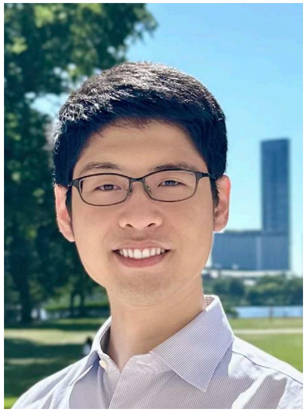
Level-Set Geometry and the Performance of Restarted-PDHG for Conic LP

Zikai Xiong and Robert Freund



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Paper available on arXiv

Huge-scale Optimization is "Everywhere"

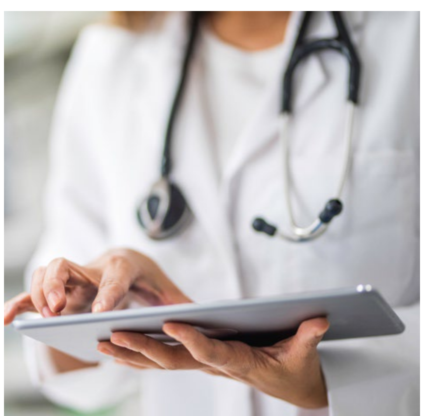
Manufacturing

Machine Learning

Energy

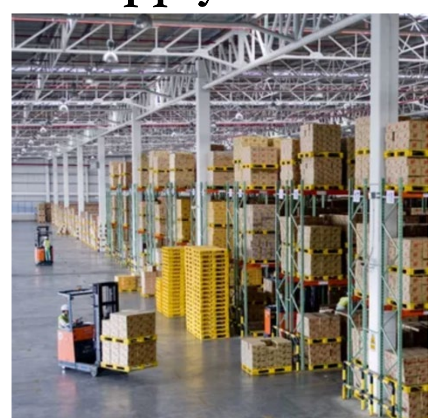
Transportation

Healthcare



Markets and Auctions

Supply Chains



Agriculture

History of Linear Optimization ("LO" or "LP")

1947

Simplex Method

[George Dantzig, 1947]
75+ years ago

1984

Interior Point Method

[Narendra Karmarkar, 1984]
40 years ago

One slide on Interior-Point Methods (IPMs)

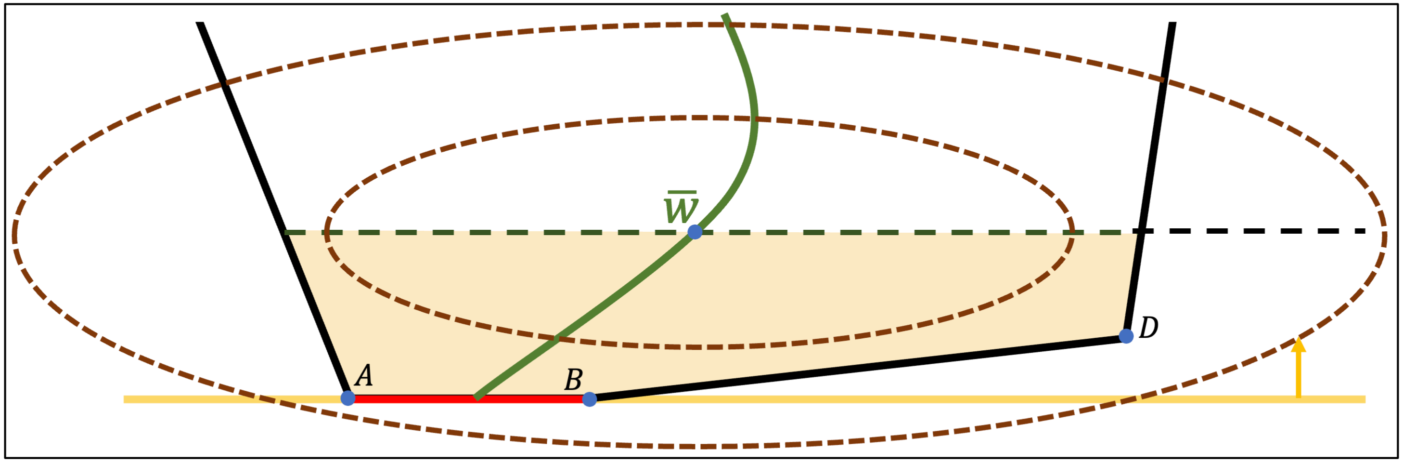
Central-Path ellipsoids have remarkable properties

Central-path solutions are:

$$x_\mu := \arg\min_x \; c^\top x + \mu \cdot F(x) \quad \text{s.t. } Ax = b,\; x \in \mathcal{K}$$

$$y_\mu, s_\mu := \arg\max_{y,s} \; b^\top y - \mu \cdot F^*(s) \quad \text{s.t. } A^\top y + s = c,\; s \in \mathcal{K}^*$$

Example for LP: $F(x) \coloneqq -\sum_{i=1}^{n} \ln(x_i)\,,\;\; \vartheta_F = n$



Simplex and IPMs require expensive matrix factorizations

Consider an LP instance with n decision variables and $\frac{n}{2}$ linear constraints, whose constraint matrix has sparsity = 0.05

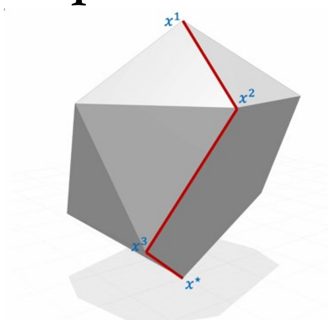
Number of variables (n)	Cost of one IPM iteration

Hence the emergence of FOMs for solving huge (and also not-so-huge) LP instances

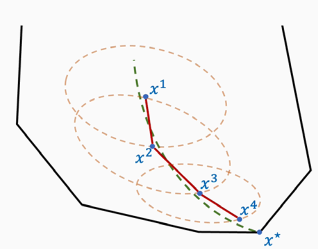
Recent Advances on Huge-Scale LP Solvers in the Industry

Classic methods

Simplex method



Interior-point method



First-order methods

- Primal-Dual Hybrid Gradient ("PDHG", "Chambolle-Pock method")
- Tackles huge-scale problems
- Benefits from modern computational architecture (such as GPU)

Google — 2021

COPT Cardinal Optimizer — State-of-art commercial optimization solver — February, 2024

NVIDIA — March, 2024

HiGHS — An open-source optimization software — March, 2024

FICO — State-of-art commercial optimization solver — April, 2024

We are witnessing a dramatic shift from classic methods to first-order methods

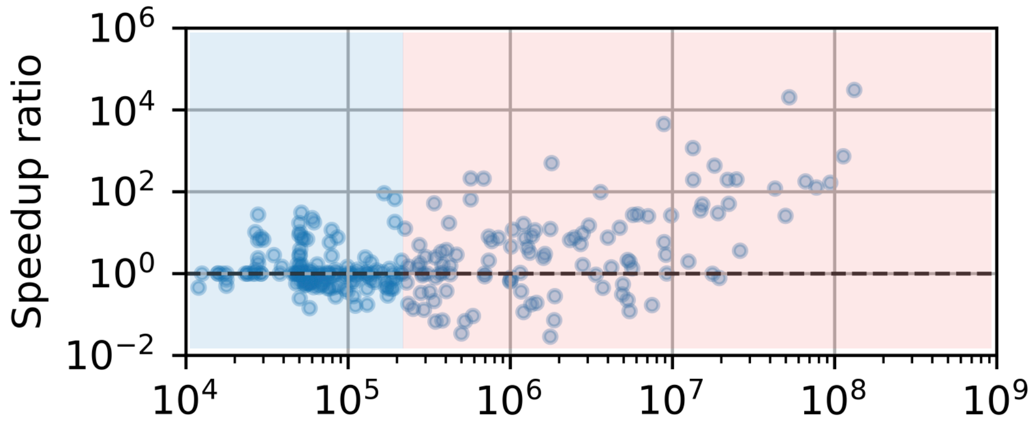
Huge-Scale LP Research

- SCS: Operator splitting/ADMM [O'Donoghue, Chu, Parikh, Boyd, 2016]
- ABIP+: ADMM-based interior-point method [Lin, Ma, Ye, Zhang, 2021] & [Deng, et al., 2022]
- Semi-smooth Newton augmented Lagrangian [Li, Sun, Toh, 2020]
- **Primal-Dual Hybrid Gradient (PDHG)** with restarts, applied directly to the primal-dual saddle point problem [Applegate, Hinder, Lu, Lubin, 2023] & [Applegate, et al., 2021] **(2024 Beale-Orchard-Hays Prize)**
- **GPU implementations** of PDHG in Julia and C [Lu and Yang, 2023] & [Lu, et al., 2023]
- **Guarantees for PDHG for LP** using "Limiting Error Ratios" and LP Sharpness [Xiong and F 2023]
- **Guarantees for PDHG for CLP** – using level-set geometry [Xiong and F 2024]

Sneak Preview:

Distribution of Speedups of our method **PDHG-AHR**

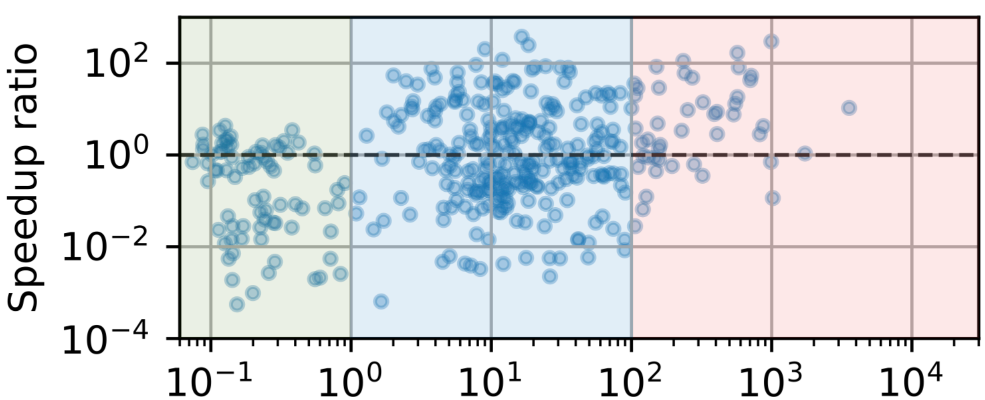
Speedups compared with **PDHG(RuizPC)**



Matrix-vector products of **PDHG(RuizPC)**

Speedups compared with a "home-grown" **IPM**

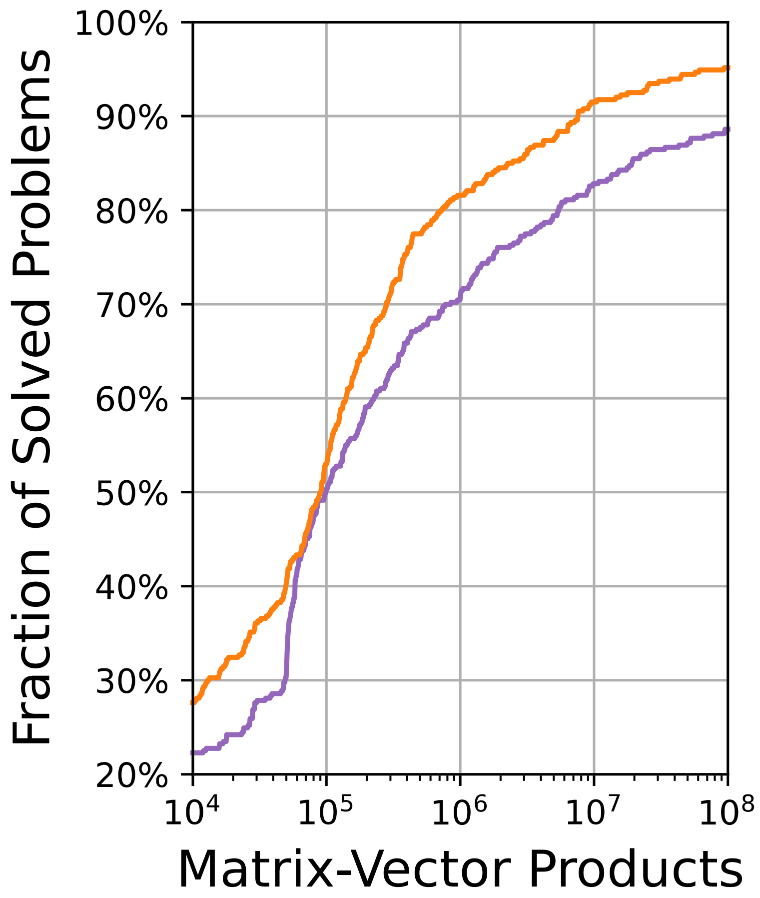
(Predictor-corrector path-following interior-point method in Nocedal and Wright *Numerical Optimization* (2006))



