Three Geometric Condition Numbers

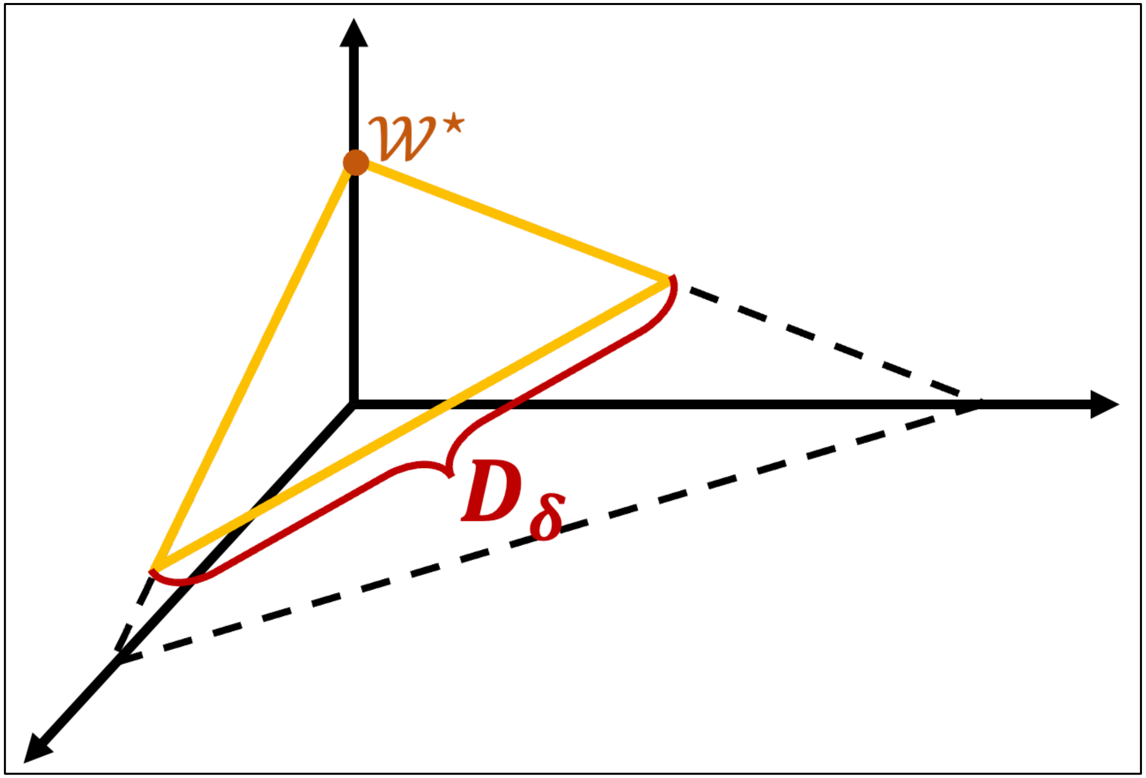
1. D_δ -- the diameter of $\mathcal{W}_\delta$
2. r_δ -- conic radius of $\mathcal{W}_\delta$
3. d_δ^H -- Haussdorf distance between $\mathcal{W}_\delta$ and $\mathcal{W}^\star$



D_δ: Diameter of δ-sublevel set $\mathcal{W}_\delta$

$$D_\delta := \max_{\bar{w}, \hat{w} \in \mathcal{W}_\delta} \|\bar{w} - \hat{w}\|$$

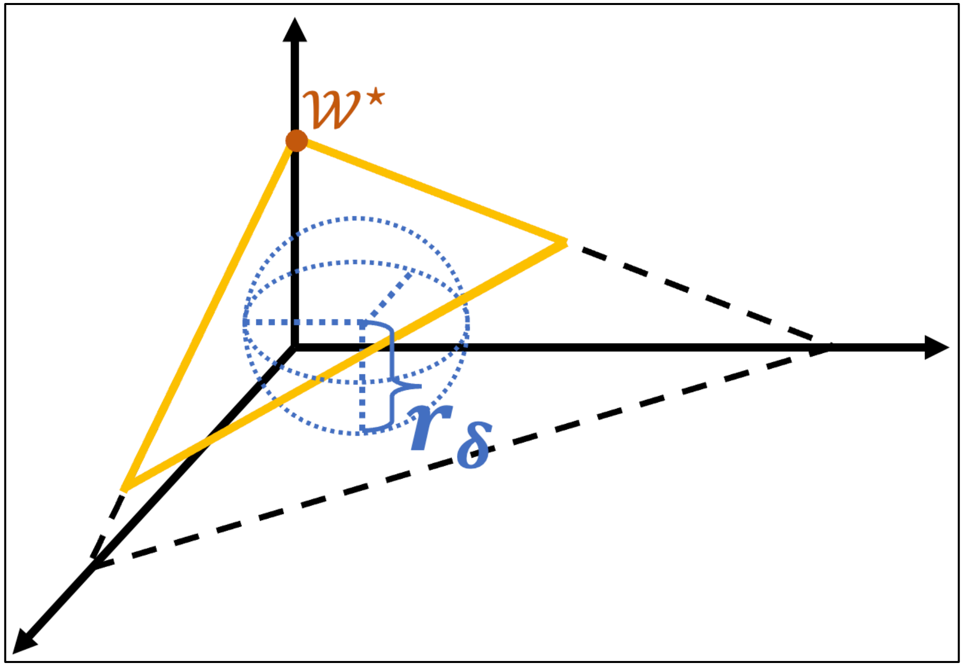
(D_δ is smaller when δ is small)



r_δ: "Conic Radius" of $\mathcal{W}_\delta$

$$r_\delta := \max_{r \geq 0,\, w \in \mathcal{W}_\delta} r \quad \text{s.t.} \quad B_w(r) \subset \mathcal{K} \times \mathcal{K}^*$$

