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ONLY AVAILABLE IN ELECTRONIC FORM

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Online Companion for
"Regret in the Newsvendor Model with Partial Information"

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Proof of Proposition 1

(a) The function $g(x, z; y) = \min\{x, z\} - \min\{x, y\}$ is concave in (x, z) on the domain $x \geq y$ and $z \geq y$. Since concavity is preserved under nonnegative weighted integration, $\int_{\Omega:x\geq y} g(x, z; y)dF(x)$ is concave in z when $z \geq y$, and so is $\int_{\Omega} g(x, z; y)dF(x)$ because $g(x, z; y) = 0$ when $x \leq y$ and $z \geq y$. Subtracting $\beta(z-y)$ maintains the concavity property. Since concavity is preserved under maximization over a convex set (Boyd and Vandenberghe 2004), and given that $\mathcal{D}$ is convex, $G(z; y) = \max_{F\in\mathcal{D}} \int_{\Omega} g(x, z; y)dF(x) - \beta(z-y)$ is concave in z when $z \geq y$. The case where $z \leq y$ is analyzed in a similar way.

However, $G(z; y)$ is not necessarily concave on $[0, \infty)$. For instance, consider $\mathcal{D}$ as the set of all distributions with support Ω. Then, $G(z; y)$ equals $(1-\beta)(z-y)$ when $z \geq y$, and $-\beta(z-y)$ otherwise, and is therefore piecewise linear convex.

(b) Let $G^-(y) = \max_{z\in[0,y]} G(z; y)$ and $G^+(y) = \max_{z\in[y,\infty)} G(z; y)$. For a fixed demand distribution $F \in \mathcal{D}$, $-\Pi_F(y)$ is a convex function of y. Because convexity is preserved under maximization (Boyd and Vandenberghe 2004), $G^-(y)$ and $G^+(y)$ are convex functions of y, as well as $\rho(y)$ (maximum of two convex functions). The function $G^-(y)$ is convex nondecreasing since it is nonnegative and $\lim_{y\to 0} G^-(y) = 0$; similarly, $G^+(y)$ is nonincreasing since it is nonnegative and $\lim_{y\to\infty} G^+(y) = 0$. Therefore, there exists some y^* such that $G^-(y^*) = G^+(y^*)$, and this quantity y^* minimizes $\rho(y)$. ∎

Proof of Theorem 1

Consider problem (4) with only the normalization constraint $\int_A^B dF(x) = 1$. From Proposition 1, two cases need to be considered: when $y \leq z$ and when $y \geq z$.

When $y \leq z$, the distribution that solves (4) is a unit impulse at z and leads to a regret of $(1-\beta)(z-y)$. Maximizing the regret over all feasible $z \in [y, B]$, we obtain a regret equal to $(1-\beta)(B-y)$.

When $y \geq z$, the worst-case distribution is also a unit impulse at z and leads to a regret of $-\beta(z-y)$. The maximum regret, taken over all feasible $z \in [A, y]$, equals $-\beta(A-y)$.

From Proposition 1, the optimal order quantity y equates the two maximum regrets. ∎

Proof of Theorem 2

When the mean is known, problem (5) can be formulated as the following semi-infinite linear optimization problem:

$$\begin{array}{lc} \min_{\alpha_0,\alpha_1} & \alpha_0 + \alpha_1 \mu, \\ \text{s.t.} & \alpha_0 + \alpha_1 x \geq \min\{x, z\} - \min\{x, y\}, \quad \forall x \geq 0. \end{array} \tag{12}$$

(a) If $z \geq y$, a dual feasible function is any straight line with ordinate α_0 and slope α_1 that is nonnegative for all $x \geq 0$, lies above the line $x - y$ between y and z, and above the line $z - y$ for all $x \geq z$. There are two possible optimal solutions: either the straight line that goes through the origin and $(z, z-y)$, or the horizontal line at $z - y$. In the first case, $\alpha_0 = 0$, $\alpha_1 = (z-y)/z$, and the optimal value of (12) is equal to $(z-y)\mu/z$. In the second case, $\alpha_0 = z - y$, $\alpha_1 = 0$, and the optimal value of (12) is equal to $(z-y)$. Therefore, the first case is optimal if and only if $\mu \leq z$.

(a.1) If $\mu \leq z$, the regret is equal to $(z-y)(\mu/z - \beta)$, which is concave in z. The regret, optimized over all possible values of $z \geq \mu$, is maximized at $z^* = \sqrt{y\mu/\beta}$, if $z^* \geq \mu$, and at μ if $z^* \leq \mu$. Replacing z with its optimal value simplifies the regret to

$$\begin{cases} (\mu - y)(1-\beta), & \text{if } y \leq \beta\mu, \\ \beta y (\sqrt{\frac{\mu}{\beta y}} - 1)^2, & \text{if } y \geq \beta\mu. \end{cases}$$

(a.2) If $z \leq \mu$, the regret is equal to $(z-y)(1-\beta)$. The maximum regret, when it is optimized over $z \leq \mu$, is attained at $z = \mu$ and equal to $(\mu - y)(1-\beta)$.

(b) On the other hand, if $y \geq z$, the right hand side of the constraints is nonincreasing. The optimal solution of (12) is $\alpha_0 = \alpha_1 = 0$ and is associated with a regret of $-\beta(z-y)$. The maximum regret, optimized over all values of $z \leq y$, is equal to βy.

From Proposition 1, the quantity y balances the opportunity cost from ordering too much with the opportunity cost from ordering too little. If $y \leq \beta\mu$, y^* minimizes the maximum of the two following convex functions:

$$\min_{y\geq 0} \max\{\beta y, (1-\beta)(\mu - y)\}.$$

In this case, it is optimal to order $y^* = (1-\beta)\mu$. The condition $y^* \leq \beta\mu$ reduces to $\beta \geq 1/2$.

If $\beta\mu \leq y$, y^* minimizes the following expression:

$$\min_{y\geq 0} \max\{\beta y, \beta y(\sqrt{\frac{\mu}{\beta y}} - 1)^2\}.$$

The minimum is attained at $y^* = \mu/(4\beta)$. The condition $\beta\mu \leq y^*$ translates into $1/2 \geq \beta$.

∎

Proof of Theorem 3

Consider problem (4) with the normalization constraint $\int_0^\infty dF(x) = 1$, the constraint on the mean $\int_0^\infty x dF(x) = \mu$, and constraint on the median $\int_0^m dF(x) = 1/2$. By strong duality, this problem is equivalent to the following semi-infinite linear optimization problem:

$$\begin{array}{lc} \min_{\alpha_0,\alpha_1,\alpha_2} & \alpha_0 + \alpha_1\mu + \alpha_2/2, \\ \text{s.t.} & \alpha_0 + \alpha_1 x + \alpha_2 1_{\{x\leq m\}} \geq \min\{x, z\} - \min\{x, y\}, \quad \forall x \geq 0, \end{array} \tag{13}$$

where $1_{\{x\leq m\}}$ equals 1 whenever $x \leq m$ and zero otherwise. Because the median is known to be m, $z \leq m$ if and only if $\beta \geq 1/2$. As a result, it is optimal to order $y \leq m$ if and only if $\beta \geq 1/2$.