Runtime of **IPM**

Sneak Preview:

Performance Comparison on MIPLIB 2017



PDHG-AHR
rPDHG with our adaptive rescaling

PDHG (RuizPC)
rPDHG with heuristic Ruiz/PC rescaling (same with "PDLP")

Conic Linear Optimization (“CLO” or “CLP”)

CLP in standard form

(primal)

$$\begin{aligned} \min \ & c^\top x \\ \text{s.t.} \ & Ax = b \\ & x \in \mathcal{K} \end{aligned}$$

(dual)

$$\begin{aligned} \max \ & b^\top y \\ \text{s.t.} \ & c - A^\top y \in \mathcal{K}^* \end{aligned}$$

Decision variables

- $x \in R^n$ (for primal problem)
- $y \in R^m$ (for dual problem)

CLP saddlepoint formulation

$$\min_{x \in \mathcal{K}} \max_{y} \ c^\top x - y^\top A x + b^\top y$$

Primal-Dual Hybrid Gradient Method (PDHG)

Conic Optimiztion in Saddlepoint Form

$$\min_{x\in\mathcal{K}} \max_{y} \; c^\top x + b^\top y - y^\top A$$

PDHG

$$x^{k+1} \leftarrow \mathrm{Proj}_{\mathcal{K}}\left(x^k - \tau\left(c - A^\top y^k\right)\right)$$

Gradient w.r.t. x^k

$$y^{k+1} \leftarrow y^k + \sigma\left(b - Ax^{k+1}\right) - \sigma A\left(x^{k+1} - x^k\right)$$

Gradient w.r.t. y^k Momentum Term

Primal-Dual Hybrid Gradient for Conic Optimization

PDHG

$$x^{k+1} \leftarrow \mathrm{Proj}_{\mathcal{K}}\left(x^k - \tau\left(c - A^\top y^k\right)\right)$$
$$y^{k+1} \leftarrow y^k + \sigma\left(b - Ax^{k+1}\right) - \sigma A\left(x^{k+1} - x^k\right)$$

- **Inexpensive iterations**:
  Only requires matrix-vector multiplications
- **“Fast” convergence rates**:
  Adaptive restarts based on average iterates yield global linear convergence on LP [Applegate, Hinder, Lu, Lubin, 2023]

We use “PDHG” to denote “PDHG with adaptive restarts”

Motivation for Restarts for PDHG: “Visualization”

$$\min_{x} \max_{y} x \cdot y$$

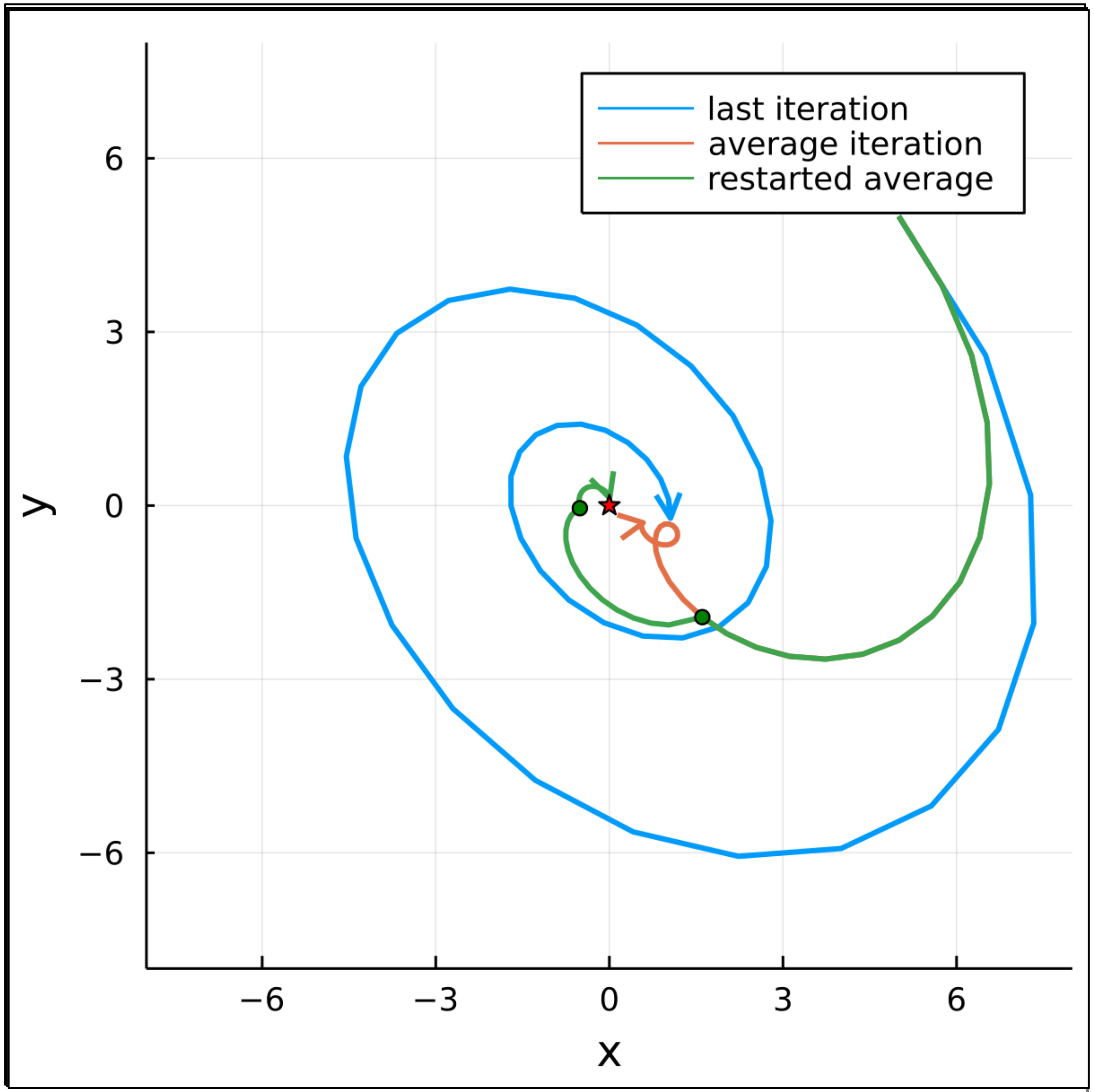
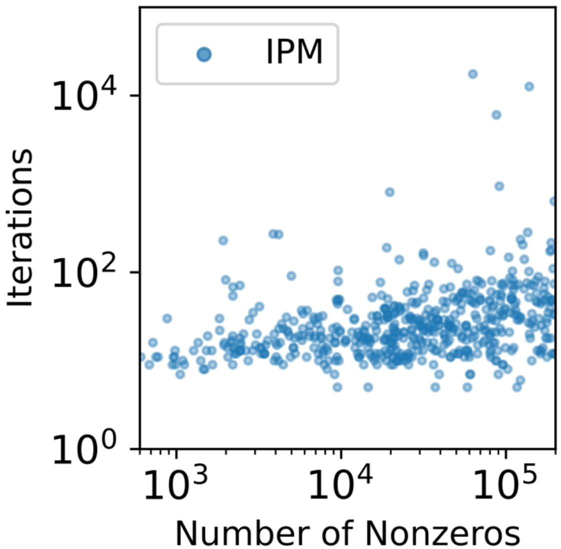


*figure courtesy Haihao Lu

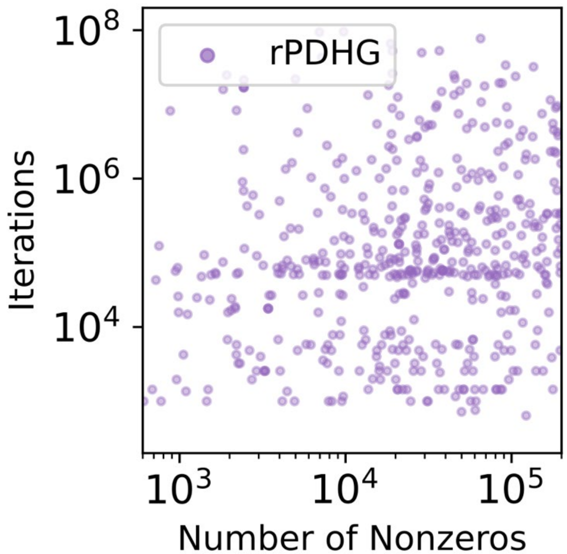
Challenge I: Variability in the Performance of PDHG

- PDHG uses many more iterations than an IPM
  makes sense, it is a first-order method … IPM iterations are hugely expensive while PDHG iterations are very cheap
- Some small problem instances require a very large number of PDHG iterations
  a real challenge for PDHG

IPM Iterations needed for
LP relaxations from MIPLIB 2017



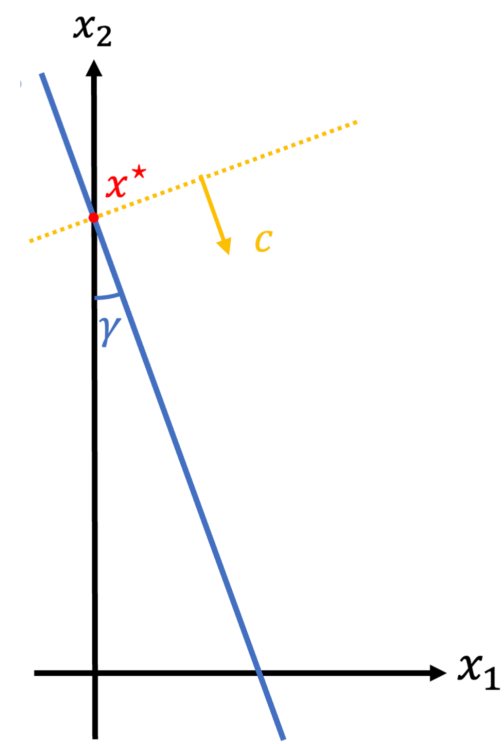
PDHG iterations
LP relaxations from MIPLIB 2017



A seemingly easy LP instance

For $\gamma \in \left(0, \frac{\pi}{2}\right)$ define:

$$P(\gamma): \quad \min_{x_1, x_2} \ \sin(\gamma)\, x_1 - \cos(\gamma)\, x_2 \quad \text{s.t.} \ \sin(\gamma)\, x_1 + \cos(\gamma)\, x_2 = 1, \quad x_1 \geq 0, x_2 \geq 0$$



$P(\gamma)$ is always easy for the simplex method and interior-point methods

However, when γ is small, PDHG requires at least 1,000,000 iterations. What **conditions** of $P(\gamma)$ make it so hard for PDHG?

Challenge II: Loose/unworkable computational guarantees

Existing computational guarantees:

Theorem [Applegate, Hinder, Lu, Lubin, 2023] PDHG computes an ε-optimal solution within

$$O\left((\|x^\star\| + \|y^\star\|) \cdot ||A|| \cdot \boldsymbol{H(K)} \cdot \log\left(\frac{\|x^\star\| + \|y^\star\|}{\varepsilon} \right) \right)$$

iterations.