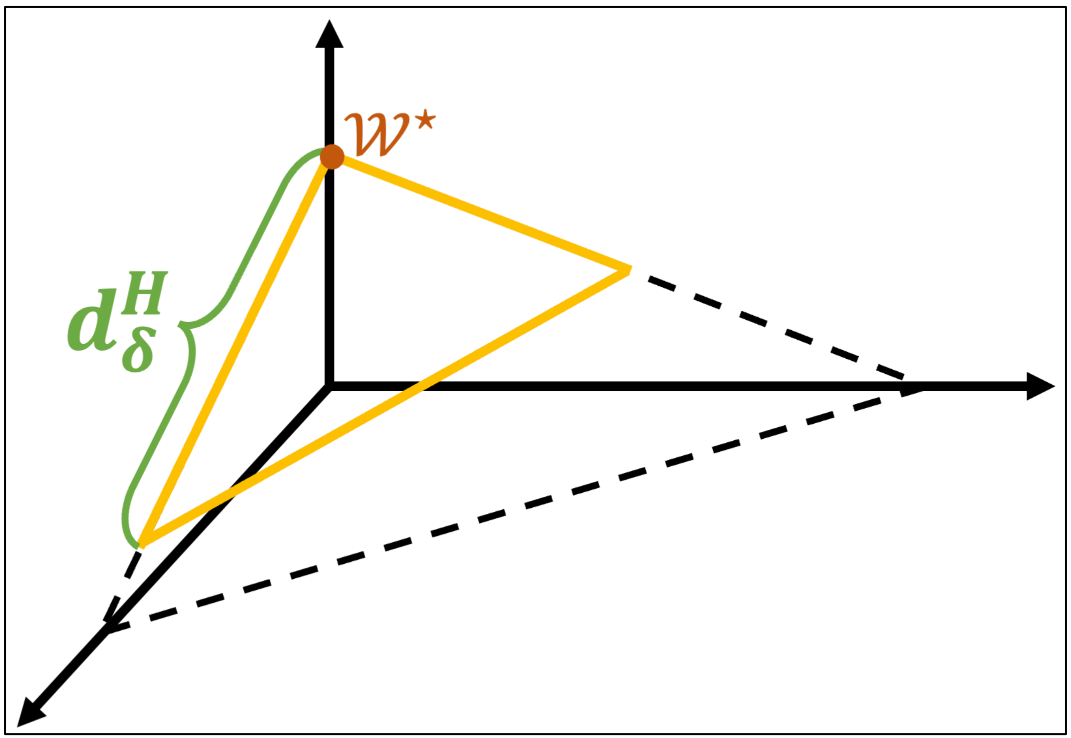
r_δ is the radius of the maximum ball inscribed in $\mathcal{K} \times \mathcal{K}^\star$ and centered at a point in $\mathcal{W}_\delta$



d_δ^H: Hausdorff distance from $\mathcal{W}_\delta$ to $\mathcal{W}^\star$

$$d_\delta^H := \max_{w \in \mathcal{W}_\delta} \mathrm{Dist}(w, \mathcal{W}^\star)$$

d_δ^H is small when δ is small



Convergence Guarantee for rPDHG

Worst-case complexity of PDHG

To get some intuition first, let's assume for the moment that the instance has unique optimum. Then:

Theorem [Xiong and F 2024]: Suppose $w^\star$ is unique. PDHG computes an $\boldsymbol{\varepsilon}$-optimal solution within

$$\tilde{O}\left(\kappa \cdot \lim_{\delta\to 0} \frac{\boldsymbol{D_\delta}}{\boldsymbol{r_\delta}} \cdot \ln\left(\frac{1}{\varepsilon}\right)\right)$$

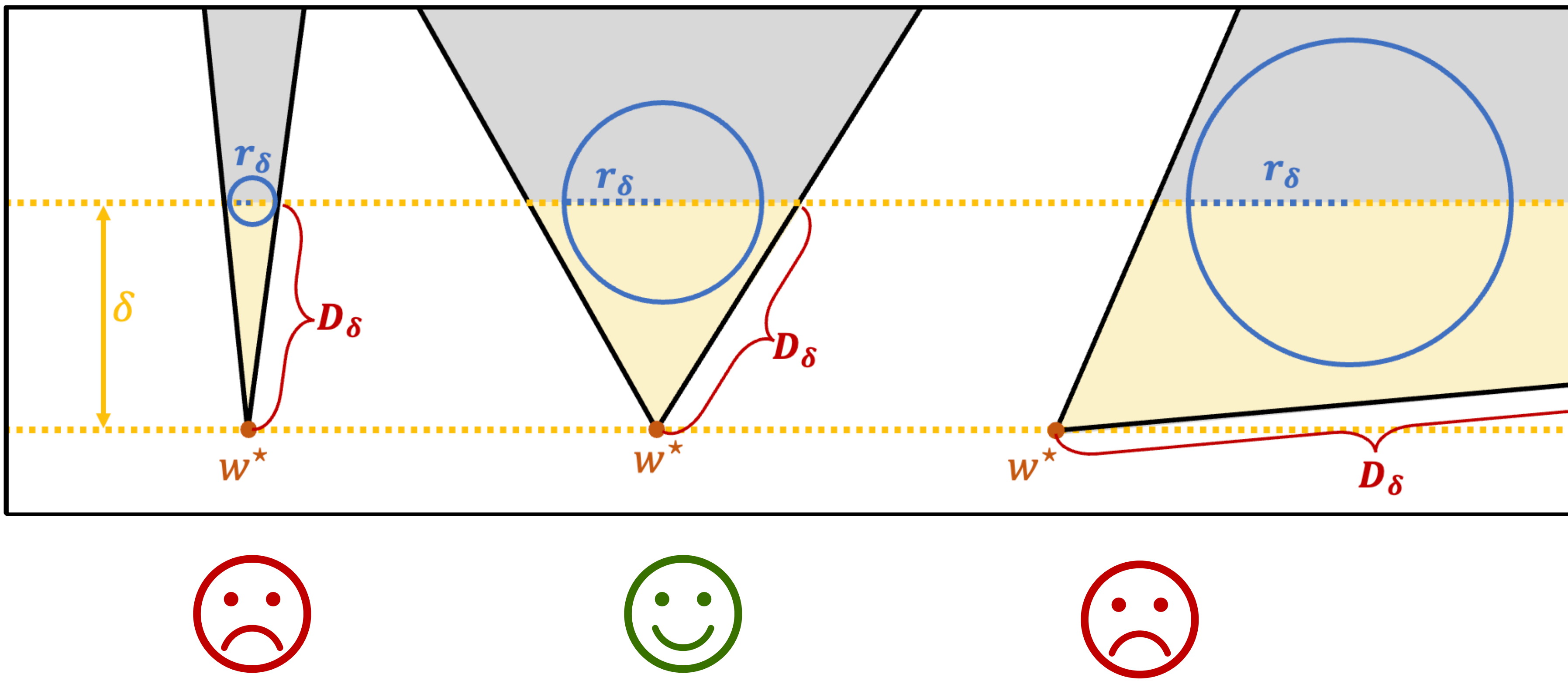
iterations.