Case 1: $\beta \geq 1/2$**.**

(a) If $z \geq y$, a dual feasible solution of (13) is a straight line with ordinate α_0, slope α_1, and possibly vertically shifted at m by an amount α_2, that is nonnegative for all $x \geq 0$, lies above the line $x - y$ between y and z, and above the line $z - y$ for all $z \leq x$. There are two candidate optimal solutions: either a straight line going from $(0, 0)$ to $(z, z - y)$ and vertically shifted by $(z-y)(1-m/z)$ at m, or a horizontal line at $z-y$. In the first solution, $\alpha_0 = 0$, $\alpha_1 = (z - y)/z$, and $\alpha_2 = (z - y)(1 - m/z)$, and the objective value of (13) is equal to $(z - y)(\mu/z + (1 - m/z)/2)$. In the second solution, $\alpha_0 = z - y$, $\alpha_1 = \alpha_2 = 0$, and the objective value of (13) is equal to $(z - y)$. Therefore, the first solution is optimal if and only if $z \geq 2\mu - m$.

(a.1) If $z \geq 2\mu - m$, the regret is equal to $(z - y)(1/2 - \beta + (\mu - m/2)/z)$, which is a concave function of z. The regret, optimized over $z \in [\max\{y, 2\mu - m\}, m]$, attains its maximum at $z^* = \sqrt{y(2\mu - m)/(2\beta - 1)}$ if $z^* \in [\max\{y, 2\mu - m\}, m]$, at m is $z^* > m$, and at $\max\{y, 2\mu - m\}$ if $z^* < \max\{y, 2\mu - m\}$. Replacing z with its optimal value gives rise to the following regrets when $\mu \leq m \leq \mu/\beta$:

$$\begin{cases} (2\mu - m - y)(1-\beta), & \text{if } y \leq (2\mu - m)(2\beta - 1), \\ y(\beta - \frac{1}{2})(\sqrt{\frac{2\mu - m}{y(2\beta-1)}} - 1)^2, & \text{if } (2\mu - m)(2\beta - 1) \leq y \leq m^2(2\beta - 1)/(2\mu - m), \\ (m - y)(\frac{\mu}{m} - \beta), & \text{if } y \geq m^2(2\beta - 1)/(2\mu - m), \end{cases}$$

and when $m \geq \mu/\beta$, the regret equals

$$\begin{cases} (2\mu - m - y)(1-\beta), & \text{if } y \leq (2\mu - m)(2\beta - 1), \\ y(\beta - \frac{1}{2})(\sqrt{\frac{2\mu - m}{y(2\beta-1)}} - 1)^2, & \text{if } (2\mu - m)(2\beta - 1) \leq y \leq (2\mu - m)/(2\beta - 1), \\ 0, & \text{if } y \geq (2\mu - m)/(2\beta - 1). \end{cases}$$

(a.2) If $z \leq 2\mu - m$, the regret equals $(1-\beta)(z - y)$ and is maximized at $z = 2\mu - m$ if $\mu \leq m$, and at $z = m$ if $\mu \geq m$.

(b) If $z \leq y$, the dual constraints are piecewise linear decreasing. The optimal dual solution is a horizontal line at zero, vertically shifted at m by an amount $\alpha_2 = z - y$. The regret is equal to $(1/2 - \beta)(z - y)$ and maximized when $z = 0$.

The minimum regret quantity equates the regret from ordering too little (a) with the regret from ordering too much (b). When $\mu \geq m$, y^* solves the following:

$$\min_{y \geq 0} \max\{(1-\beta)(m - y), y(\beta - \frac{1}{2})\}$$

and is equal to $y^* = 2m(1-\beta)$.

When $\mu \leq m \leq \mu/\beta$, and $y \leq (2\mu - m)(2\beta - 1)$, y^* solves the following problem:

$$\min_{y \geq 0} \max\{(1-\beta)(2\mu - m - y), y(\beta - \frac{1}{2})\},$$

and is equal to $y^* = 2(1-\beta)(2\mu - m)$. The condition $y^* \leq (2\mu - m)(2\beta - 1)$ simplifies to $3/4 \leq \beta$.

When $\mu \leq m \leq \mu/\beta$, and $(2\mu - m)(2\beta - 1) \leq y \leq m^2(2\beta - 1)/(2\mu - m)$, y^* solves the following problem:

$$\min_{y \geq 0} \max\{y(\beta - \frac{1}{2})(\sqrt{\frac{2\mu - m}{y(2\beta - 1)}} - 1)^2, y(\beta - \frac{1}{2})\},$$

and is equal to $y^* = (2\mu - m)/(8\beta - 4)$. The conditions $(2\mu - m)(2\beta - 1) \leq y^* \leq m^2(2\beta - 1)/(2\mu - m)$ simplify to $3/4 \geq \beta \geq 1/4 + \mu/(2m)$.

When $\mu \le m \le \mu/\beta$, and $y \ge m^2(2\beta-1)/(2\mu-m)$, y^* solves the following problem:

$$\min_{y\ge 0} \max\{(m-y)(\frac{\mu}{m}-\beta), y(\beta-\frac{1}{2})\},$$

and is equal to $y^* = 2m(\mu-\beta m)/(2\mu-m)$. The condition $y^* \ge m^2(2\beta-1)/(2\mu-m)$ simplifies to $\beta \le 1/4+\mu/(2m)$.

Similarly, when $m \ge \mu/\beta$, $y^* = 2(1-\beta)(2\mu-m)$ if $\beta \ge 3/4$ and $y^* = (2\mu-m)/(8\beta-4)$ otherwise.

Case 2: $\beta \le 1/2$.

(a) If $z \ge y$, a dual feasible solution is a straight line with ordinate α_0, slope α_1, and possibly vertically shifted at m by an amount α_2, that is nonnegative for all $x \ge 0$, lies above the line $x-y$ between y and z, and above the line $z-y$ for all $z \le x$. There are two candidate optimal solutions: either a line going with slope $(z-y)/(z-m)$, passing through $(m,0)$ and $(z, z-y)$, and shifted vertically at m by an amount $m(z-y)/(z-m)$; or a discontinuous horizontal line, starting at zero, and shifted vertically at m by an amount $z-y$. In the first solution, $\alpha_0 = -\alpha_2 = -(z-y)m/(z-m)$ and $\alpha_1 = (z-y)/(z-m)$, and the objective value of (13) is equal to $(z-y)(\mu-m/2)/(z-m)$. In the second solution, $\alpha_0 = z-y = -\alpha_2$, $\alpha_1 = 0$, and the objective value of (13) is equal to $(z-y)/2$. Therefore, the first solution is optimal if and only $z \ge 2\mu$.