$\boldsymbol{H(K)}$ is the global Hoffman constant of the matrix $\boldsymbol{K}$ of the KKT system

$$\boldsymbol{K} = \begin{pmatrix} A & \\ -A & \\ & A^\top \\ c^\top & -b^\top \end{pmatrix} \qquad \boldsymbol{H(K)} \approx \frac{1}{\min_{J \subseteq \{1,\ldots,2m+n+1\}} \sigma_{\min}^{+}(K_J)}$$

Likely way too loose, and very hard to compute/validate/analyze.

Challenge II: Loose/unworkable computational guarantees

Existing computational guarantees:

Theorem [Applegate, Hinder, Lu, Lubin, 2023] PDHG computes an ε-optimal solution within

$$O\left((\|x^\star\| + \|y^\star\|) \cdot ||A|| \cdot \boldsymbol{H(K)} \cdot \log\left(\frac{\|x^\star\| + \|y^\star\|}{\varepsilon}\right)\right)$$

iterations.

Key questions:

- What conditions of the problem actually drive the performance of PDHG? Sublevel-set Geometry
- Can we improve these condition numbers and so improve computational performance in theory/practice?
  Yes, we will improve the geometry using Hessian rescaling

Sublevel-set geometry and new performance guarantees

Primal-Dual Slack Space

Primal	Dual
$\min_{x} c^\top x$	$\max_{y,s} \; b^\top y$
$\text{s.t. } Ax = b$	$\text{s.t. } A^\top y + s = c$
$x \in \mathcal{K}$	$s \in \mathcal{K}^*$

The "primal-dual slack-space variable" is w :

$$w \coloneqq (x, s) \text{ are primal/dual feasible slacks}$$

Duality gap: $\text{Gap}(x, s) = c^\top x - b^\top y$

(which is a linear function of x and s)

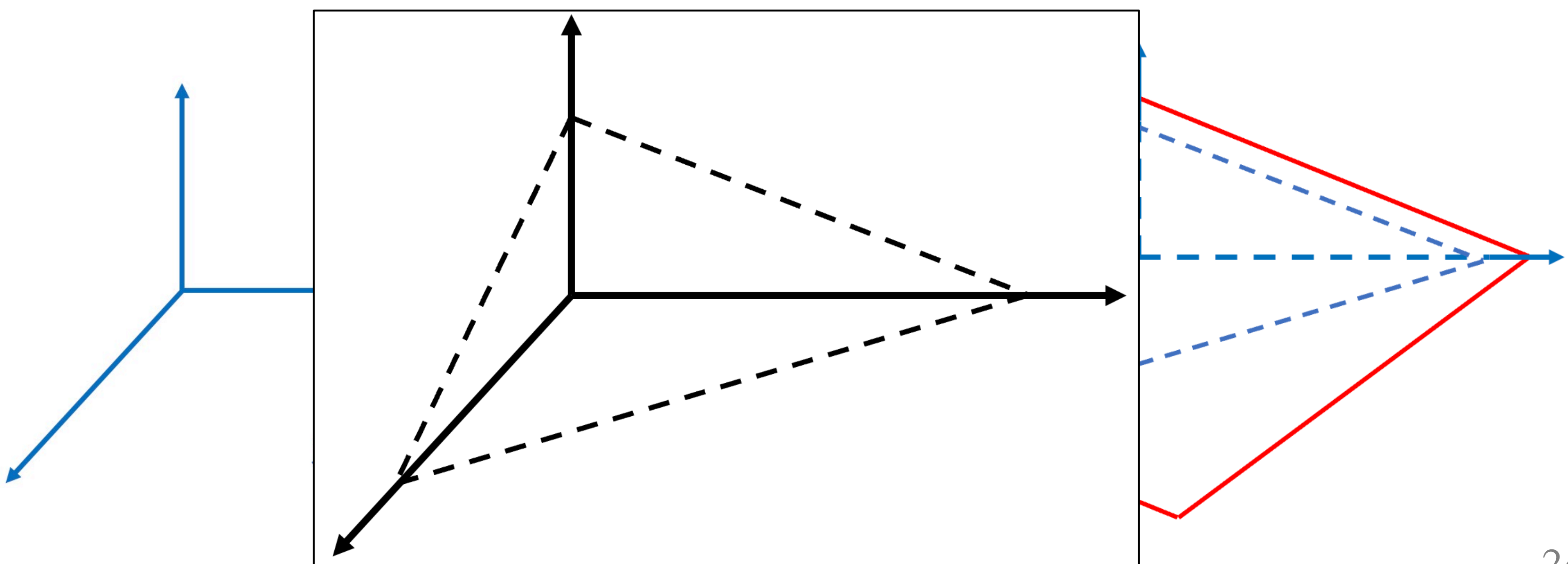
The feasible primal-dual slack-space variables

$$\min_{x} c^\top x \quad \text{s.t. } Ax = b, \; x \in \mathcal{K}$$

$$\max_{y,s} b^\top y \quad \text{s.t. } A^\top y + s = c, \; s \in \mathcal{K}^*$$

(x, s) in the primal and dual cone for CLP

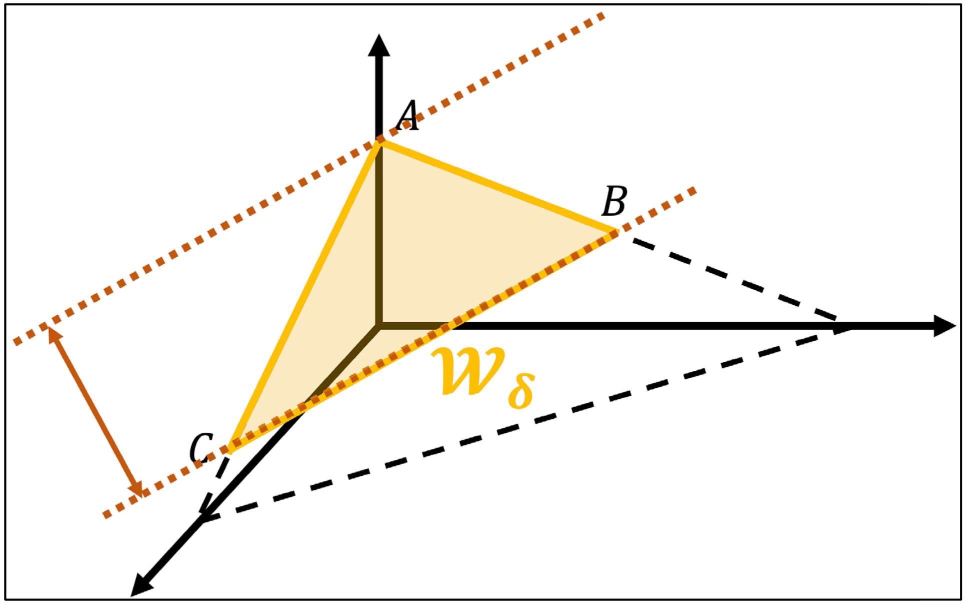
(x, s) lies in an affine subspace



Primal-Dual Slack Sublevel Set

$$\mathcal{W}_\delta := \left\{ w := (x, s) \;\middle|\; \begin{array}{c} w \text{ is primal/dual feasible} \\ \textbf{Gap}(w) \le \delta \end{array} \right\}$$

Note: $\mathcal{W}_0 = \mathcal{W}^\star$



Worst-case complexity of PDHG (under unique optima)

Theorem [Xiong and F 2024]: Suppose $w^\star$ is unique. PDHG computes an $\boldsymbol{\varepsilon}$-optimal solution within

$$\tilde{O}\left(\kappa \cdot \lim_{\delta \to 0} \frac{\boldsymbol{D_\delta}}{\boldsymbol{r_\delta}} \cdot \ln\left(\frac{1}{\varepsilon}\right)\right)$$

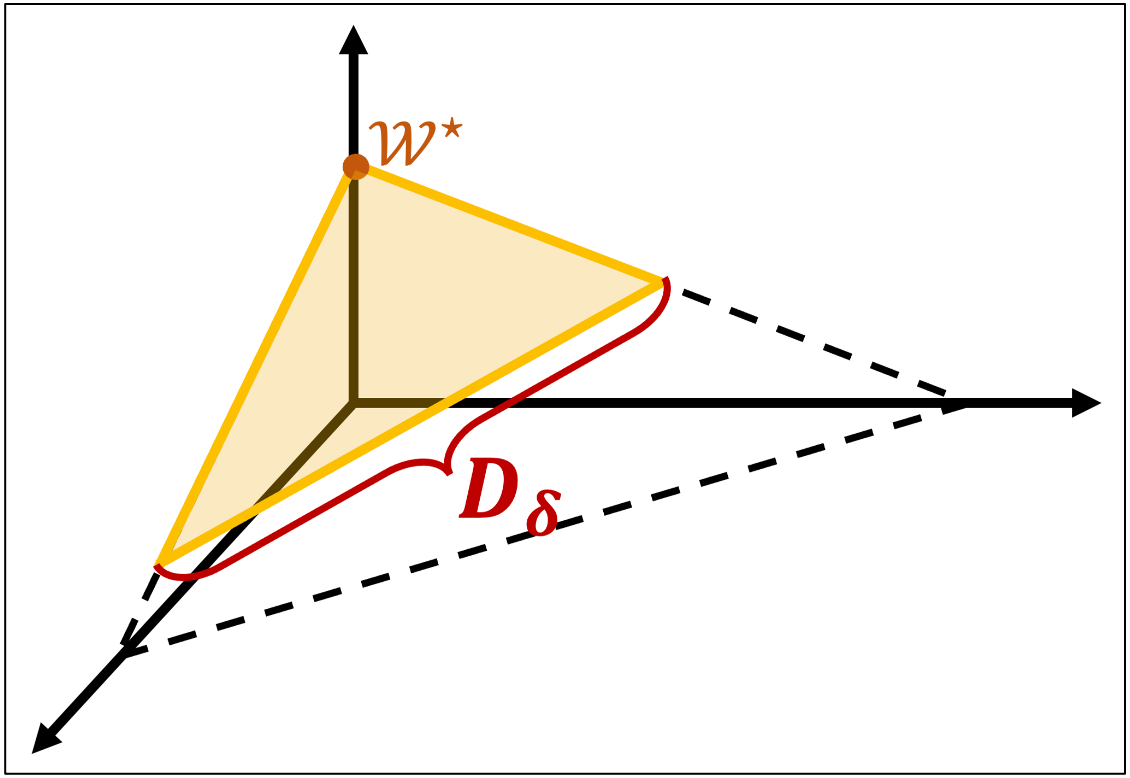
iterations.