Matrix condition number of A:
$\kappa := \sigma_{\max}^{+}(A)/\sigma_{\min}^{+}(A)$

"Sublevel-set geometry"

Local Geometry of and $\lim_{\delta \to 0} \frac{D_\delta}{r_\delta}$ in the case of LP

When $w^\star$ is unique and δ is sufficiently small, $\mathcal{W}_\delta$ is a slice of a pointed cone at $w^\star$.



Very small r_δ
Intermediate D_δ

Intermediate r_δ
Intermediate D_δ

Intermediate r_δ
Very large D_δ

Worst-case complexity of PDHG (under unique optima)

Matrix condition number of A: $\kappa = \sigma^+_{\max}(A)/\sigma^+_{\min}(A)$

Theorem [Xiong and F 2024]: Suppose $w^\star$ is unique. PDHG computes an $\boldsymbol{\varepsilon}$-optimal solution within

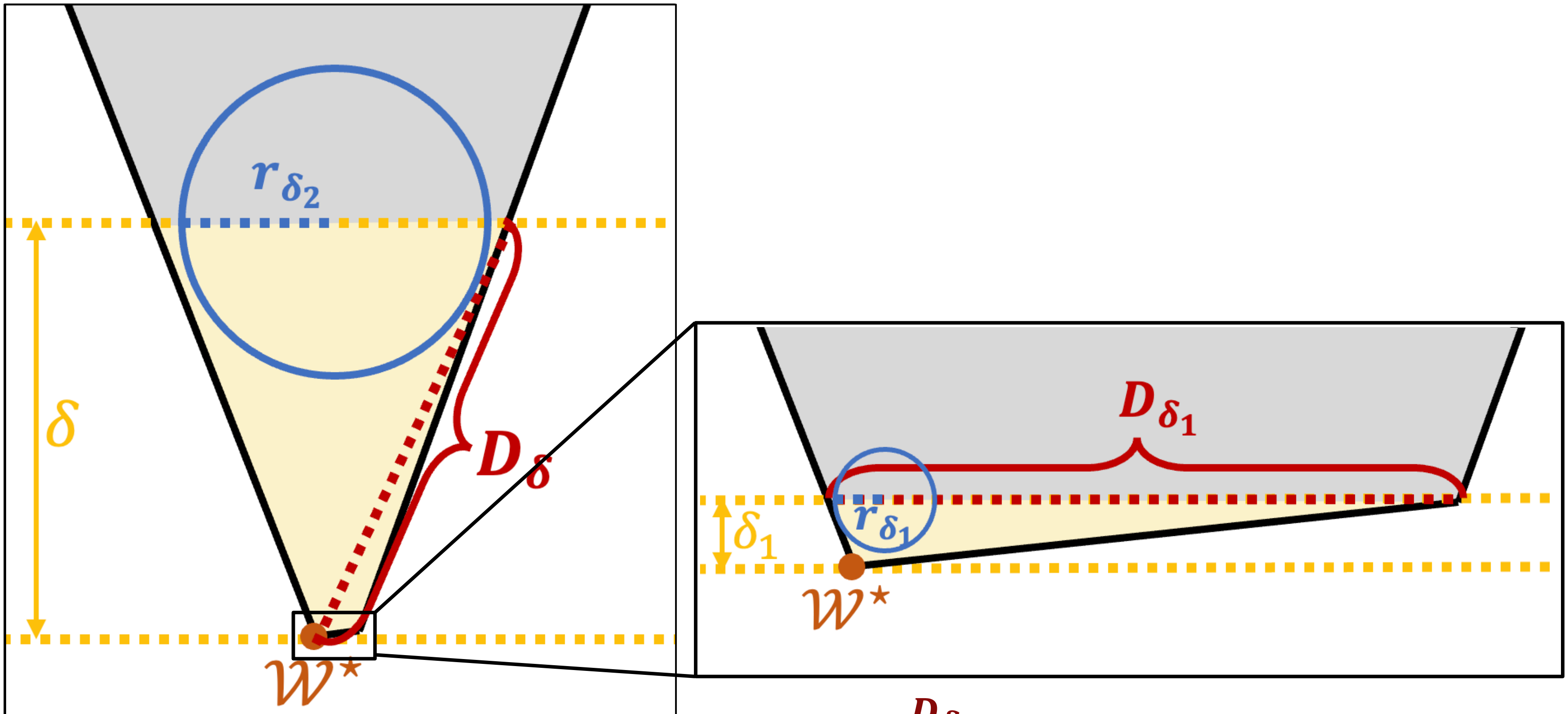
$$\tilde{O}\left(\kappa \cdot \lim_{\delta \to 0} \frac{\boldsymbol{D_\delta}}{\boldsymbol{r_\delta}} \cdot \ln\left(\frac{1}{\varepsilon}\right)\right)$$

iterations.

Matrix condition number	Local geometric condition

- This iteration bound is "superior" to the Hoffman constant
- For LP, this bound is $\tilde{O}\left(n^{2.5} \cdot \ln\left(\frac{1}{\varepsilon}\right)\right)$ with high probability [Xiong, 2024]
- For LP, this bound has a closed-form expression [Xiong, 2024]

Another example

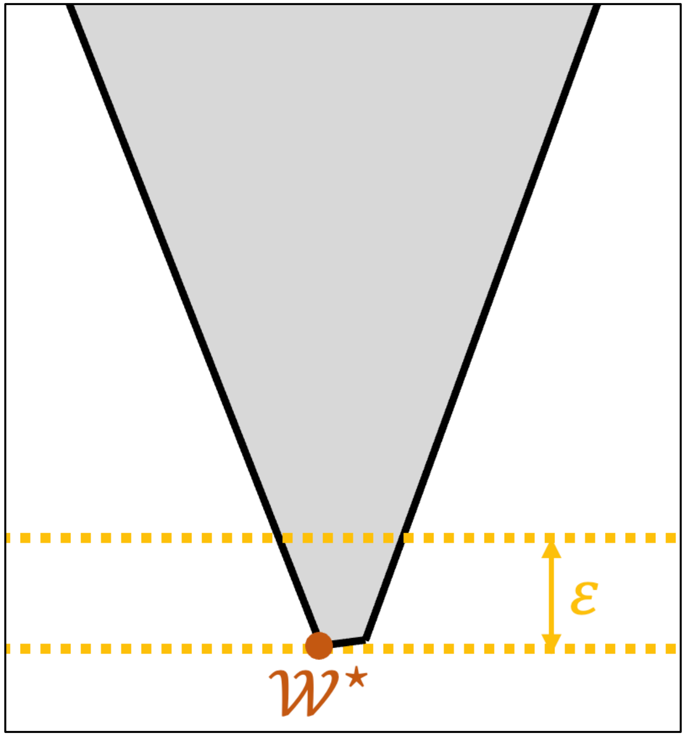


For $\delta_2 > \delta_1$, $\frac{D_{\delta_2}}{r_{\delta_2}}$ becomes smaller/better

$\frac{D_{\delta_1}}{r_{\delta_1}}$ is very large/bad (due to the small r_{δ_1})

Is $\lim_{\delta \to 0} \frac{D_\delta}{r_\delta}$ the only geometric condition?