(a.1) If $z \ge 2\mu$, the regret is equal to $(z-y)((\mu-m/2)/(z-m)-\beta)$, which is a concave function of z. The regret, optimized over $z \ge \max\{2\mu, y\}$, attains its maximum at $z^* = m + \sqrt{(\mu-m/2)(y-m)/\beta}$ if $z^* \ge \max\{y, 2\mu\}$ and at $\max\{y, 2\mu\}$ if $z^* < \max\{y, 2\mu\}$. Replacing z by its optimal value leads to the following values of regret:

$$\begin{cases} (\frac{1}{2}-\beta)(2\mu-y), & \text{if } y \min\{2\mu, \le m+4\beta(\mu-m/2)\}, \\ (y-m)\beta(\sqrt{\frac{\mu-m/2}{(y-m)\beta}}-1)^2, & \text{if } \min\{2\mu, m+4\beta(\mu-m/2)\} \le y \le \max\{2\mu, m+(\mu-m/2)/\beta\}, \\ 0, & \text{if } y \ge \max\{2\mu, m+(\mu-m/2)/\beta\}. \end{cases}$$

(a.2) If $z \le 2\mu$, then the regret is equal to $(1/2-\beta)(z-y)$ and is maximized when $z = 2\mu$.

(b) If $z \le y$, the dual constraints are piecewise linear decreasing. The optimal dual solution is a horizontal line at zero, and is associated with a regret of $-\beta(z-y)$ and maximized when $z = m$.

If $y \leq \min\{2\mu, m + 4\beta(\mu - m/2)\}$, y^* solves the following problem:

$$\min_{y \geq 0} \max\{(\frac{1}{2} - \beta)(2\mu - y), -\beta(m - y)\},$$

and is equal to $y^* = 2\mu + 2\beta(m - 2\mu)$. The condition $y^* \leq 2\mu$ is always satisfied since, by Markov's inequality, $2\mu \geq m$. Furthermore, the condition $y^* \leq m + 4\beta(\mu - m/2)$ becomes $\beta \geq 1/4$.

If $m + 4\beta(\mu - m/2) \leq y \leq m + (\mu - m/2)/\beta$, y^* solves the following problem:

$$\min_{y \geq 0} \max\{(y - m)\beta(\sqrt{\frac{\mu - m/2}{(y - m)\beta}} - 1)^2, -\beta(m - y)\},$$

and is equal to $y^* = m + (\mu - m/2)/(4\beta)$. The condition $y^* \geq m + 4\beta(\mu - m/2)$ simplifies into $\beta \leq 1/4$, and the condition $y^* \geq 2\mu$ translated into $\beta \leq 1/8$; since only one of the two conditions must hold, the condition $\beta \leq 1/4$ is sufficient. Furthermore, the condition $y^* \leq m + (\mu - m/2)/\beta$ trivially holds, which allows us to disregard the third case, when the regret equals zero. ∎

Proof of Theorem 4

Following Popescu (2005), the closed convex set of symmetric distributions $\mathcal{D}$ can be generated by pairs of symmetric Diracs. Using this characterization, the dual problem (5) can be formulated as follows (Popescu 2005):

$$\begin{array}{ll} \min_{\alpha_0,\alpha_1} & \alpha_0 + \alpha_1\mu, \\ \text{s.t.} & 2\alpha_0 + 2\mu\alpha_1 \geq \\ & \quad \min\{\mu - x, z\} + \min\{\mu + x, z\} - \min\{\mu - x, y\} - \min\{\mu + x, y\}, \quad \forall 0 \leq x \leq \mu. \end{array}$$

The dual problem can easily be solved geometrically. A dual feasible solution is a horizontal line, lying above the piecewise linear function described by the right-hand side of the constraint. By symmetry, the mean is equal to the median. As a result, $y, z \geq \mu$ if and only if $\beta \leq 1/2$.

Case 1: $\beta \leq 1/2$**.** When $z \geq y$, the ordinate of the line is equal to $z - y$. The associated regret, $(1/2 - \beta)(z - y)$, is maximized at $z = 2\mu$. When $z \leq y$, the ordinate of the line is equal to zero. The associated regret, $-\beta(z - y)$, is maximized at $z = \mu$. Equating both regrets, $(1/2 - \beta)(2\mu - y) = -(\mu - y)$, gives rise to the robust order quantity $y = 2\mu(1 - \beta)$.

Case 2: $\beta \geq 1/2$**.** When $z \geq y$, the ordinate of the line is equal to $2(z-y)$. The associated regret, $(1-\beta)(z-y)$, is maximized at $z = \mu$. When $z \leq y$, the ordinate of the line is equal to $z-y$. The associated regret, $(1/2-\beta)(z-y)$, is maximized when z is zero. Equating both regrets, $(1-\beta)(\mu-y) = -(1/2-\beta)y$, gives rise to the robust order quantity $y = 2\mu(1-\beta)$. ∎

Proof of Theorem 5

Following Popescu (2005), the closed convex set of unimodal distributions with mode M, $\mathcal{D}$, can be generated with M-rectangular distributions (i.e., uniform distributions over a segment bounded by M). With this representation, the dual problem (5) can be formulated as follows (Popescu 2005):

$$\begin{array}{lll} \min_{\alpha_0} & \alpha_0, & \\ \text{s.t.} & \alpha_0(M-x) \geq \int_x^M \min\{\xi, z\} - \min\{\xi, y\} d\xi, & \forall A \leq x \leq M, \\ & \alpha_0(x-M) \geq \int_M^x \min\{\xi, z\} - \min\{\xi, y\} d\xi, & \forall M \leq x \leq B. \end{array} \tag{14}$$

The dual problem can easily be solved geometrically. A dual feasible solution is a piecewise linear function, passing through $(M, 0)$, with slope $-\alpha_0$ before M and α_0 after M, lying above the piecewise quadratic function described by the right-hand side of the constraint. Six cases need to be considered, depending on the relative order of z, y, and M.

Case 1: $z \leq y \leq M$**.** The right-hand side is constant linear for $x \leq z$, quadratic between z and y, then increasing linear until M, and decreasing linear thereafter. The optimal dual solution is such that the constraint is tight at A. Therefore, $\alpha_0 = (z-y)(M-z/2-y/2)/(M-A)$, and the regret equals $(z-y)(M-z/2-y/2)/(M-A) - \beta(z-y)$. The maximum regret is attained at $z = M - \beta(M-A)$ or $z = y$, whichever is the smallest. It is equal to zero when $y \leq M - \beta(M-A)$, and to $(M-y-\beta(M-A))^2/(2(M-A))$ otherwise.