Matrix condition number of A: $\kappa := \sigma_{\max}^{+}(A)/\sigma_{\min}^{+}(A)$

"Sublevel-set geometry"

D_δ: Diameter of δ-sublevel set $\mathcal{W}_\delta$

$$D_\delta := \max_{\bar{w}, \widehat{w} \in \mathcal{W}_\delta} \|\bar{w} - \widehat{w}\|$$

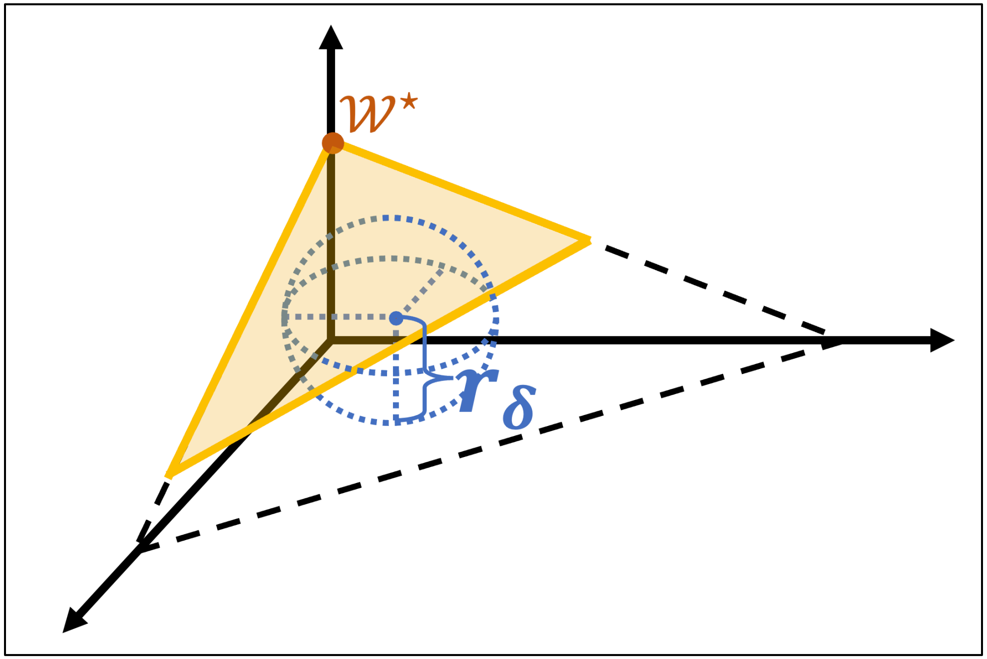


$\boldsymbol{r}_{\delta}$: "Conic Radius" of $\mathcal{W}_{\delta}$

$$\boldsymbol{r}_{\delta} := \max_{r \geq 0,\, w \in \mathcal{W}_{\delta}} r$$

$$\text{s.t.} \quad B_w(r) \subset R_+^{2n}$$

$\boldsymbol{r}_{\delta}$ is the radius of the maximum ball inscribed in R_+^{2n} and centered at a point in $\mathcal{W}_{\delta}$



Target: ε-optimal solution

(x, s) is an $\boldsymbol{\varepsilon}$-optimal solution if:

- distance to each type of constraint is no larger than ε , and
- the duality gap is not larger than ε

(x, s) is an $\boldsymbol{\varepsilon}$-optimal solution if:

- $\text{Dist}(x, \{x | Ax = b\}) \leq \varepsilon$
- $\text{Dist}(x, \mathcal{K}) \leq \varepsilon$
- $\text{Dist}(s, \{s | \exists\ y \ \text{ s. t. } A^\top y + s = c\}) \leq \varepsilon$
- $\text{Dist}(s, \mathcal{K}^\star) \leq \varepsilon$
- $c^\top x - b^\top (AA^\top)^{-1} A(c - s) \leq \varepsilon$

Worst-case complexity of PDHG (under unique optima)

<u>Theorem</u> [Xiong and F 2024]: Suppose $w^\star$ is unique. PDHG computes an $\boldsymbol{\varepsilon}$-optimal solution within

$$\tilde{O}\left(\kappa \cdot \lim_{\delta \to 0} \frac{\boldsymbol{D_\delta}}{\boldsymbol{r_\delta}} \cdot \ln\left(\frac{1}{\varepsilon}\right)\right)$$

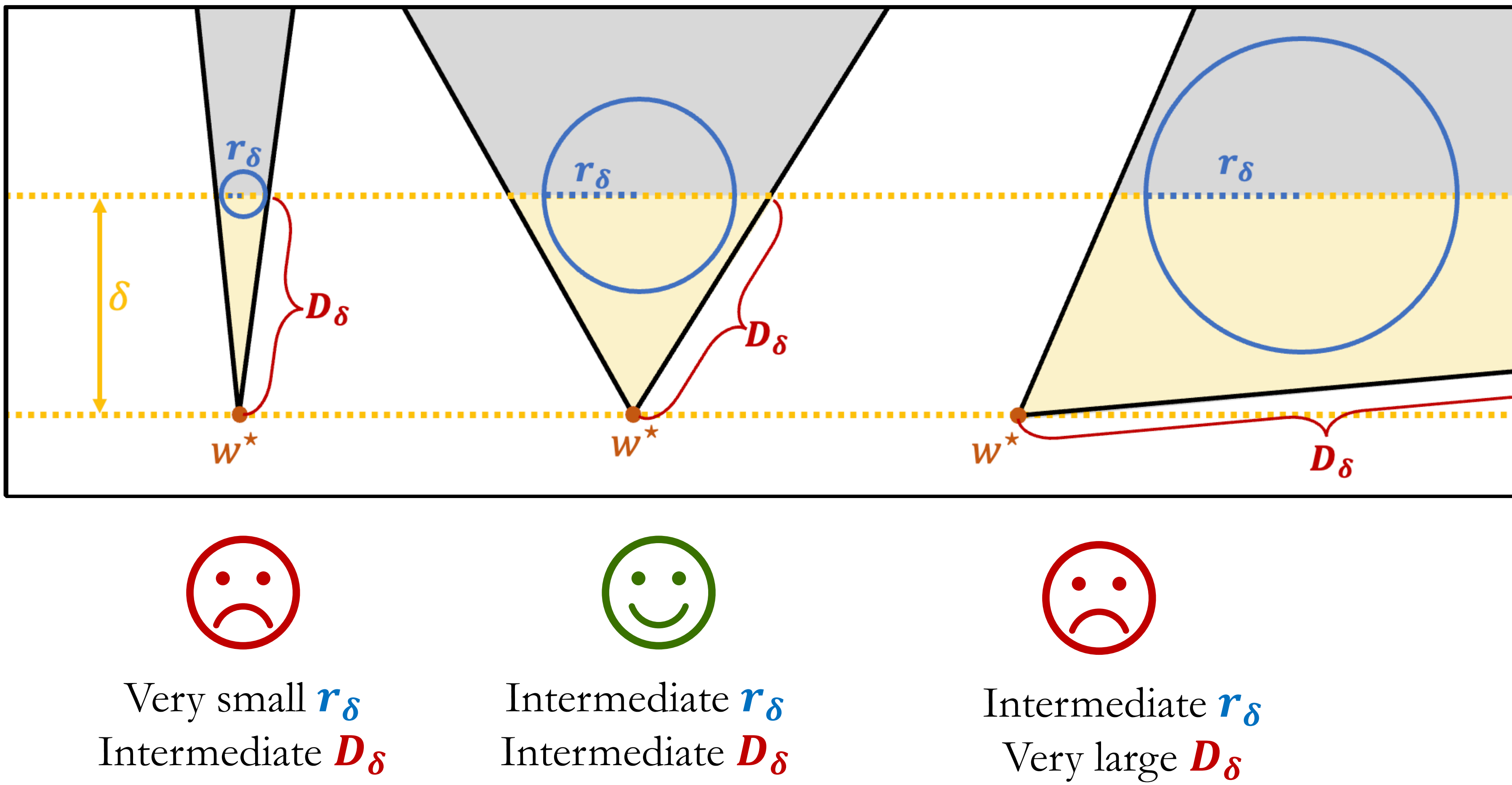
iterations.

Matrix condition number of A:
$\kappa := \sigma_{\max}^{+}(A)/\sigma_{\min}^{+}(A)$

"Sublevel-set geometry"

Local Geometry of and $\lim_{\delta \to 0} \frac{D_\delta}{r_\delta}$ in the case of LP

When $w^\star$ is unique and δ is sufficiently small, $\mathcal{W}_\delta$ is a slice of a pointed cone at $w^\star$.



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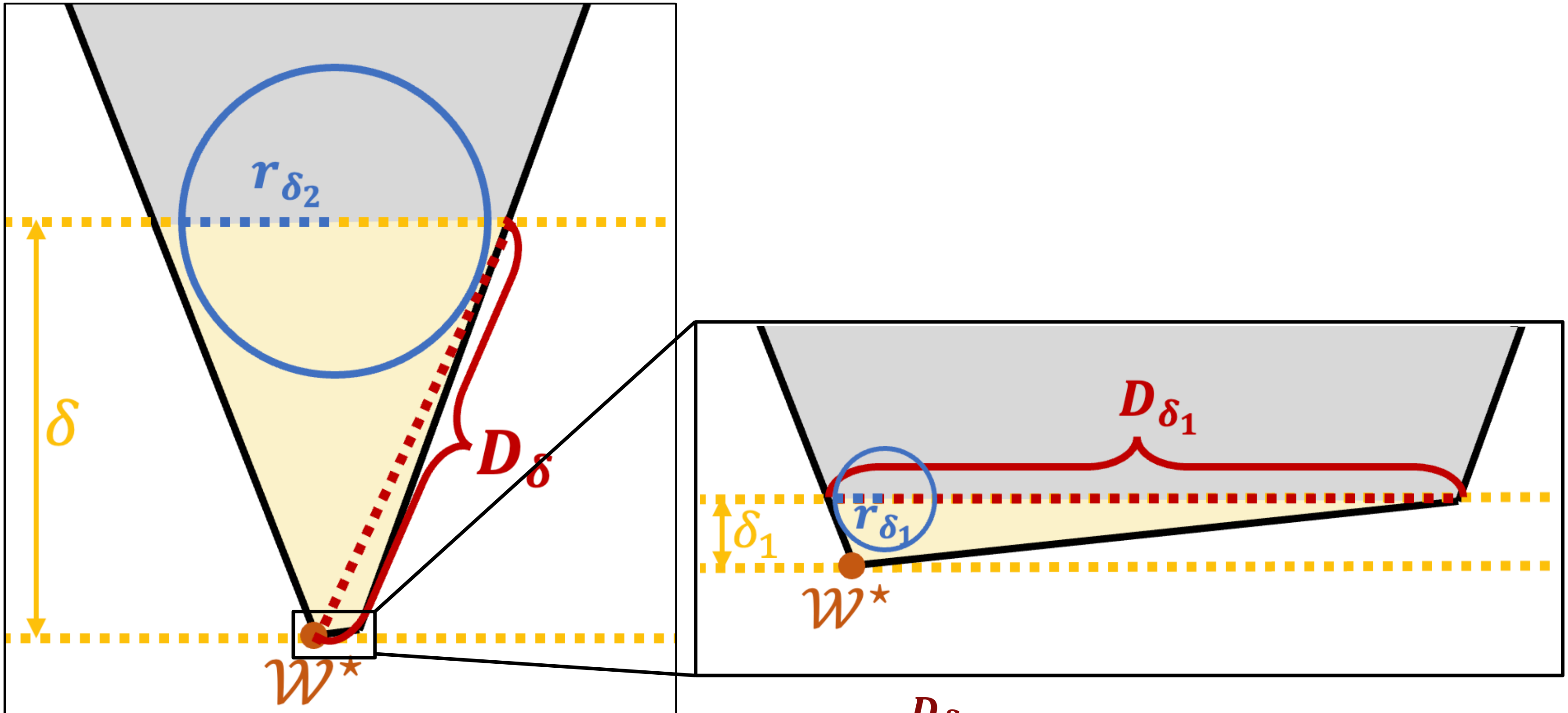
iterations.