Suppose we want an ε-optimal solution:



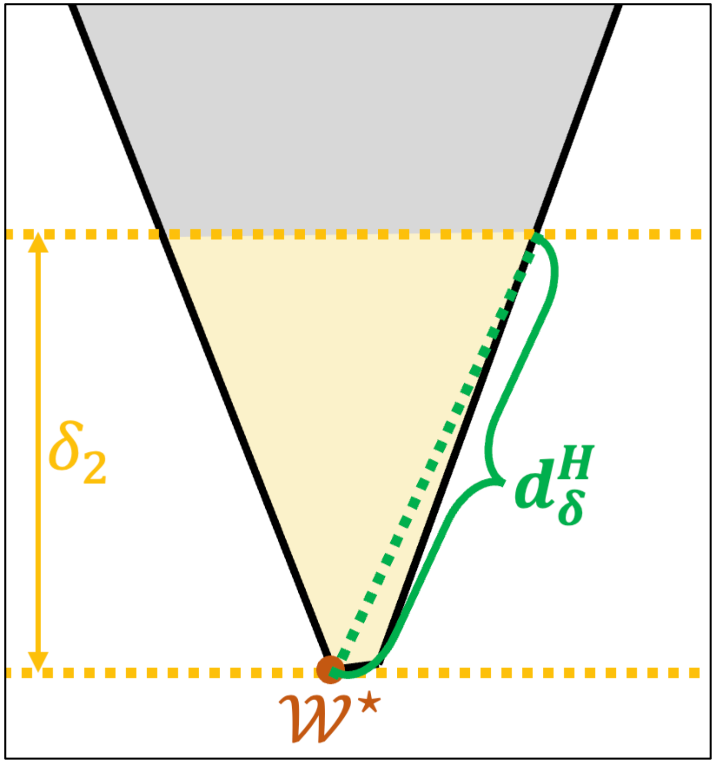
Intuition: The very-local bad geometry should not have a significant impact when the iterates of the algorithm have not yet reached the local neighborhood.

Is $\lim_{\delta \to 0} \frac{D_\delta}{r_\delta}$ the only geometric condition?

We need a third geometric measure to define "being close to $\mathcal{W}^\star$"

$$d_\delta^H \coloneqq \max_{w \in \mathcal{W}_\delta} \mathrm{Dist}(w, \mathcal{W}^\star)$$

Hausdorff distance from $\mathcal{W}_\delta$ to $\mathcal{W}^\star$



Our General Conic Optimization Computational Guarantee

Theorem [Xiong and F 2024]: The number of PDHG iterations required to compute an $\boldsymbol{\varepsilon}$-optimal solution is upper bounded by:

$$T_\delta := \tilde{O}\left(\kappa \cdot \max\left\{\frac{\boldsymbol{D_\delta}}{\boldsymbol{r_\delta}} \cdot \ln\left(\frac{1}{\varepsilon}\right), \frac{\boldsymbol{d_\delta^H}}{\varepsilon}(1 + \mathrm{Dist}(0, \mathcal{W}^\star))\right\}\right)$$

for each $\delta > 0$.

How good the geometry of $\mathcal{W}_\delta$ is

How close $\mathcal{W}_\delta$ is to $\mathcal{W}^\star$

Remark: This result holds under multiple optima and general conic optimization.

$\boldsymbol{D_\delta}$: Diameter of $\mathcal{W}_\delta$

$\boldsymbol{r_\delta}$: Conic radius of $\mathcal{W}_\delta$

$\boldsymbol{d_\delta^H}$: Hausdorff distance from $\mathcal{W}_\delta$ to $\mathcal{W}^\star$

$\boldsymbol{\kappa} = \sigma_{\max}^+(A)/\sigma_{\min}^+(A)$ (the standard condition number of A)

Our General Conic Optimization Computational Guarantee

Theorem [Xiong and F 2024]: The number of PDHG iterations required to compute an $\boldsymbol{\varepsilon}$-optimal solution is upper bounded by:

$$\tilde{O}\left(\inf_{\delta>0} T_\delta := \kappa \cdot \max\left\{\frac{\boldsymbol{D_\delta}}{\boldsymbol{r_\delta}} \cdot \ln\left(\frac{1}{\varepsilon}\right), \frac{\boldsymbol{d_\delta^H}}{\varepsilon}\left(1 + \mathrm{Dist}(0, \mathcal{W}^\star)\right)\right\}\right)$$

- Linear convergence part
- Note that $\boldsymbol{D_\delta}/\boldsymbol{r_\delta}$ might/not be large when δ is small