Case 2: $z \leq M \leq y$**.** The right-hand side is constant up to z, then quadratic between z and y, with a change of concavity at M, and linear after y. An optimal dual solution is such that the constraint is tight at A. Therefore, $\alpha_0 = -1/2(M-z)^2/(M-A)$, and the regret equals $-1/2(M-z)^2/(M-A) - \beta(z-y)$. The maximum regret equals $\beta/2(\beta(M-A)+2(y-M))$ and is attained at $z = M - \beta(M-A)$.

Case 3: $M \leq z \leq y$**.** The right-hand side is zero for $x \leq z$ and negative thereafter. The optimal dual solution is equal to zero, and the regret equals $-\beta(z-y)$. The regret is maximized when $z = M$; as a result, Case 3 is dominated by Case 2.

Case 4: $M \leq y \leq z$**.** The right-hand side is zero for $x \leq y$ and increasing after; it is convex between y and z, and linear beyond z. The optimal solution is such that the function $\alpha_0(x - M)$ crosses the constraint set at B, i.e., $\alpha_0 = (z - y)(B - z/2 - y/2)/(B - M)$. The regret equals $((B - z/2 - y/2)/(B - M) - \beta)(z - y)$ and is maximized at $z = B - \beta(B - M)$. At its maximum, the regret equals $(B - y - \beta(B - M))^2/(2(B - M))$.

Case 5: $y \leq M \leq z$**.** The right-hand side is constant for $x \leq y$, concave decreasing between y and M, convex increasing between M and z and linear increasing after. The optimal solution is such that the function $\alpha_0(x - M)$ crosses the constraint set at B, i.e., $\alpha_0 = ((z - M)(z/2 + M/2 - y) + (z - y)(B - z))/(B - M)$. The maximum regret, attained at $z = B - \beta(B - M)$, is equal to $(1 - \beta)((B + M)/2 - \beta(B - M)/2 - y)$.

Case 6: $y \leq z \leq M$**.** The right-hand side is constant for $x \leq y$, concave decreasing between y and z, linear decreasing until M, and linear increasing thereafter. The optimal solution is to have the slope of the function equal to the slope of the linear piece of the constraint. Accordingly, $\alpha_0 = z - y$; the regret equals $(1 - \beta)(z - y)$ and is maximized at $z = M$; as a result, Case 6 is dominated by Case 5.

The robust order quantity equates the regrets. Taking $y \leq M - \beta(M - A)$ is suboptimal, since the regret when $z \geq y$, equal to $(1 - \beta)((B + M)/2 - \beta/2(B - M) - y)$ is larger than the regret when $z \leq y$, equal to zero. When $M - \beta(M - A) \leq y \leq M$, y minimizes the maximum regrets:

$$\min_{y \geq 0} \max\{\frac{(M - y - \beta(M - A))^2}{2(M - A)}, (1 - \beta)(\frac{B + M}{2} - \beta\frac{B - M}{2} - y)\},$$

and is equal to $y^* = A + \sqrt{(M - A)(1 - \beta)(B(1 - \beta) - A(1 + \beta) + 2\beta M)}$. The condition $y \geq M - \beta(M - A)$ is always satisfied. The condition $y^* \leq M$ simplifies to $M(1 - 2\beta(1 - \beta)) \geq \beta^2 A + (1 - \beta)^2 B$.

When $y \geq M$, the robust order quantity minimizes the maximum regrets:

$$\min_{y \geq 0} \max\{\frac{\beta}{2}(\beta(M - A) + 2(y - M)), \frac{(B - y - \beta(B - M))^2}{2(B - M)}\}$$

and is equal to $B - \sqrt{\beta(B - M)(B(2 - \beta) - \beta A - 2M(1 - \beta))}$. The condition $y^* \geq M$ simplifies to $M(1 - 2\beta(1 - \beta)) \leq \beta^2 A + (1 - \beta)^2 B$. ∎

Proof of Theorem 6

The dual problem (5) is similar to (14), with an additional variable associated with the median:

$$
\begin{array}{ll}
\min_{\alpha_0,\alpha_1} & \alpha_0 + \alpha_1/2, \\
\text{s.t.} & \alpha_0(M-x) + \alpha_1(\min\{M,m\} - \min\{x,m\}) \geq \int_x^M \min\{\xi,z\} - \min\{\xi,y\} d\xi, \quad \forall x \leq M, \\
 & \alpha_0(x-M) + \alpha_1(\min\{x,m\} - \min\{M,m\}) \geq \int_M^x \min\{\xi,z\} - \min\{\xi,y\} d\xi, \quad \forall x \geq M.
\end{array}
$$

The constraint set is the same as that in problem (14). A dual feasible solution is a piecewise linear function, with slope changing at M and possibly at m, lying above the piecewise quadratic function described by the right-hand side of the constraint. Four cases must be considered, depending on the relative order of m and M, and whether $\beta \geq 1/2$ or not.

Case 1: $m \leq M, \beta \leq 1/2$. (a) When $z \leq y \leq M$, the constraints are tight at zero in an optimal dual solution, i.e., $\alpha_0 M + \alpha_1 m = (z-y)(M - z/2 - y/2)$, and at m, i.e., $\alpha_0(M-m) = (z-y)(M-z/2-y/2)$. The objective value of this solution is less than the optimal solution proposed in Case 1 of Theorem 5 if and only if $2m \geq M$. The regret, equal to $(z-y)((M-z/2-y/2)/(2(M-m)) - \beta)$, is maximized at $z^* = M - 2\beta(M-m)$ or $z^* = y$, whichever is the smallest. Thus, the maximum regret equals $(M - 2\beta(M-m) - y)^2/(4(M-m))$ if $y \geq M - 2\beta(M-m)$, zero otherwise.

(b) When $z \leq M \leq y$, the constraints are tight at zero in the optimal dual solution, i.e., $\alpha_0 M + \alpha_1 m = -1/2(M-z)^2$, and at m, i.e., $\alpha_0(M-m) = -1/2(M-z)^2$. The objective value of this solution is less than the optimal solution proposed in Case 2 of Theorem 5 if and only if $2m \geq M$. The regret, equal to $-1/4(M-z)^2/(M-m) - \beta(z-y)$, is maximized at $z^* = M - 2\beta(M-m)$. Thus, the maximum regret equals $\beta^2(M-m) - \beta(M-y)$.