Matrix condition number	Local geometric condition

- This iteration bound is "superior" to the Hoffman constant
- For LP, this bound is $\tilde{O}\left(n^{2.5} \cdot \ln\left(\frac{1}{\varepsilon}\right)\right)$ with high probability [Xiong, 2024]
- For LP, this bound has a closed-form expression [Xiong, 2024]

Another example

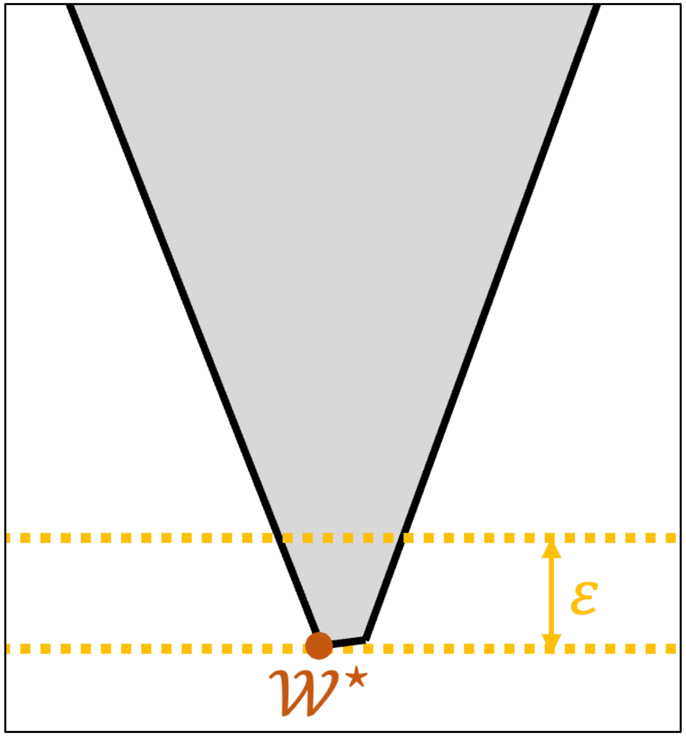


For $\delta_2 > \delta_1$, $\frac{D_{\delta_2}}{r_{\delta_2}}$ becomes smaller/better

$\frac{D_{\delta_1}}{r_{\delta_1}}$ is very large/bad (due to the small r_{δ_1})

Is $\lim_{\delta \to 0} \frac{D_\delta}{r_\delta}$ the only geometric condition?

Suppose we want an ε-optimal solution:



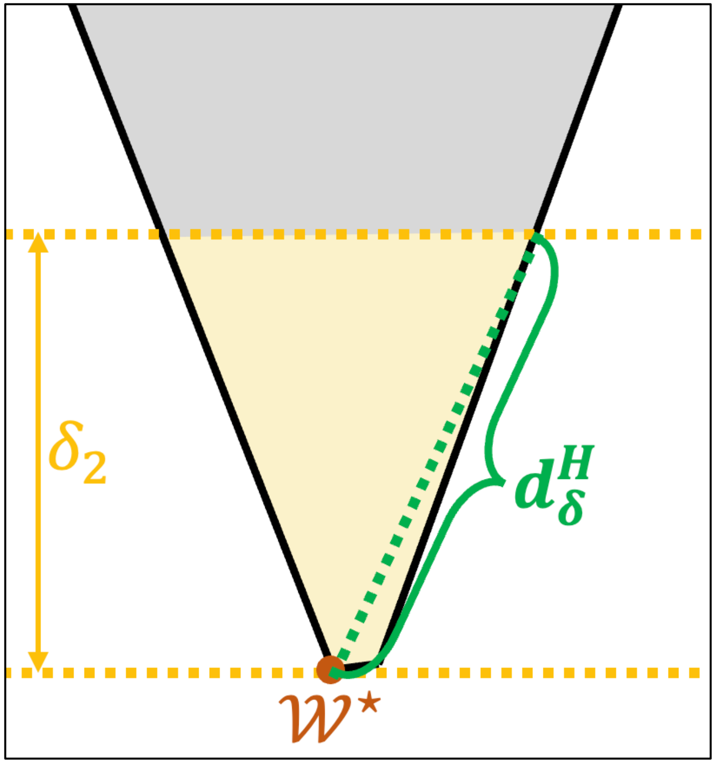
Intuition: The very-local bad geometry should not have a significant impact when the iterates of the algorithm have not yet reached the local neighborhood.

Is $\lim_{\delta \to 0} \frac{D_\delta}{r_\delta}$ the only geometric condition?

We need a third geometric measure to define "being close to $\mathcal{W}^\star$"

$$d_\delta^H := \max_{w \in \mathcal{W}_\delta} \text{Dist}(w, \mathcal{W}^\star)$$

Hausdorff distance from $\mathcal{W}_\delta$ to $\mathcal{W}^\star$



Our General Conic Optimization Computational Guarantee

Theorem [Xiong and F 2024]: The number of PDHG iterations required to compute an ε-optimal solution is upper bounded by:

$$T_\delta := \tilde{O}\left(\kappa \cdot \max\left\{\frac{D_\delta}{r_\delta} \cdot \ln\left(\frac{1}{\varepsilon}\right), \frac{d_\delta^H}{\varepsilon}(1 + \mathrm{Dist}(0, \mathcal{W}^\star))\right\}\right)$$

for each $\delta > 0$.

How good the geometry of $\mathcal{W}_\delta$ is

How close $\mathcal{W}_\delta$ is to $\mathcal{W}^\star$

Remark: This result holds under multiple optima and general conic optimization.

D_δ: Diameter of $\mathcal{W}_\delta$

r_δ: Conic radius of $\mathcal{W}_\delta$

d_δ^H: Hausdorff distance from $\mathcal{W}_\delta$ to $\mathcal{W}^\star$

$\kappa = \sigma_{\max}^+(A)/\sigma_{\min}^+(A)$ (the standard condition number of A)

Our General Conic Optimization Computational Guarantee

Theorem [Xiong and F 2024]: The number of PDHG iterations required to compute an $\boldsymbol{\varepsilon}$-optimal solution is upper bounded by:

$$\tilde{O}\left(\inf_{\delta>0} T_\delta := \kappa \cdot \max\left\{\frac{\boldsymbol{D_\delta}}{\boldsymbol{r_\delta}} \cdot \ln\left(\frac{1}{\varepsilon}\right), \frac{\boldsymbol{d_\delta^H}}{\varepsilon}\left(1 + \mathrm{Dist}(0, \mathcal{W}^\star)\right)\right\}\right)$$

- Linear convergence part
- Note that $\boldsymbol{D_\delta}/\boldsymbol{r_\delta}$ might/not be large when δ is small

- Sublinear convergence part
- Note $\boldsymbol{d_\delta^H}$ is small when δ is small

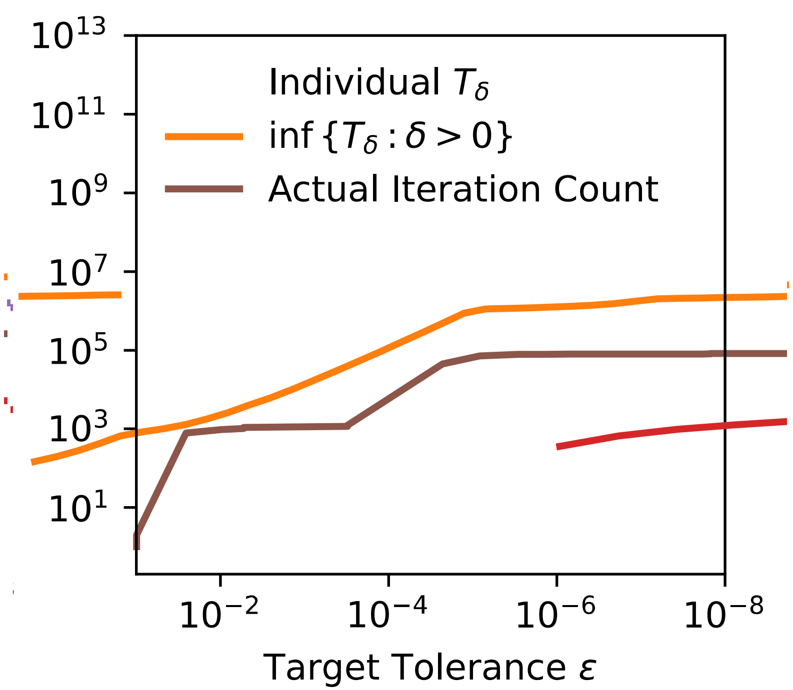
The tightest bound is given by the T_δ that minimizes the bound ☺

$\boldsymbol{D_\delta}$: Diameter of $\mathcal{W}_\delta$
$\boldsymbol{r_\delta}$: Conic radius of $\mathcal{W}_\delta$
$\boldsymbol{d_\delta^H}$: Hausdorff distance between $\mathcal{W}_\delta$ and $\mathcal{W}^\star$
$\boldsymbol{\kappa} = \sigma_{\max}^+(A)/\sigma_{\min}^+(A)$

T_δ and $\inf\{ T_\delta : \ \delta > 0\}$

$$\min_{x=(x_1,x_2,x_3)} \quad 0.20001 \cdot x_1 + x_2 + 1.0001 \cdot x_3$$
$$\text{s.t.} \quad -10x_1 + x_2 + x_3 = 1$$
$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0$$

- Different δ yield different bounds T_δ
- For each tolerance ε, $\inf\{T_\delta : \delta > 0\}$ provides the best bound
- In the beginning, PDHG converges sublinearly
- Since $\lim_{\delta \searrow 0} \frac{D_\delta}{r_\delta} < \infty$ and $\lim_{\delta \searrow 0} d_\delta^H = 0$, we eventually obtain linear convergence
- Validated by actual iteration count



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Corollary: If there exists $\delta > 0$ whose δ-sublevel set satisfies:

- $\frac{D_\delta}{r_\delta}$ is small ($\mathcal{W}_\delta$ has good geometry), and
- d_δ^H is small ($\mathcal{W}_\delta$ is close to $\mathcal{W}^\star$),

then PDHG will converge faster. (If not, rPDHG might be slow…)

Q: Can we possibly improve $\frac{D_\delta}{r_\delta}$ and d_δ^H ?

A: Yes, we can do so by using Hessian Rescaling

D_δ: Diameter of …. r_δ: Conic radius of … d_δ^H : Hausdorff distance…

Using IPM theory to develop practical computational speed-ups of PDHG

Recall the two slides on IPMs

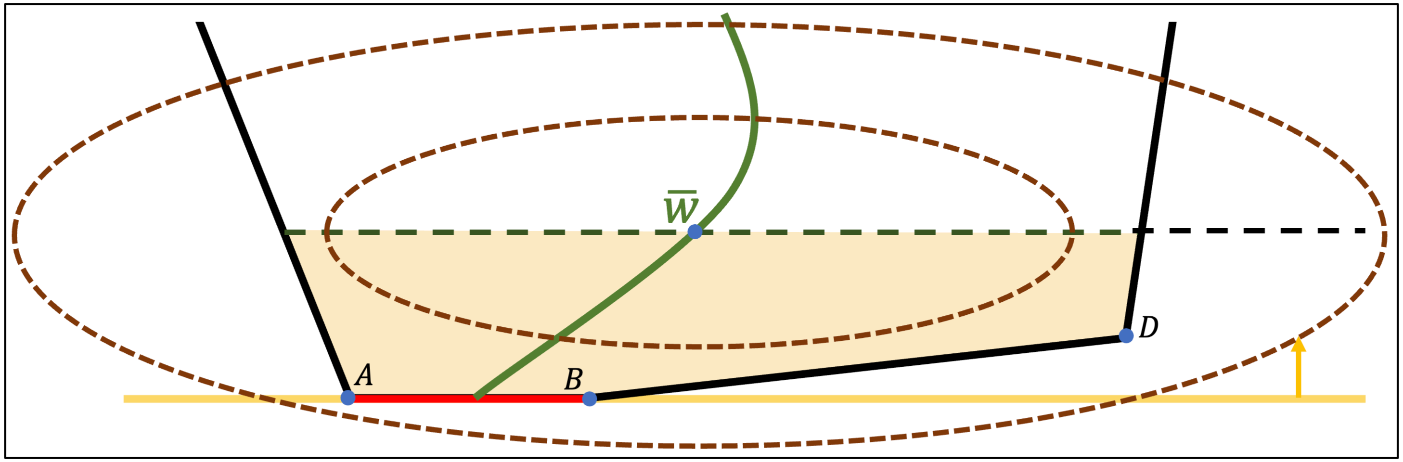
Central-Path ellipsoids have remarkable properties

Central-path solutions are:

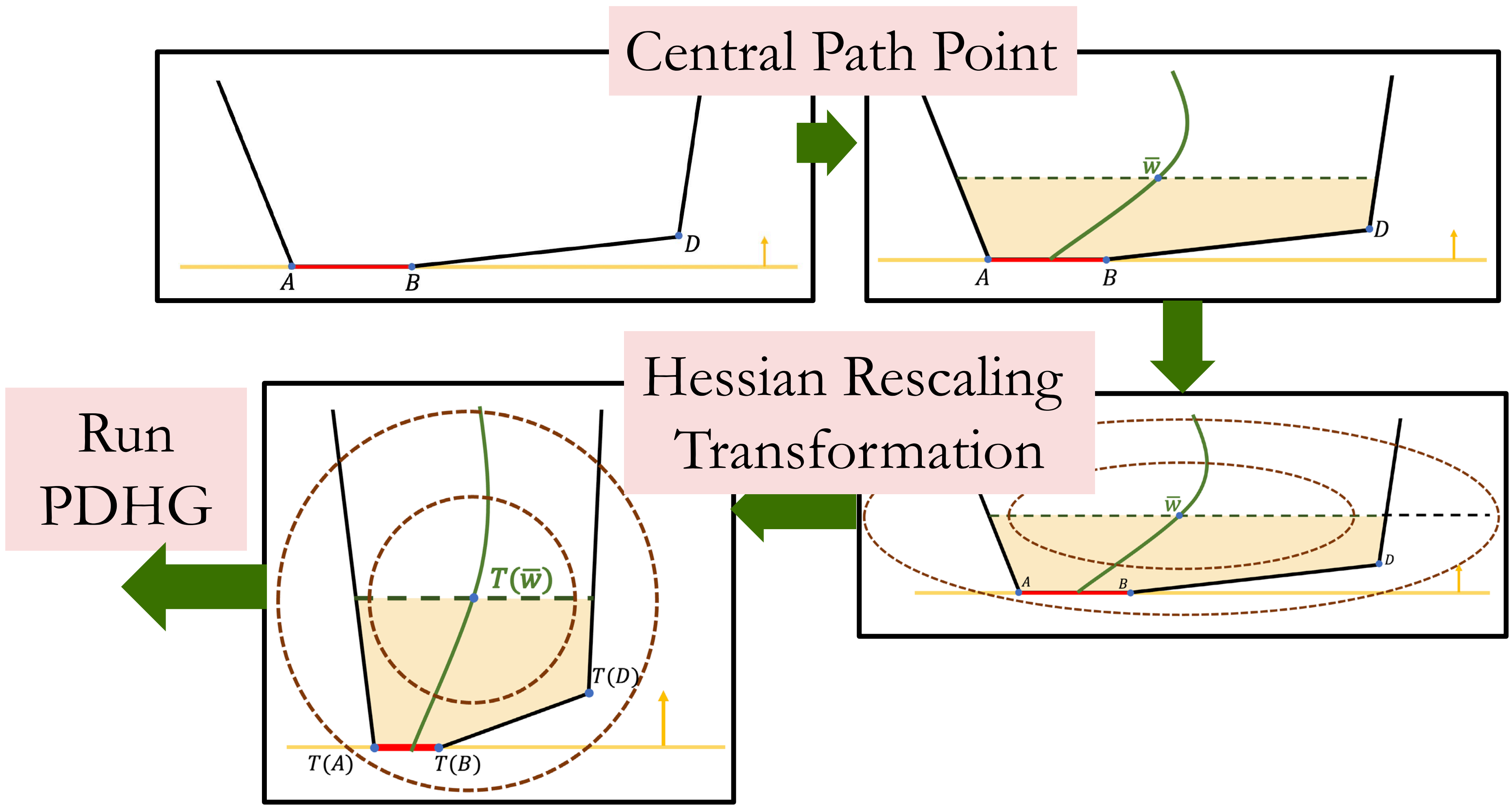
$$x_\mu := \arg\min_x \; c^\top x + \mu \cdot F(x) \quad \text{s.t. } Ax = b,\; x \in \mathcal{K}$$

$$y_\mu, s_\mu := \arg\max_{y,s} \; b^\top y - \mu \cdot F^*(s) \quad \text{s.t. } A^\top y + s = c,\; s \in \mathcal{K}^*$$

Example for LP: $F(x) \coloneqq -\sum_{i=1}^{n} \ln(x_i)\,,\;\; \vartheta_F = n$



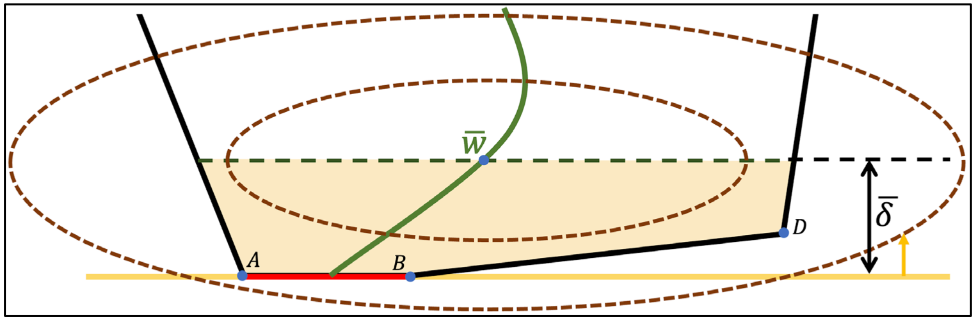
This suggests the following strategy:



Transformation based on a central-path solution

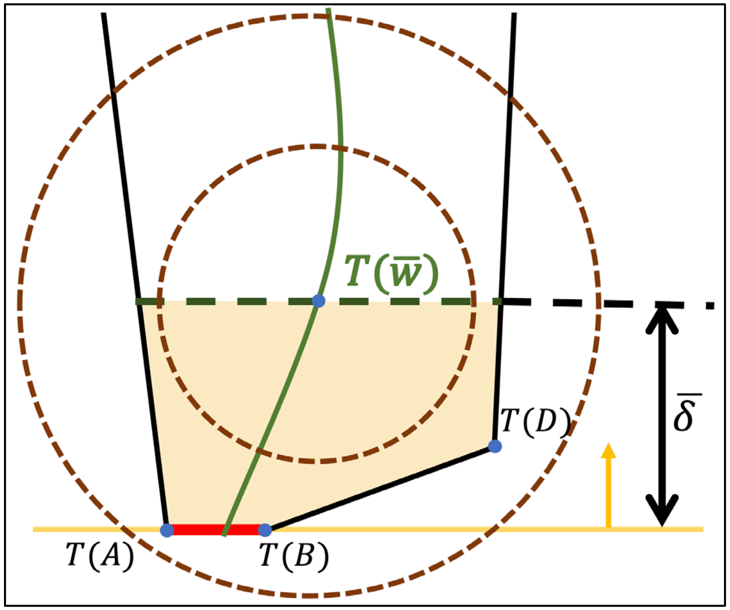
Let $H := \mu \cdot \nabla^2 F(x_\mu)$, then

- $\bar{\boldsymbol{\delta}} := \boldsymbol{n\mu} = \mathrm{Gap}(\bar{w})$
- $\boldsymbol{D_{\bar{\delta}} \leq 2n \cdot \sigma_{\min}^{+}(H)^{-1/2}}$
- $\boldsymbol{r_{\bar{\delta}} \geq \sigma_{\max}^{+}(H)^{-1/2}}$
- $\boldsymbol{d_{\bar{\delta}}^{H} \leq 2n \cdot \sigma_{\min}^{+}(H)^{-1/2}}$

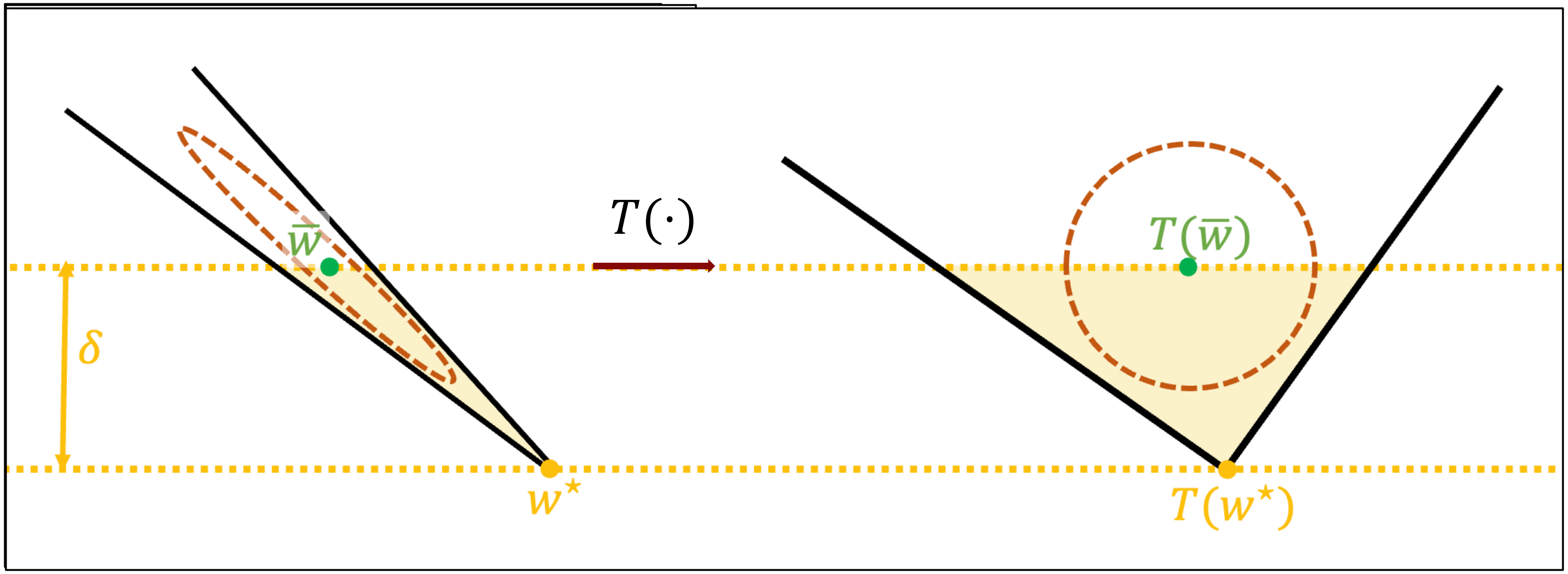


After a rescaling transformation (that maps the local-norm ball to a Euclidean norm ball) we have:

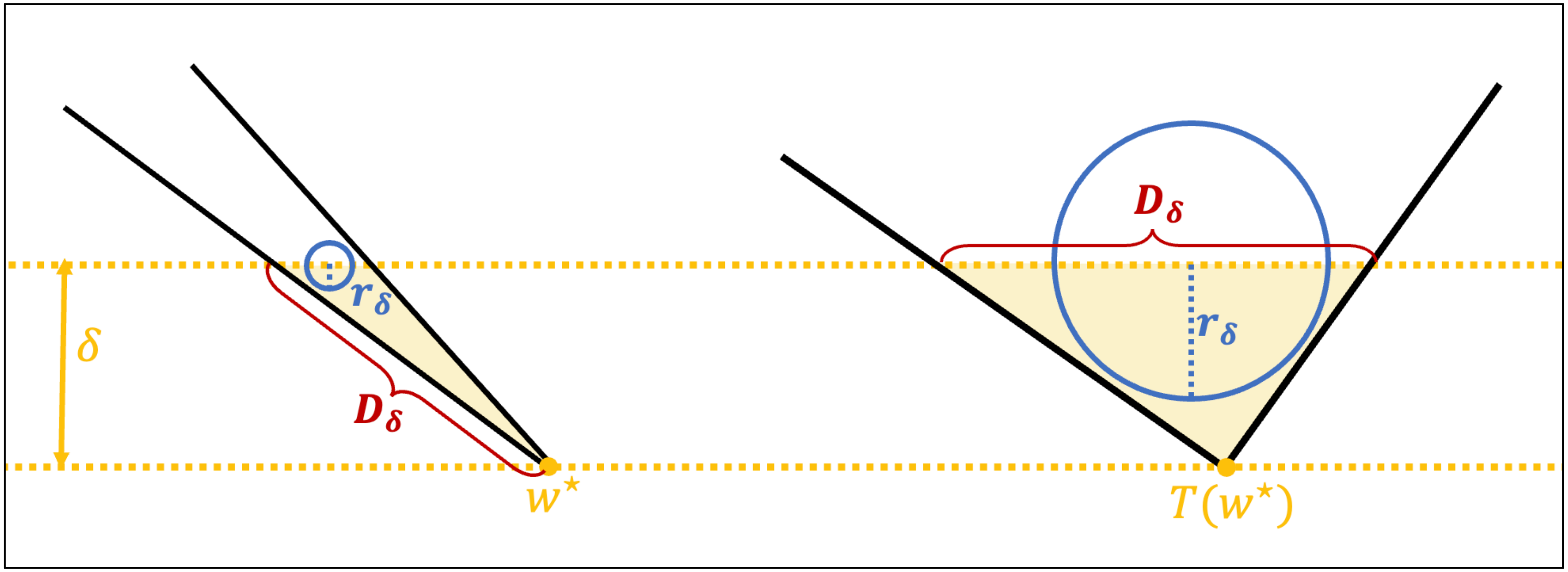
- $\boldsymbol{D_{\bar{\delta}} \leq 2\sqrt{n} \cdot \sqrt{\bar{\delta}}}$
- $\boldsymbol{r_{\bar{\delta}} \geq \sqrt{\frac{1}{n}} \cdot \sqrt{\bar{\delta}}}$
- $\boldsymbol{d_{\bar{\delta}}^{H} \leq 2\sqrt{n} \cdot \sqrt{\bar{\delta}}}$
- $\boldsymbol{\frac{D_{\bar{\delta}}}{r_{\bar{\delta}}} \leq 2n}$
- $d_{\bar{\delta}}^{H}$ is small if $\bar{\boldsymbol{\delta}}$ is small
- "Very nice theory"



How does Hessian rescaling improve the geometry?



How Hessian rescaling improves the geometry?



Very Small r_δ
Intermediate D_δ
Large $\frac{D_\delta}{r_\delta}$

Intermediate r_δ
Intermediate D_δ
Small $\frac{D_\delta}{r_\delta}$

Complexity guarantee after central-path Hessian rescaling

Suppose we do the following:

1. rescaling transformation using a central-path solution **with duality gap** $\bar{\delta}$
2. row transformation to try to decrease κ closer to $\kappa = 1$

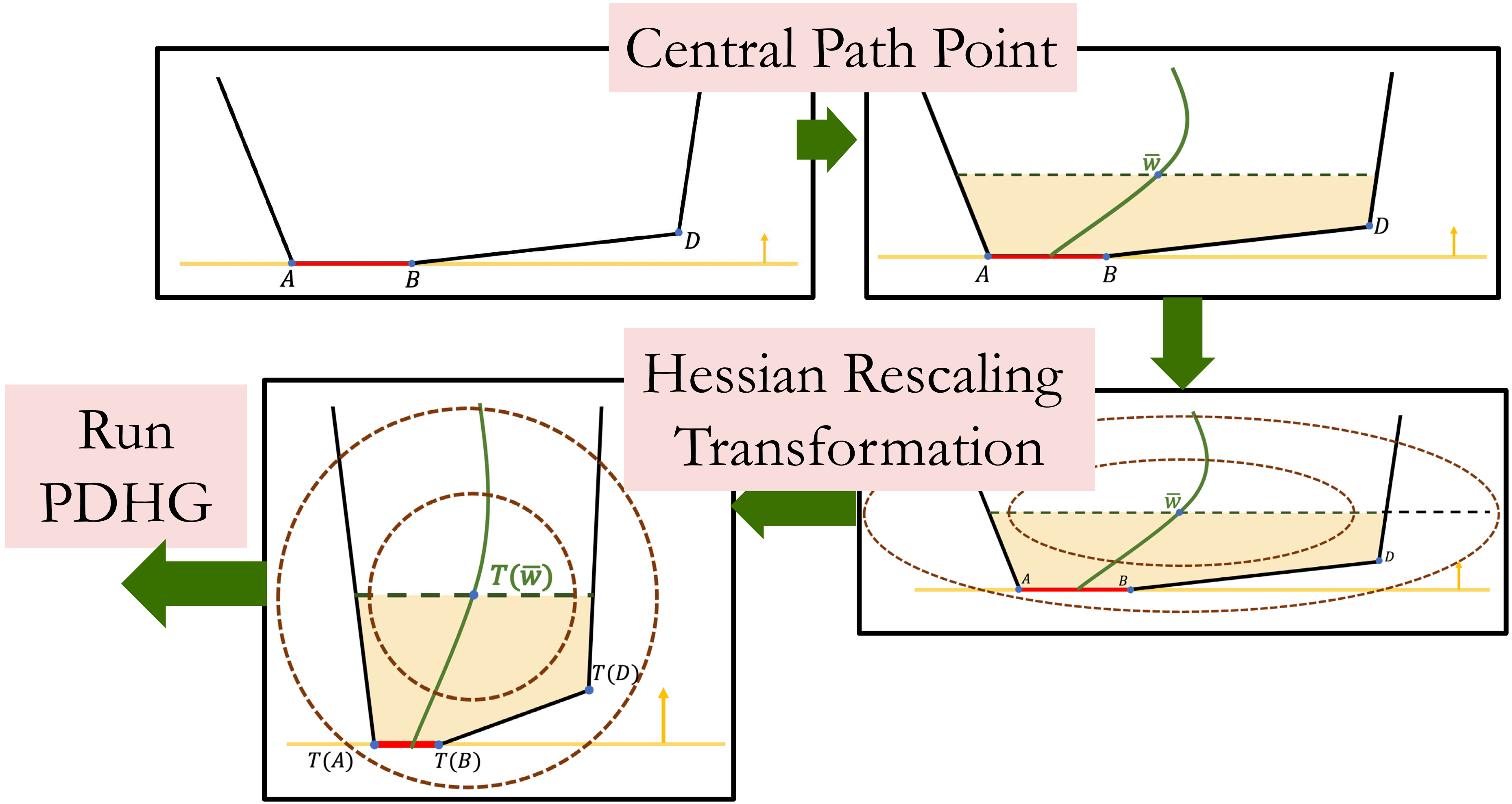
Then the number of PDHG iterations required to compute an ε-optimal solution of the original problem is upper bounded by:

$$\tilde{O}\left(n \cdot \left(\ln\left(\frac{1}{\varepsilon}\right) + \frac{D_{\bar{\delta}} + \bar{\delta}}{\varepsilon}\right)\right)$$

We have replaced $\frac{D_\delta}{r_\delta}$ by n

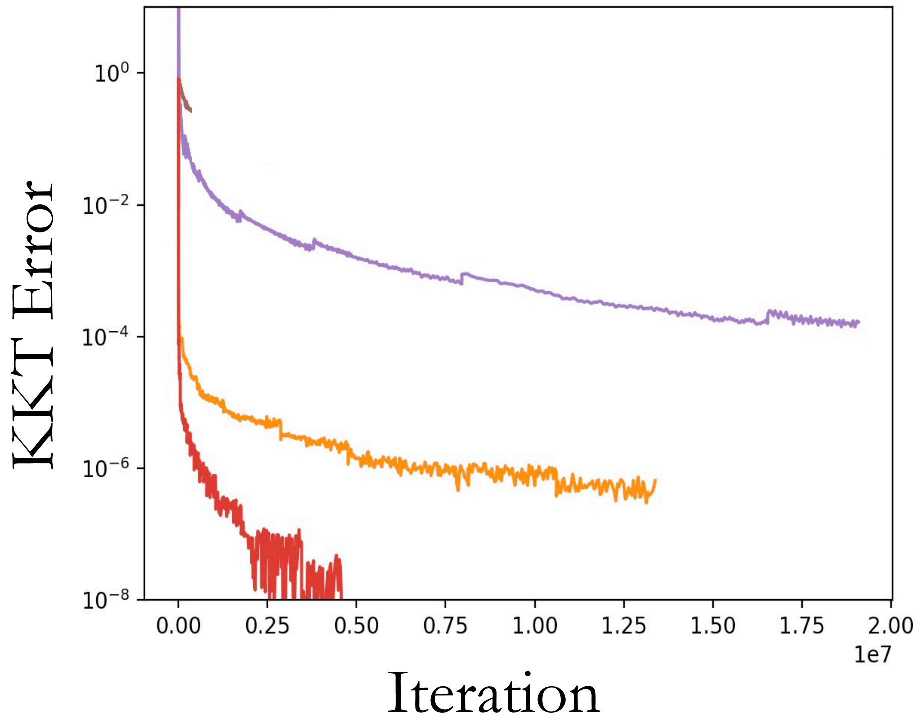
The smaller $\bar{\delta}$ is, the faster the convergence

Summary



Central Path Point
Hessian Rescaling
Transformation
Run
PDHG
A
B
D
$\bar{w}$
$T(\bar{w})$
$T(A)$
$T(B)$
$T(D)$

“Proof of concept” applied to problem instance **bmoipr2**



Here we pre-multiply A by $(AA^\top)^{-1/2}$ to yield $\kappa = 1$

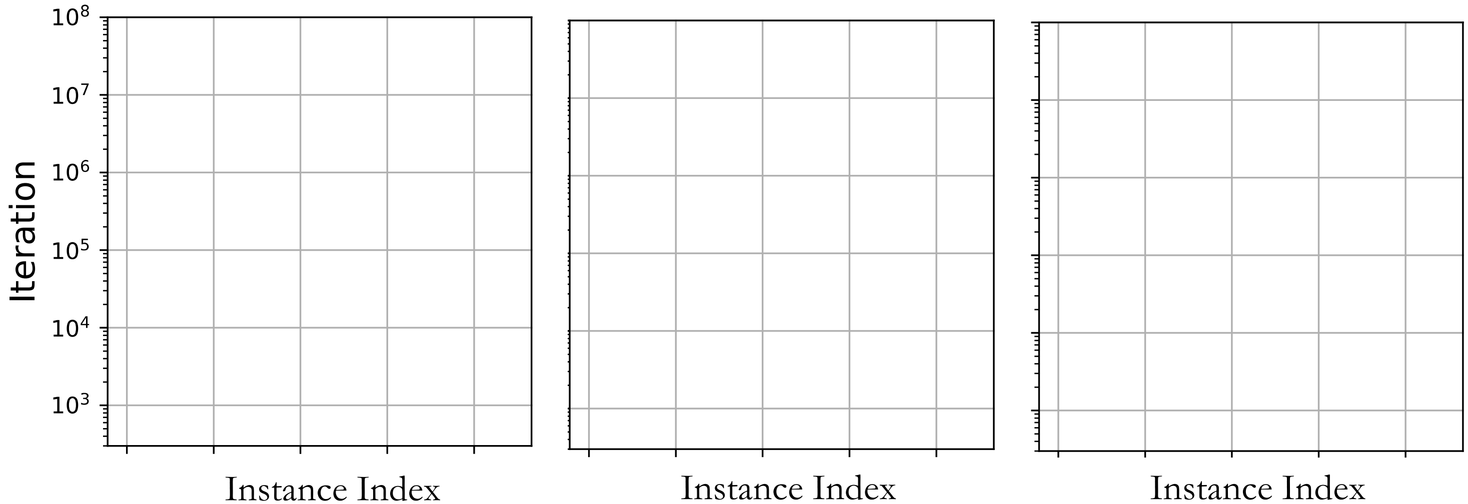
PDHG-EasyColumn
Column rescaling to **normalize** L_∞ column norms
PDHG-Central δ**=0.1**
Column rescaling using **central-path solution** with KKT error **0.1**
PDHG-Central δ**=0.01**
Similar as above, with KKT error **0.01**