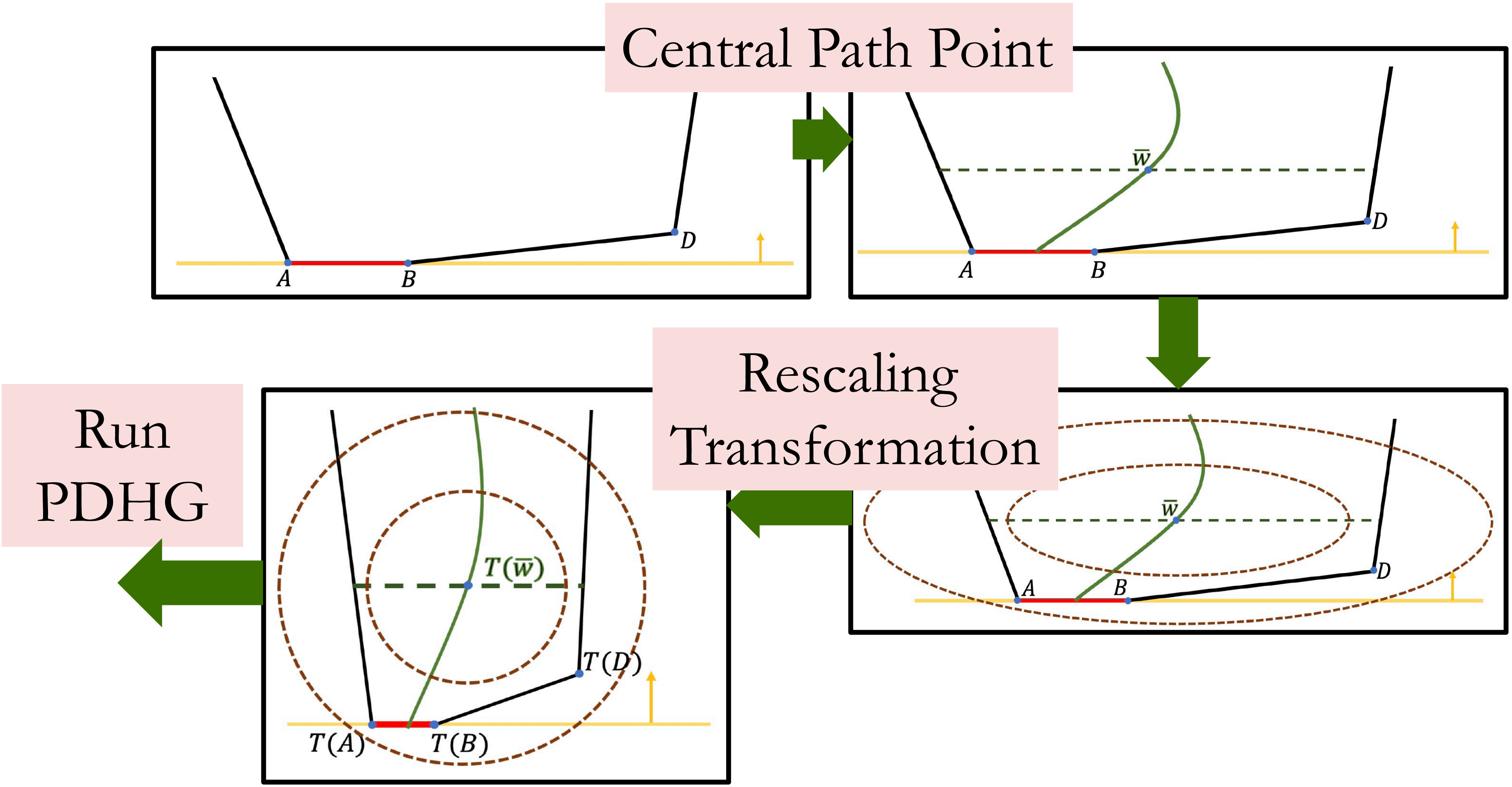
- Sublinear convergence part
- Note $\boldsymbol{d_\delta^H}$ is small when δ is small

The tightest bound is given by the T_δ that minimizes the bound ☺

$\boldsymbol{D_\delta}$: Diameter of $\mathcal{W}_\delta$
$\boldsymbol{r_\delta}$: Conic radius of $\mathcal{W}_\delta$
$\boldsymbol{d_\delta^H}$: Hausdorff distance between $\mathcal{W}_\delta$ and $\mathcal{W}^\star$
$\boldsymbol{\kappa} = \sigma_{\max}^+(A)/\sigma_{\min}^+(A)$

Using theory to develop practical computational speed-ups of PDHG

Brief idea


Central Path Point
Rescaling Transformation
Run PDHG
A
B
D
$\bar{w}$
$T(\bar{w})$
$T(A)$
$T(B)$
$T(D)$

Central-Path solutions are good interior-points

Let $F(\cdot)$ be a logarithmically homogeneous self-concordant barrier for $\mathcal{K}$ with complexity value ϑ_F

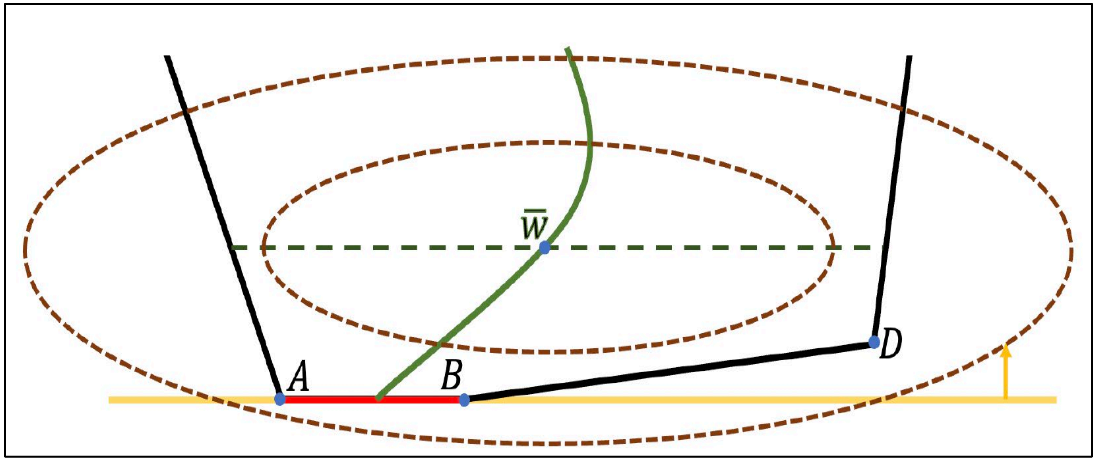
Central-path solutions are:

$$x_\mu := \arg\min_{x} \ c^\top x + \mu \cdot F(x) \quad \text{s.t. } Ax = b,\ x \in \mathcal{K}$$

$$y_\mu, s_\mu := \arg\max_{y,s} \ b^\top y - \mu \cdot F^*(s) \quad \text{s.t. } A^\top y + s = c,\ s \in \mathcal{K}^*$$

Example for LP: $F(x) := -\sum_{i=1}^n \ln(x_i)$, $\vartheta_F = n$

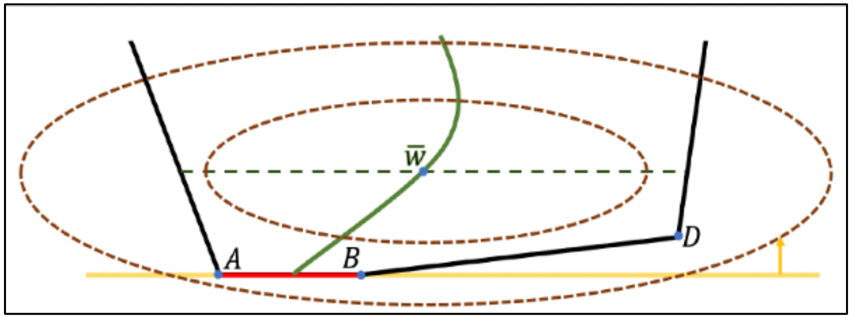
At $\overline{w} = (x_\mu, s_\mu)$, the local-norm ball nicely approximates the shape of sublevel sets:



Transformation based on a central-path solution

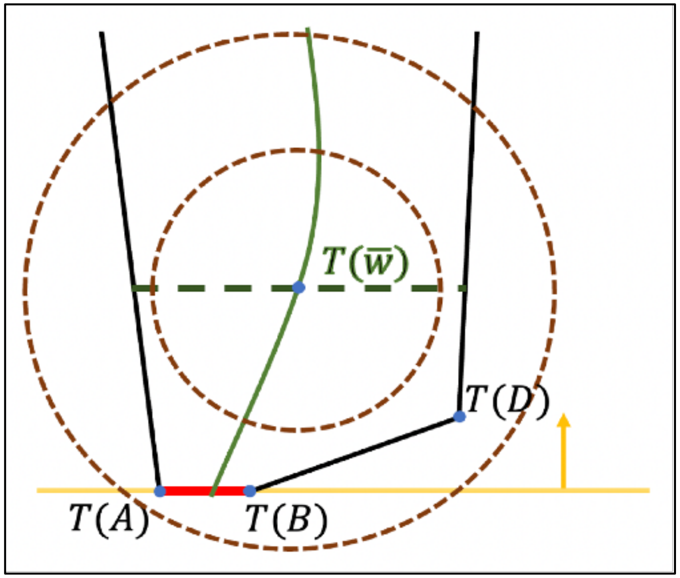
Let $H := \mu \cdot \nabla^2 F(x_\mu)$, then for $\bar{\boldsymbol{\delta}} := \boldsymbol{\vartheta_F} \cdot \boldsymbol{\mu}$ we have:

- $D_{\bar{\delta}} \leq 2\vartheta_F \cdot \sigma_{\min}^{+}(H)^{-1/2}$
- $r_{\bar{\delta}} \geq \sigma_{\max}^{+}(H)^{-1/2}$
- $d_{\bar{\delta}}^{H} \leq 2\vartheta_F \cdot \sigma_{\min}^{+}(H)^{-1/2}$



After a rescaling transformation (turns the local-norm ball into a Euclidean norm ball) we have:

- $D_{\bar{\delta}} \leq 2\sqrt{\vartheta_F} \cdot \sqrt{\bar{\delta}}$
- $r_{\bar{\delta}} \geq \sqrt{\frac{1}{\vartheta_F}} \cdot \sqrt{\bar{\delta}}$
- $d_{\bar{\delta}}^{H} \leq 2\sqrt{\vartheta_F} \cdot \sqrt{\bar{\delta}}$
- $\frac{D_{\bar{\delta}}}{r_{\bar{\delta}}} \leq 2\vartheta_F$
- $d_{\bar{\delta}}^{H}$ is small if $\bar{\boldsymbol{\delta}}$ is small
- "Very nice theory"



Proposed strategy

Suppose we do the following:

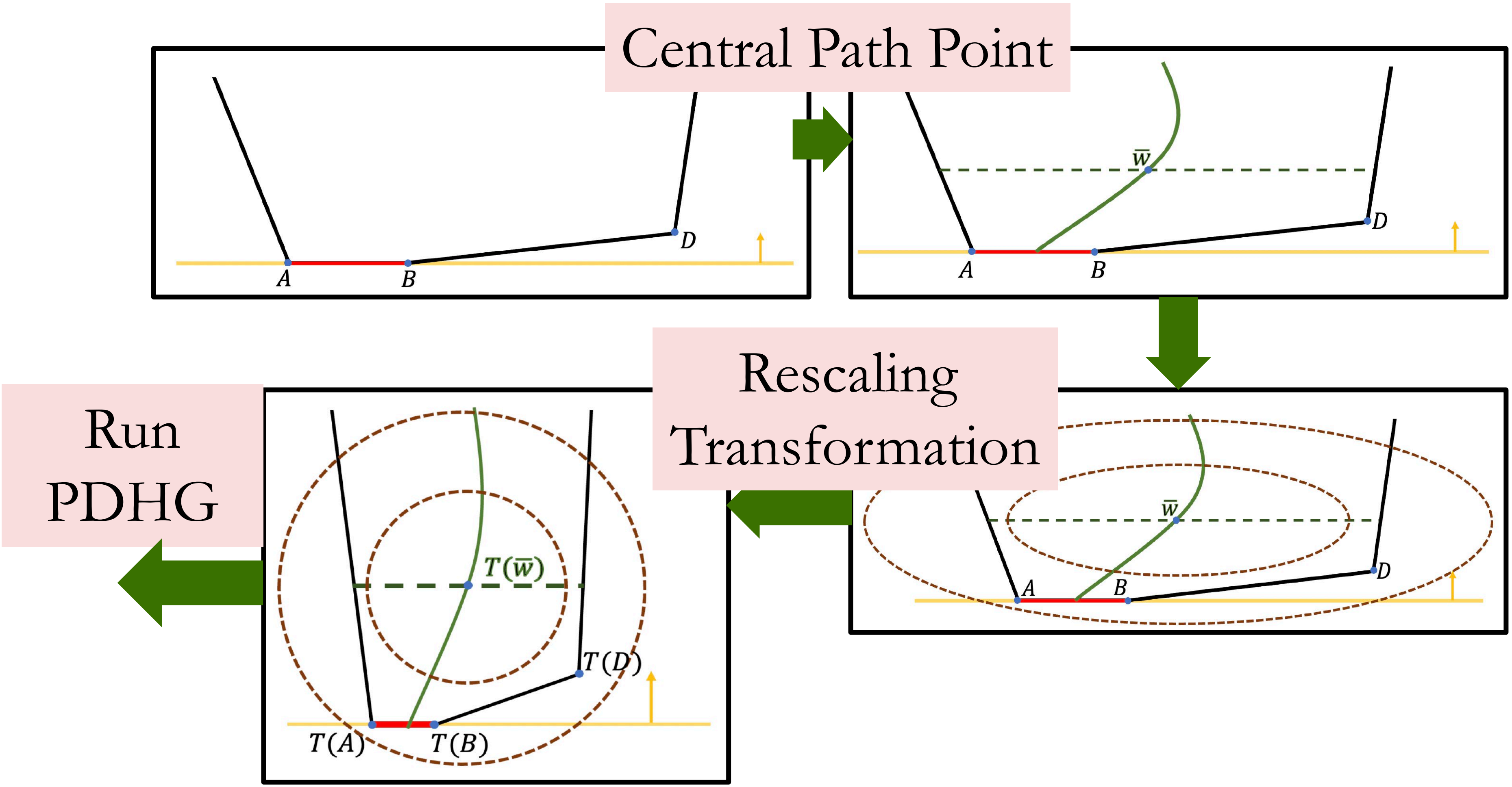
1. rescaling transformation using a central-path solution **with duality gap $\bar{\boldsymbol{\delta}}$**
2. row transformation to try to decrease closer to $\kappa = 1$