(c) When $M \leq z \leq y$, the optimal dual solution is a horizontal line at zero. The regret, equal to $-\beta(z-y)$, is maximized at $z^* = M$. Thus, the maximum regret equals $-\beta(M-y)$.

(d) When $M \leq y \leq z$, the constraints are tight at zero in an optimal dual solution, i.e., $\alpha_0 M + \alpha_1 m = 0$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)(1 - M/(2m) - \beta)$, is maximized at $z^* = y$ if $\beta \geq 1 - M/(2m)$ and grows to infinity when $z \to \infty$ otherwise. Thus, the maximum regret equals zero if $\beta \geq 1 - M/(2m)$ and tends to infinity otherwise.

(e) When $y \leq M \leq z$, the constraints are tight at zero in an optimal dual solution, i.e., $\alpha_0 M + \alpha_1 m = (M-y)^2/2$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)(1 - M/(2m) - \beta) + (M-y)^2/(4m)$, is maximized at M if $\beta \geq 1 - M/(2m)$, and increases with z otherwise. Thus, the maximum regret equals $(M-y)(1-M/(2m)-\beta) + (M-y)^2/(4m)$ if $\beta \geq 1 - M/(2m)$, and tends to infinity otherwise.

(f) When $y \leq z \leq M$, the constraints are tight at zero in the optimal dual solution, i.e., $\alpha_0 M + \alpha_1 m = (z-y)(M - z/2 - y/2)$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)((2m - z/2 - y/2)/(2m) - \beta)$, is maximized at $z^* = 2m(1-\beta)$ or at M, whichever is the smallest. Thus, the maximum regret equals $(2m(1-\beta)-y)^2/(4m)$ when $\beta \geq 1 - M/(2m)$, and $(M-y)((2m - M/2 - y/2)/(2m) - \beta)$ otherwise.

When $\beta \leq 1 - M/(2m)$, the regret associated with ordering less than optimal is infinite, and the minimax regret order quantity is not defined. If on the other hand, $\beta \geq 1 - M/(2m)$, having $y > M$ is suboptimal because the regret of ordering too much (b) is increasing with y while the regret of ordering too little (d) remains constant. When $y \leq M$, the minimax regret order quantity solves the following problem:

$$\min_{y \geq 0} \max\{(2m(1-\beta)-y)^2 \frac{1}{4m}, (M - 2\beta(M-m) - y)^2 \frac{1}{4(M-m)}\},$$

and is equal to $y^* = m + \sqrt{m(M-m)}(1-2\beta)$; the maximum regret equals $(1-2\beta)^2(M - 2\sqrt{m(M-m)})/4$. One can check that the conditions $y^* \geq M - 2\beta(M-m)$ (a), and $y^* \leq 2m(1-\beta)$ (f) are satisfied under the assumption that $M \leq 2m$. The order quantity y^* is less than M because $\sqrt{(M-m)/m} \geq (M-m)/m$ and $\beta \geq 1 - M/(2m)$.

Case 2: $m \leq M, \beta \geq 1/2$. (a) When $z \leq y$, the constraints are tight at zero in the optimal dual solution, i.e., $\alpha_0 M + \alpha_1 m = (z-y)(M - z/2 - y/2)$, and at m, i.e., $\alpha_0 = (z-y)$. The objective value of this solution is less than the optimal solution proposed in Case 1 of Theorem 5 if and only if $2m \geq M$. The regret, equal to $(z-y)((2m - z/2 - y/2)/(2m) - \beta)$ is maximized at $z^* = 2m(1-\beta)$ or $z^* = y$, whichever is the smallest. Thus, the maximum regret equals $(2m(1-\beta)-y)^2/(4m)$ if $y \geq 2m(1-\beta)$, zero otherwise.

(b) When $y \leq z$, the optimal dual solution is $\alpha_0 = z - y$, and $\alpha_1 = 0$. The regret, equal to $(z-y)(1-\beta)$, is maximized at $z^* = m$. Thus, the maximum regret equals $(m-y)(1-\beta)$.

The minimax regret order quantity solves the following problem:

$$\min_{y \geq 0} \max\{(2m(1-\beta)-y)^2 \frac{1}{4m}, (m-y)(1-\beta)\},$$

and is equal to $y^* = 2m\sqrt{\beta(1-\beta)}$; the maximum regret equals $m(1-\beta)(1-2\sqrt{\beta(1-\beta)})$. The condition $y^* \geq 2m(1-\beta)$ (a) is satisfied because $\beta \geq 1/2$.

Case 3: $m \geq M, \beta \leq 1/2$. (a) When $z \leq y$, the optimal dual solution is a horizontal line at zero. The regret, equal to $-\beta(z-y)$, is maximized at $z^* = m$. Thus, the maximum regret equals $-\beta(m-y)$.

(b) When $y \leq z$, the constraints are tight at zero in an optimal dual solution, i.e., $\alpha_0 M + \alpha_1 M = 0$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)(1/2-\beta)$, is increasing with z. Thus, the maximum regret tends to infinity as $z \to \infty$. Therefore, the minimax regret is not well defined.

Case 4: $m \geq M, \beta \geq 1/2$. (a) When $z \leq y \leq M$, the constraints are tight at zero in an optimal dual solution, i.e., $\alpha_0 M + \alpha_1 M = (z-y)(M - z/2 - y/2)$, and the last segment is parallel to the constraint set, i.e., $\alpha_0 = z - y$. The regret, equal to $(z-y)((2M - z/2 - y/2)/(2M) - \beta)$ is maximized at $z^* = 2M(1-\beta)$ or $z^* = y$, whichever is the smallest. Thus, the maximum regret equals $(2M(1-\beta)-y)^2/(4M)$ if $y \geq 2M(1-\beta)$, zero otherwise.

(b) When $z \leq M \leq y$, the constraints are tight at zero in an optimal dual solution, i.e., $\alpha_0 M + \alpha_1 M = -1/2(M-z)^2$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)/2 - (M-z)^2/(4M) - \beta(z-y)$, is maximized at $z^* = 2M(1-\beta)$. Thus, the maximum regret equals $(2\beta-1)(2y-3M+2\beta M)/4$.

(c) When $M \leq z \leq y$, the constraints are tight at zero in the optimal dual solution, i.e., $\alpha_0 M + \alpha_1 M = 0$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)(1/2-\beta)$, is maximized at $z^* = M$. Thus, the maximum regret equals $(M-y)(1/2-\beta)$, and is dominated by (b).

(d) When $M \leq y \leq z$, the constraints are tight at m in an optimal dual solution, i.e., $(\alpha_0+\alpha_1)(m-M) = (z-y)(m-z/2-y/2)$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)((2m - M - z/2 - y/2)/(2(m-M)) - \beta)$, is maximized at $z^* = 2m - M - 2\beta(m-M)$ or at y, whichever is the largest. Thus, the maximum regret equals $(2m-M-y-2\beta(m-M))^2/(4(m-M))$ if $y \geq 2m - M - 2\beta(m-M)$, zero otherwise.