- Here the central path solutions were computed by MOSEK
- This is an admittedly “unfair” comparison, but it validates the potential of the overall approach…

Proof of concept, continued

PDHG-EasyColumn
Column rescaling to **normalize L_∞** column norms
PDHG-Central δ=0.1
Column rescaling using **central-path solution** with KKT error **0.1**
PDHG-Central δ=0.01
Column rescaling using **central-path solution** with KKT error **0.01**
Note: we pre-multiply A by $(AA^\top)^{-1/2}$ to yield $\kappa = 1$ for all rescalings

PDHG iterations needed to compute a solution with KKT error **10^{-8}**

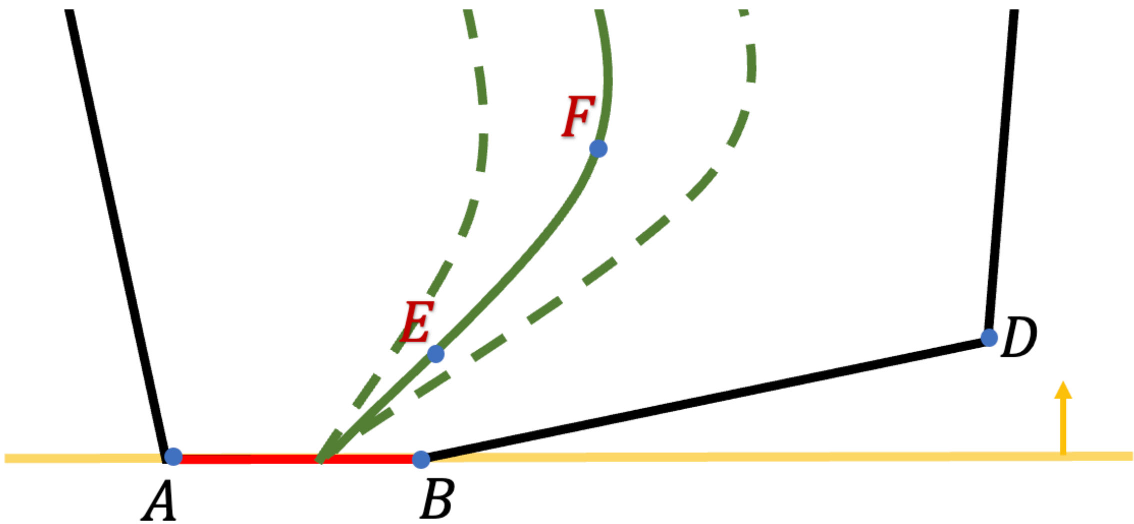


222 LP relaxation instances from the MIPLIB 2017 dataset

PDHG-AHR (“Adaptive Hessian Rescaling”) and Computational Experiments

Main Strategy

Main strategy: use a "CG-IPM" to compute a low-accuracy central-path solution to obtain a good rescaling, and then use PDHG

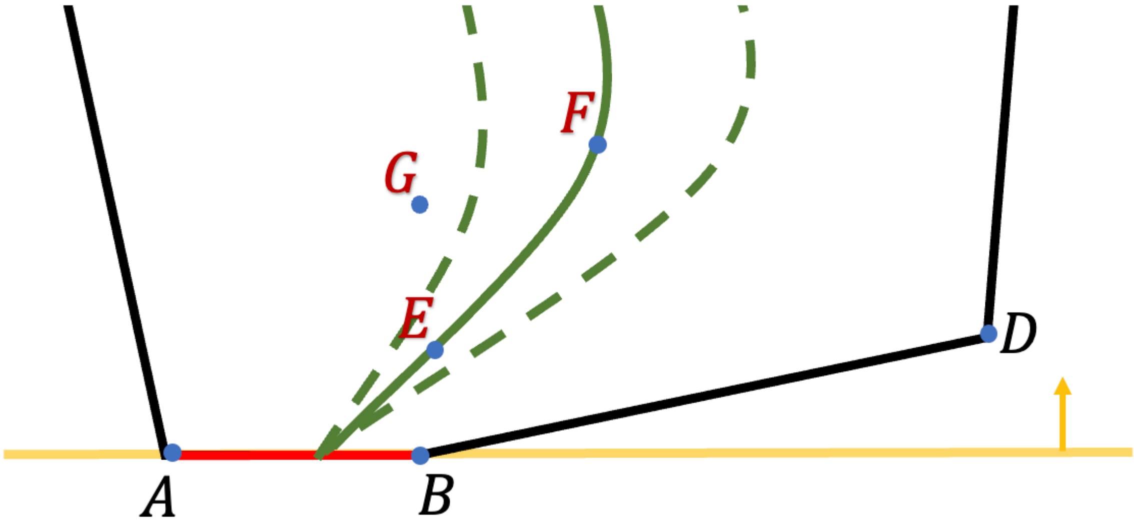


Main Strategy, continued

Main strategy: use a "CG-IPM" to compute a low-accuracy central-path solution to obtain a good rescaling, and then use PDHG

- **We use a first-order method to compute the Newton steps of a "central path" solution**
  - we use a conjugate-gradient-method-based IPM (**CG-IPM**)
  - otherwise, implementation exactly follows Nocedal and Wright *Numerical Optimization* (2006)
  - but we solve the normal equations using conjugate gradient method
- **We employ only diagonal row rescaling to try to improve $\boldsymbol{\kappa}$**
  - Column rescaling uses the central-path Hessian of w_{int} followed by PDLP's rescaling (Ruiz rescaling and Pock-Chambolle rescaling), which we call the "$\boldsymbol{w_{int}}$-rescaled problem"

Using Adaptive Hessian Rescaling



Motivating concepts of Adaptive Hessian Rescaling:

- Adaptively balance the cost of computing the rescaling (CG-IPM) with the savings from running PDHG on the rescaled instance
- Try to identify a "good-enough" rescaling <u>as early as possible</u> and use the good-enough rescaling

Computational Experiments with **PDHG-AHR**

We compare:

1. **PDHG-AHR** : PDHG with Adaptive central-path Hessian Rescaling (using CG-IPM for the central-path computations)
2. **PDHG(RuizPC)** : use heuristic Ruiz rescaling on ***A***, followed by Pock-Chambolle rescaling. (This is the same as the rescaling used in PDLP.)
3. **IPM** : a home-grown standard primal-dual predictor-corrector interior point method, straight from Nocedal and Wright *Numerical Optimization* (2006).

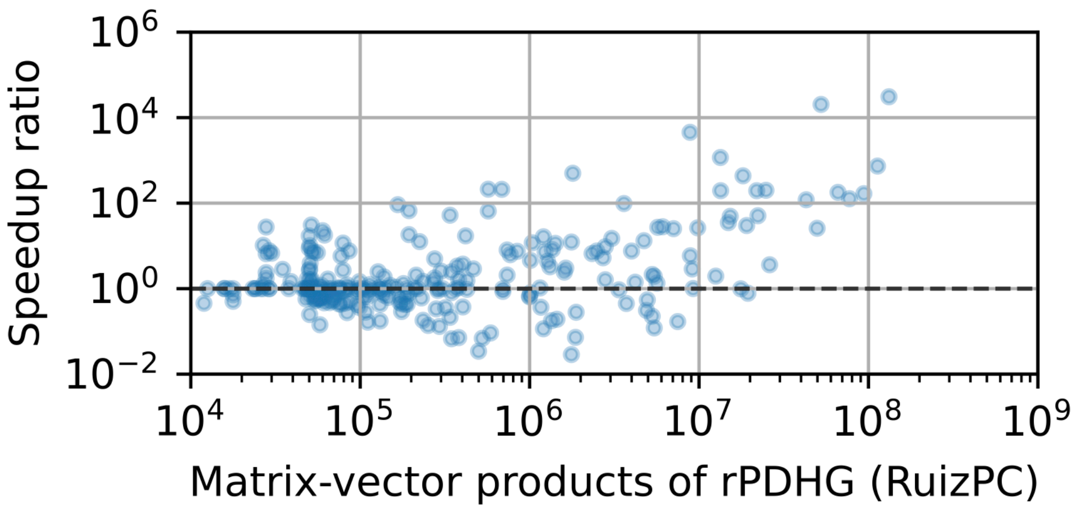
We performed tests on all the LP relaxations from MIPLIB 2017 dataset that are:

- large enough ($m \times n > 10^6$)
- but not too large for the IPM (number of non-zeros $< 10^5$)
- This yielded 413 instances in total

Computational Comparison: **PDHG-AHR** and **PDHG(RuizPC)**

Speedup Ratio:

$$\frac{\text{Matrix-vector products } \mathbf{PDHG(RuizPC)}}{\text{Matrix-vector products } \mathbf{PDHG\text{-}AHR}}$$



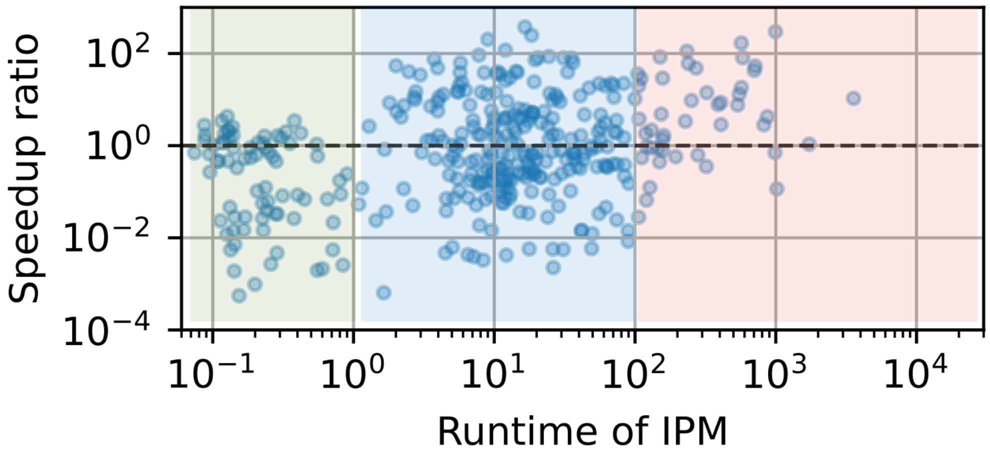
		Fraction solved by **PDHG-AHR**	
		S	**N**
Fraction solved by **PDHG (RuizPC)**	**S**	**87.7%**	**1.7%**
	N	**7.5%**	**3.1%**

- In general, the harder the problem is for **PDHG(RuizPC)**, the larger the speedups from using **PDHG-AHR**
- **PDHG-AHR** solves about 95.2% of problems and is more reliable than **PDHG(RuizPC)** (solves about 89.4% of problems)

Computational Comparison: **PDHG-AHR** and **IPM**

Speedup Ratio:

$$\frac{\text{Runtime } \textbf{IPM}}{\text{Runtime } \textbf{PDHG−AHR}}$$



		Fraction solved by **PDHG−AHR**	
		S	**N**
Fraction solved by **IPM**	**S**	**93.0%**	**1.5%**
	N	**2.2%**	**3.4%**

- Generally speaking, the harder the problem is for **IPM**, the larger the speedups from using **PDHG-AHR**.
- **PDHG-AHR** is also (slightly) more reliable than **IPM**

Recap, Takeaways, and Remarks

Recap and takeaways:

- The convergence rate of PDHG on CLP is related to the geometry of primal-dual sublevel sets measured with $\boldsymbol{D_\delta}$, $\boldsymbol{r_\delta}$, $\boldsymbol{d_\delta^H}$
- Rescaling using a central-path solution can improve the geometry of the primal-dual sublevel sets
- Our strategy: **PDHG-AHR** uses a "CG-IPM" to compute a low-accuracy central-path solution to obtain a good rescaling, and then use PDHG
- FOMs can compete and outperform IPMs

Remarks:

- Our results relied only on PDHG's average iterate convergence and non-expansiveness properties. Similar results might also hold for other FOMs, in particular ADMM, EGM, …

Thank you!