Then the number of PDHG iterations required to compute an ε-optimal solution of the original CLP problem is upper bounded by:

$$\tilde{O}\left(\boldsymbol{\vartheta_F} \cdot \left(\ln\left(\frac{1}{\varepsilon}\right) + \frac{D_{\bar{\delta}} + \bar{\boldsymbol{\delta}}}{\varepsilon}\right)\right)$$

- We have replaced $\dfrac{D_\delta}{r_\delta}$ by ϑ_F
- The smaller $\bar{\boldsymbol{\delta}}$ is, the faster the convergence
- Of course we will need to "pay" to compute the point on the central path…

Summary



Central Path Point
A
B
D
$\overline{w}$
Rescaling Transformation
Run PDHG
$T(\overline{w})$
$T(D)$
$T(A)$
$T(B)$

rPDHG-AHR (“Adaptive Hessian Rescaling”) and Computational Experiments

Proof of concept

PDHG-EasyColumn

Column rescaling to **normalize L_∞** column norms

PDHG-Central δ=0.1

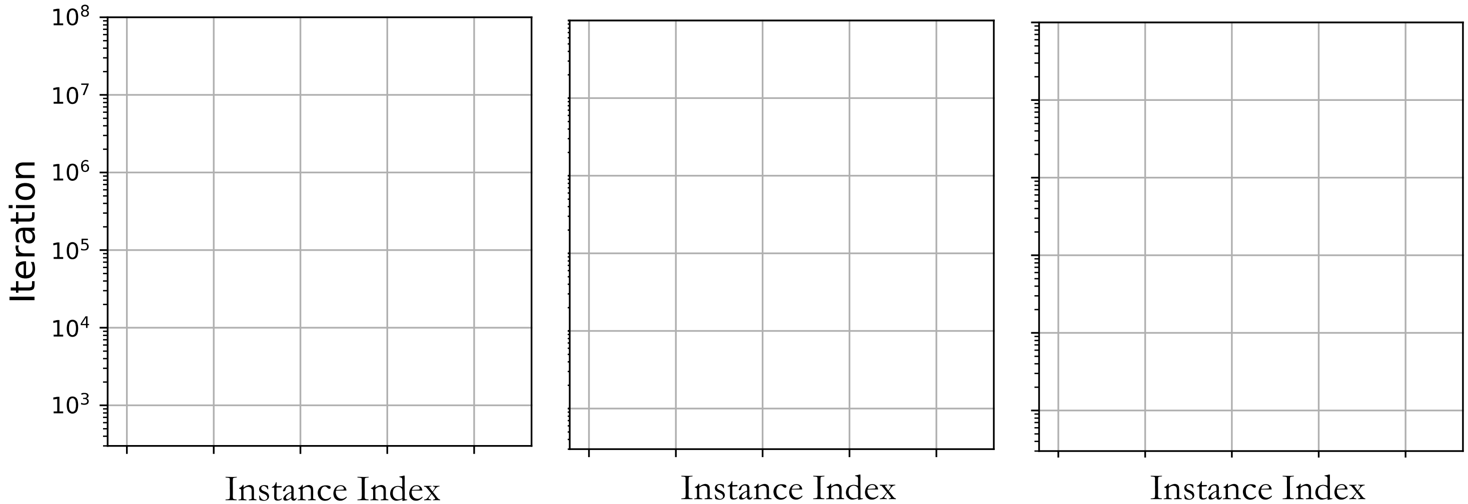
Column rescaling using **central-path solution** with KKT error **0.1**

PDHG-Central δ=0.01

Column rescaling using **central-path solution** with KKT error **0.01**

Note: we pre-multiply A by $(AA^\top)^{-1/2}$ to yield $\kappa = 1$ for all rescalings

PDHG iterations needed to compute a solution with KKT error **10^{-8}**



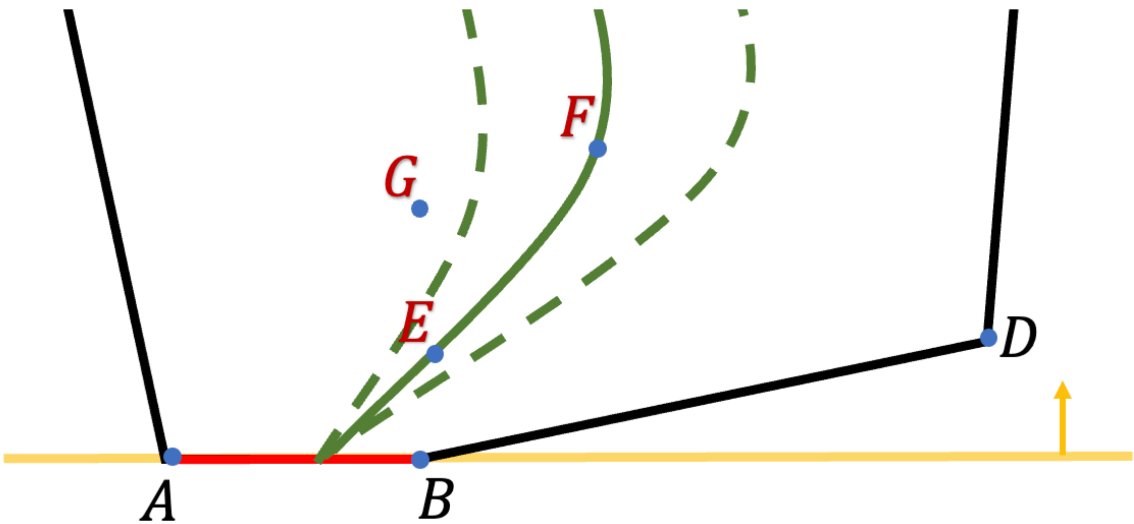
222 LP relaxation instances from the MIPLIB 2017 dataset

Main Strategy, continued

Main strategy: use a "CG-IPM" to compute a low-accuracy central-path solution to obtain a good rescaling, and then use PDHG

- **We use a first-order method to compute the Newton steps of a "central path" solution**
  - we use a conjugate-gradient-method-based IPM (**CG-IPM**)
  - otherwise, implementation exactly follows Nocedal and Wright *Numerical Optimization* (2006)
  - but we solve the normal equations using conjugate gradient method
- **We employ only diagonal row rescaling to try to improve $\boldsymbol{\kappa}$**
  - Column rescaling uses the central-path Hessian of w_{int} followed by PDLP's rescaling (Ruiz rescaling and Pock-Chambolle rescaling), which we call the "$\boldsymbol{w_{int}}$-rescaled problem"