(e) When $y \leq M \leq z$, the constraints are tight at m in an optimal dual solution, i.e., $(\alpha_0+\alpha_1)(m-M) = (z-y)(m-z) + (z-M)(z/2+M/2-y)$, and the last line segment is parallel to the constraint set, i.e., $\alpha_0 = (z-y)$. The regret, equal to $(z-y)(1/2-\beta) + ((z-$

$y)(m-z)+(z-M)(z/2+M/2-y))/(2(m-M))$, is maximized at $z^* = m+(1-2\beta)(m-M)$. Thus, the maximum regret equals $(1-\beta)(m-\beta(m-M)-y)$.

(f) When $y \leq z \leq M$, the optimal dual solution is $\alpha_0 = z - y$, and $\alpha_1 = 0$. The regret, equal to $(z-y)(1-\beta)$, is maximized at $z^* = M$. Thus, the maximum regret equals $(M-y)(1-\beta)$, and it is dominated by (e).

When $y \leq M$, the maximum regrets are given by (a) and (e). Thus, the minimax regret order quantity solves the following problem:

$$\min_{y \geq 0} \max\{(2M(1-\beta)-y)^2 \frac{1}{4M}, (1-\beta)(m-\beta(m-M)-y)\},$$

and is equal to $y^* = 2\sqrt{(1-\beta)M(2\beta M - \beta m + m - M)}$; the maximum regret equals $(1-\beta)(m-\beta(m-M)-y^*)$. The condition $y^* \geq 2M(1-\beta)$ (a) is satisfied because $\beta \geq 1/2$. The order quantity y^* is less than M if $m \leq M(8\beta^2 - 12\beta + 5)/(4(\beta-1)^2)$.

When $y \geq M$, the maximum regrets are given by (b) and (d). Thus, the minimax regret order quantity solves the following problem:

$$\min_{y \geq 0} \max\{(2\beta-1)(2y-3M+2\beta M)\frac{1}{4}, (2m-M-y-2\beta(m-M))^2 \frac{1}{4(m-M)}\},$$

and is equal to $y^* = m - \sqrt{(m-M)(2\beta-1)(4\beta M - 2\beta m - 4M + 3m)}$; the maximum regret equals $(2\beta-1)(2y^*-3M+2\beta M)/4$. The condition $y^* \geq 2m-M-2\beta(m-M)$ (d) is satisfied because $\beta \geq 1/2$. The order quantity y^* is greater than M if $m \geq M(8\beta^2-12\beta+5)/(4(\beta-1)^2)$. ∎

Proof of Theorem 7

Following Popescu (2005), the set of unimodal and symmetric distributions with mean μ can be generated using a mixture of μ-centered rectangular distributions (i.e., uniform distributions centered around μ). Using this representation, the dual problem (5) can be formulated as follows:

$$\begin{array}{ll} \min_{\alpha_0,\alpha_1} & \alpha_0 + \alpha_1\mu, \\ \text{s.t.} & 2t(\alpha_0 + \mu\alpha_1) \geq \int_{\mu-t}^{\mu+t} \min\{\xi, z\} - \min\{\xi, y\} d\xi, \quad \forall 0 \leq t \leq \mu. \end{array}$$

A dual feasible solution is any linear function, passing through the origin, and lying above the piecewise quadratic function defined by the right-hand side. Because the mean equals the median (by symmetry), $z \geq \mu$ whenever $\beta \leq 1/2$.

Case 1: $\beta \geq 1/2$. When $z \geq y$, the right-hand side of the dual constraint is increasing, linearly with slope $2(z-y)$ for $t \leq \mu - z$, then concavely between $\mu - z$ and $\mu - y$, and then linearly with slope $z - y$. The dual optimal solution is a straight line with slope equal to the first piece of the constraint, i.e., $\alpha_0 + \alpha_1 \mu = z - y$. The regret, $(1-\beta)(z-y)$, is maximized at $z = \mu$.

If on the other hand $z \leq y$, the right-hand side of the dual constraint is decreasing, first linearly with slope $2(z-y)$ until $t = \mu - y$, then convexly between $\mu - y$ and $\mu - z$, and finally linearly with slope $z - y$. The optimal dual solution is a straight line intersecting the constraint at the origin and at $t = \mu$. Accordingly, $\alpha_0 + \alpha_1 \mu = (z-y)(2\mu - z/2 - y/2)/(2\mu)$, and the regret equals $(z-y)((2\mu - z/2 - y/2)/(2\mu) - \beta)$. The maximum regret, attained at $z = 2\mu(1-\beta)$, is then equal to $(y - 2\mu(1-\beta))^2/(4\mu)$.

The robust order quantity minimizes the maximum of the following regrets:

$$\min_{y \geq 0} \max\{(y - 2\mu(1-\beta))^2 \frac{1}{4\mu}, (1-\beta)(\mu - y)\}$$

and is then equal to $y = 2\mu\sqrt{\beta(1-\beta)}$.

Case 2: $\beta \leq 1/2$. When $z \geq y$, the right-hand side of the dual constraint is zero for $x \leq y$ and increasing thereafter; it is convex between y and z, and linearly increasing beyond z. The optimal dual solution is a straight line, intersecting the curve defined by the right-hand side at zero and at $t = \mu$. Therefore, $\alpha_0 + \alpha_1 \mu = (z-y)(2\mu - z/2 - y/2)/(2\mu)$, and the regret equals $(z-y)((2\mu - z/2 - y/2)/(2\mu) - \beta)$. The maximum regret, attained at $z = 2\mu(1-\beta)$, is then equal to $(y - 2\mu(1-\beta))^2/(4\mu)$.

On the other hand, when $z \leq y$, the right-hand side is zero for $x \leq z$ and decreasing thereafter. The optimal dual solution is a horizontal line. Therefore, $\alpha_0 + \alpha_1 \mu = 0$, and the regret equals $-\beta(z-y)$. The maximum regret, attained at $z = \mu$, is then equal to $-\beta(\mu - y)$.