Using Adaptive Hessian Rescaling



Motivating concepts of Adaptive Hessian Rescaling:

- Adaptively balance the cost of computing the rescaling (CG-IPM) with the savings from running PDHG on the rescaled instance
- Try to identify a "good-enough" rescaling <u>as early as possible</u> and use the good-enough rescaling

Computational Experiments with **PDHG-AHR**

We compare:

1. **PDHG-AHR** : PDHG with Adaptive central-path Hessian Rescaling (using CG-IPM for the central-path computations)
2. **PDHG(RuizPC)** : use heuristic Ruiz rescaling on ***A***, followed by Pock-Chambolle rescaling. (This is the same as the rescaling used in PDLP.)
3. **IPM** : a home-grown standard primal-dual predictor-corrector interior point method, straight from Nocedal and Wright *Numerical Optimization* (2006).

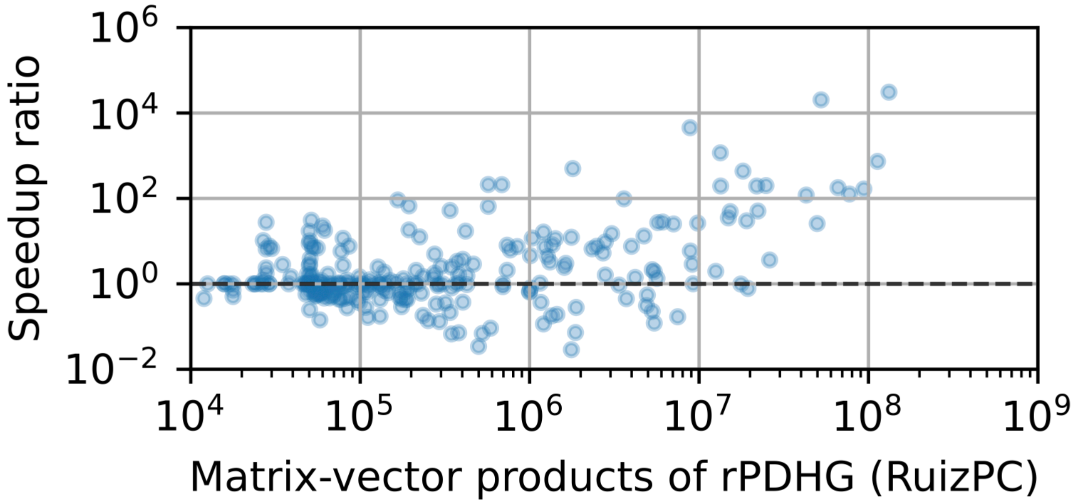
We performed tests on all the LP relaxations from MIPLIB 2017 dataset that are:

- large enough ($m \times n > 10^6$)
- but not too large for the IPM (number of non-zeros $< 10^5$)
- This yielded 413 instances in total

Computational Comparison: PDHG-AHR and PDHG(RuizPC)

Speedup Ratio:

$$\frac{\text{Matrix−vector products PDHG(RuizPC)}}{\text{Matrix−vector products PDHG−AHR}}$$



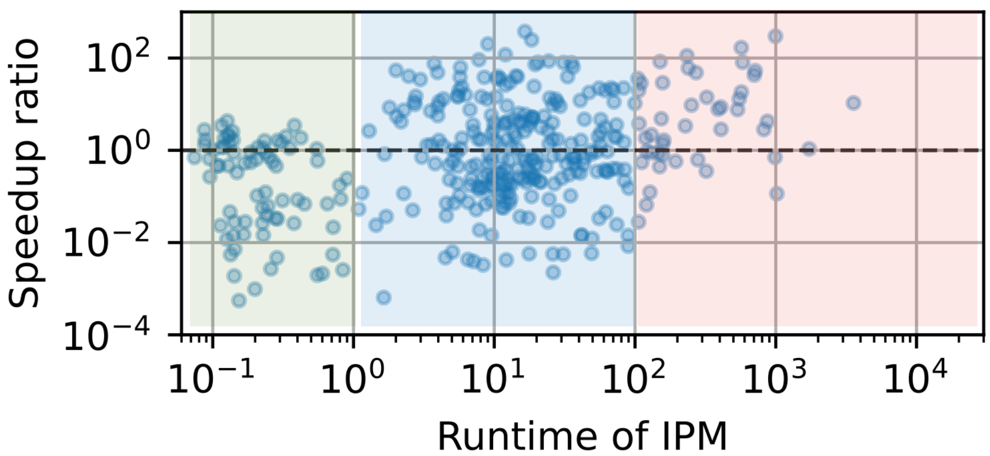
		Fraction solved by PDHG-AHR	
		S	**N**
Fraction solved by **PDHG (RuizPC)**	**S**	**87.7%**	**1.7%**
	N	**7.5%**	**3.1%**

- In general, the harder the problem is for **PDHG(RuizPC)**, the larger the speedups from using **PDHG-AHR**
- **PDHG-AHR** solves about 95.2% of problems and is more reliable than **PDHG(RuizPC)** (solves about 89.4% of problems)

Computational Comparison: PDHG-AHR and IPM

Speedup Ratio:

$$\frac{\text{Runtime IPM}}{\text{Runtime PDHG-AHR}}$$



		Fraction solved by PDHG−AHR: S	Fraction solved by PDHG−AHR: N
Fraction solved by IPM	S	93.0%	1.5%
	N	2.2%	3.4%

- Generally speaking, the harder the problem is for **IPM**, the larger the speedups from using **PDHG-AHR**.
- **PDHG-AHR** is also (slightly) more reliable than **IPM**

Recap, Takeaways, and Remarks

Recap and takeaways:

- The convergence rate of PDHG on CLP is related to the geometry of primal-dual sublevel sets measured with $\boldsymbol{D_\delta}$, $\boldsymbol{r_\delta}$, $\boldsymbol{d_\delta^H}$
- Rescaling using a central-path solution can improve the geometry of the primal-dual sublevel sets
- Our strategy: **PDHG-AHR** uses a "CG-IPM" to compute a low-accuracy central-path solution to obtain a good rescaling, and then use PDHG
- FOMs can compete and outperform IPMs

Remarks:

- Our results relied only on PDHG's average iterate convergence and non-expansiveness properties. Similar results might also hold for other FOMs, in particular ADMM, EGM, …

Thank you!