The robust order quantity minimizes the following maximum regrets:

$$\min_{y \geq 0} \max\{(y - 2\mu(1-\beta))^2 \frac{1}{4\mu}, -\beta(\mu - y)\}$$

and is then equal to $y = 2\mu(1 - \sqrt{\beta(1-\beta)})$. ∎

Proof of Theorem 8

When only the mean μ and the variance σ^2 are known, the dual problem (5) is the following:

$$\min \qquad \alpha_0 + \alpha_1 \mu + \alpha_2(\sigma^2 + \mu^2),$$

$$s.t. \quad \alpha_0 + \alpha_1 x + \alpha_2 x^2 \geq \min\{x, z\} - \min\{x, y\}, \quad \forall x \geq 0. \tag{15}$$

Case 1: $z \geq y$. The right-hand side of the constraints of the dual problem is piecewise linear increasing. A dual feasible function is any quadratic function $h(x)$ that, on the positive orthant, is nonnegative, lies above the line $x - y$ between y and z, and above the line $z - y$ after z. If the quadratic term of the function is zero, the problem reduces to finding a straight line, as in Theorem 2. Otherwise, in an optimal solution, either the parabola is tangent to $x-y$ at some point between y and z or it passes through the kink point $(z, y-z)$. Combining these possibilities gives rise to three different cases that we analyze next.

Case 1.1: $h(x)$ is a straight line. When $z \leq \mu$, the optimal solution is a horizontal line. The associated dual objective value is equal to $z - y$. On the other hand, when $z \geq \mu$, the optimal solution is a straight line passing through the origin. The optimal dual objective equals in this case $(z - y)\mu/z$. (See Theorem 2.)

Case 1.2: $h(x)$ is tangent to $x - y$. If the quadratic function is tangent to $x - y$, it can be expressed as $h(x) = a(x - b)^2 + x - y$ for some $a \geq 0$. The minimum of the function, denoted by x_0, equals $b - 1/(2a)$.

If $b \geq 1/(2a)$, $x_0 \geq 0$. Since $h(x_0) = 0$, $a = 1/(4b - 4y)$. After plugging this value for a into the dual objective function, we minimize the function over all $b \in [\max\{y, 1/(2a)\}, z]$. The minimum value equals $1/2(\mu - y) + (1/2)\sqrt{\sigma^2 + (\mu - y)^2}$ and is attained at $b = y + \sqrt{\sigma^2 + (\mu - y)^2}$. Trivially, $b \geq y$. On the other hand, $b \geq 1/(2a)$ if and only if $y \geq (\mu^2 + \sigma^2)/(2\mu)$, and $b \leq z$ if and only if $y + \sqrt{\sigma^2 + (\mu - y)^2} \leq z$.

If $b \leq 1/(2a)$, $x_0 \leq 0$. Therefore, $h(x)$ must pass through the origin. Thus, $a = y/b^2$. After plugging this value for a into the dual objective function, we minimize the function over all $b \in [y, \min\{1/(2a), z\}]$. The minimum value equals $\mu - y\mu^2/(\mu^2 + \sigma^2)$ and is attained at $b = (\sigma^2 + \mu^2)/\mu$. The condition that $b \leq 1/(2a)$ simplifies to $y \leq (\mu^2 + \sigma^2)/(2\mu)$; therefore, the condition $b \geq y$ is automatically satisfied. Moreover, with this value, requiring that $b \leq z$ is equivalent to requiring that $(\mu^2 + \sigma^2)/\mu \leq z$.

Case 1.3: $h(x)$ passes through $(z, z - y)$. Since $h(x)$ must lie above the piecewise linear function defined by the constraint, the derivative of $h(x)$ at the kink point must be less than 1, i.e., $2\alpha_2 z + \alpha_1 \leq 1$. The minimum of the function, denoted by x_0, equals $-\alpha_1/(2\alpha_2)$.

If $-\alpha_1/(2\alpha_2) \leq 0$, the quadratic function passes through the origin, i.e. $h(0) = 0$. Accordingly, the dual objective function simplifies to $\alpha_2(\sigma^2+\mu^2-\mu z)+\mu(z-y)/z$. Minimizing the objective with respect to α_2 gives rise to three possible cases. When $z \leq (\sigma^2+\mu^2)/\mu$, the minimum is attained at $\alpha_2 = 0$ (straight line) and equals $\mu(z-y)/z$ (see Case 1.1). If $z \geq (\sigma^2+\mu^2)/\mu$ and $z \leq 2y$, the minimum equals $(\sigma^2+\mu^2)(z-y)/z^2$, and $x_0 = 0$. It is easy to see that this solution is always greater than or equal to that found in Case 1.2. If $z \geq (\sigma^2+\mu^2)/\mu$ and $z \geq 2y$, the minimum equals $(\sigma^2+\mu^2)y/z^2+\mu(z-2y)/z$ with $h'(z) = 1$, which is also greater than the solution found in Case 1.2.

On the other hand, if $-\alpha_1/(2\alpha_2) \geq 0$, the quadratic function is minimized at some $x_0 \geq 0$. Thus, $h(x)$ is assumed to pass through $(z, z-y)$ with a derivative less than 1, and to have a minimum value of zero, attained on the interval $[0, y]$. Under these conditions, the dual feasible function can be expressed as a the following function of α_2 only: $\alpha_2(\sigma^2+\mu^2-2\mu z+z^2)+2\sqrt{\alpha_2}\sqrt{z-y}(\mu-z)+z-y$. Minimizing the dual objective over all nonnegative values of α_2, such that the above conditions are met, gives rise to the following cases. If $z \leq (\sigma^2+\mu^2)/\mu$, the minimum dual objective value equals $(\sigma^2+\mu^2)(z-y)/z^2$, which is greater than $(z-y)\mu/z$ (Case 1.1) and can be disregarded from consideration. If $z \geq y+\sqrt{\sigma^2+(y-\mu)^2}$, the dual objective value equals $\mu-y+(\sigma^2+(\mu-z)^2)/(4(z-y))$, with $h'(z) = 1$, which cannot be smaller than the solution of Case 1.2 since the solution is constrained to pass through $(z, z-y)$. Therefore, this solution can be ignored in the sequel. Finally, if $z \geq (\sigma^2+\mu^2)/\mu$ and if $z \leq y+\sqrt{\sigma^2+(y-\mu)^2}$, the minimum dual objective value equals $\sigma^2(z-y)/(\sigma^2+(\mu-z)^2)$, attained when $\sqrt{\alpha_2} = (z-\mu)\sqrt{z-y}/(\sigma^2+(\mu-z)^2)$. Notice that α_2 is well defined only if $z \geq \mu$.

Summarizing, the optimal value of (15) is equal to the minimum among $z-y$ (Case 1.1), $(z-y)\mu/z$ (Case 1.1), $(z-y)\sigma^2/(\sigma^2+(z-\mu)^2)$ if $z \leq y+\sqrt{\sigma^2+(y-\mu)^2}$ and $z \leq (\sigma^2+\mu^2)/\mu$ (Case 1.3), $\mu-y\mu^2/(\sigma^2+\mu^2)$ if $y \leq (\sigma^2+\mu^2)/(2\mu)$ and $z \geq (\sigma^2+\mu^2)/\mu$ (Case 1.2), and $(\mu-y)/2+\sqrt{\sigma^2+(\mu-y)^2}/2$ if $y \geq (\sigma^2+\mu^2)/(2\mu)$ and $z \geq y+\sqrt{\sigma^2+(\mu-y)^2}$ (Case 1.2). The regret is obtained by subtracting $\beta(z-y)$ from these functions.

The maximum regret will never be attained by the last two expressions (derived in Case 1.2). Indeed, in these cases, the expected difference in sales is independent of z, and the regret is therefore decreasing with z. For instance, if the expected difference in sales equals $(\mu-y)/2+\sqrt{\sigma^2+(\mu-y)^2}/2$, the regret is maximized when z is the smallest, i.e., when $z = y+\sqrt{\sigma^2+(\mu-y)^2}$. But when $z \leq y+\sqrt{\sigma^2+(\mu-y)^2}$, the expression for the difference in sales is given by $(z-y)\sigma^2/(\sigma^2+(z-\mu)^2)$. Similar, the maximum regret is never achieved by a sales difference equal to $z-y$ (Case 1.1, horizontal line). Indeed, in this case, the regret is increasing with z, and is therefore maximized when $z=\mu$. But when $z \geq \mu$, the expression for the difference in sales is given by $(z-y)\mu/z$.

Therefore, the regret is equal to the minimum between $(\mu/z-\beta)(z-y)$ when $\mu \leq z \leq (\sigma^2+\mu^2)/\mu$, and $(\sigma^2/(\sigma^2+(\mu-z)^2)-\beta)(z-y)$, when $(\sigma^2+\mu^2)/\mu \leq z \leq y+\sqrt{\sigma^2+(y-\mu)^2}$. Both expressions are concave functions of z in the intervals of definition.

Case 2: $z \leq y$. The right hand sides of the constraints of the dual problem is piecewise linear decreasing. A dual feasible function is any quadratic function $h(x)$ that, on the positive orthant, is nonnegative, lies above the line $x-z$ between z and y, and above the line $z-y$ after y. If the quadratic term of the function is zero, the problem reduces to finding a straight line, as in Theorem 2. Otherwise, in an optimal solution, either the parabola is tangent to $x-z$ at some point between z and y or it passes through the kink point $(z,0)$. In the last two cases, its minimum value is $z-y$ on the half line $x \geq y$. Combining these possibilities gives rise to three different cases that we analyze next.

Case 2.1: $h(x)$ is a straight line. Since the constraint right-hand side are decreasing, a horizontal line at zero is a candidate solution, and the optimal dual objective equals zero.

Case 2.2: $h(x)$ is tangent to $x-z$. In this case, the function can be expressed as $h(x)=a(x-b)^2+z-x$, for some $a \geq 0$. The minimum of the function, denoted by x_0, is then equal to $b+1/(2a)$. Since $h(x_0)=z-y$, $b=y-1/(4a)$. The dual objective function can then be expressed as a function of a only, namely $a(\sigma^2+\mu^2)-2ay\mu+z-\mu/2+a(1/(4a)-y)^2$. When minimized over all nonnegative a, the function attains its minimum at $a=1/(4\sqrt{\sigma^2+(\mu-y)^2})$ and is equal to $\sqrt{\sigma^2+(\mu-y)^2}/2+z-\mu/2-y/2$. The point of tangency between the quadratic function and the constraint is equal to $b=y-\sqrt{\sigma^2+(\mu-y)^2}$, and is less than y. It is greater than z if $z \leq y-\sqrt{\sigma^2+(\mu-y)^2}$.

Case 2.3: $h(x)$ passes through $(z,0)$. Since $h(x)$ must lie above the piecewise linear function defined by the constraint, the derivative of $h(x)$ at the kink point must be greater than -1, i.e., $2\alpha_2 z+\alpha_1 \geq -1$. The minimum of the function, denoted by x_0, equals $-\alpha_1/(2\alpha_2)$. Thus, $h(x)$ is assumed to pass through $(z,0)$ with a derivative greater than -1, and to have a minimum value of $z - y$, attained on the interval $[y, \infty)$. From these conditions, the dual feasible function can be expressed as the following function of α_2 only: $\alpha_2(\sigma^2 + \mu^2 - 2\mu z + z^2) + 2\sqrt{\alpha_2}\sqrt{z-y}(\mu - z)$. Minimizing the dual objective over all nonnegative values of α_2, such that the above conditions are met, gives rise to the following cases. If $z \geq \mu$, the derivative of the objective function with respect to α_2 is always positive. The optimal solution is to take α_2 as small as possible, i.e., $\alpha_2 = 0$. In this case, the dual optimal solution is a horizontal line at zero, similarly to Case 2.1. If $z \leq \mu$ and $z \geq y - \sqrt{\sigma^2 + (y-\mu)^2}$, the minimum objective is attained when $\sqrt{\alpha_2} = (\mu - z)\sqrt{y-z}/(\sigma^2 + (\mu - z)^2)$, and equals $(z-y)(\mu - z)^2/(\sigma^2 + (\mu - z)^2)$. The other cases lead to an objective value that is greater than the solutions obtained in Case 2.2, and can be discarded from future consideration.

Summarizing, the optimal value of the dual problem (15) is equal to the minimum among 0 if $z \geq \mu$, $(z-y)(\mu-z)^2/(\sigma^2 + (z-\mu)^2)$ if $\mu \geq z \geq y - \sqrt{\sigma^2 + (y-\mu)^2}$, and $\sqrt{\sigma^2 + (\mu - y)^2}/2 + z - \mu/2 - y/2$ if $z \leq y - \sqrt{\sigma^2 + (y-\mu)^2}$. The regret is obtained by subtracting $\beta(z-y)$ from these functions. The maximum regret will not be attained when the expected sale difference are equal to zero (Case 2.1). Indeed, in this case, the regret is decreasing with z and is therefore maximized when $z = \mu$. But when $z \leq \mu$, the sale difference is given by $(z-y)(\mu-z)^2/(\sigma^2 + (z-\mu)^2)$. Similar, the maximum regret is not attained by the last expression (Case 2.2). Indeed, in this case, the regret is increasing with z, and is therefore maximized when $z = y - \sqrt{\sigma^2 + (y-\mu)^2}$. But when $z \geq y - \sqrt{\sigma^2 + (y-\mu)^2}$, the difference in sales is given by $(z-y)(\mu-z)^2/(\sigma^2 + (z-\mu)^2)$. Thus the maximum regret equals $(z-y)((z-\mu)^2/(\sigma^2 + (z-\mu)^2) - \beta)$ for $\min\{\mu, y\} \geq z \geq y - \sqrt{\sigma^2 + (\mu - y)^2}$. This function is concave on its domain of definition.

Equating both regrets gives rise to the theorem statement. ∎