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Comparison of Bertrand and Cournot Profits - Electronic Companion

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Appendix A: M-Matrices

A square matrix whose diagonal elements are positive and off-diagonal elements are non-positive is called a Z-matrix. One definition of an M-matrix is that it is a Z-matrix with the additional property that all leading principal minors are positive. It suffices to note that column (or row) diagonally dominant Z-matrices are M-matrices. A symmetric Z-matrix is an M-matrix if and only if it is positive definite.

M-matrices enjoy a number of structural properties. We refer the reader to Horn and Johnson (1991) for a detailed treatment. The following two properties in particular are used extensively in our proofs. Let $\mathbf{X}$ be an M-matrix and $\mathbf{Y}$ be a Z-matrix such that $\mathbf{X} \leq \mathbf{Y}$. Then:

1. $\mathbf{X}^{-1}$ exists and $\mathbf{X}^{-1} \geq \mathbf{0}$;
2. $\mathbf{Y}$ is an M-matrix and $\mathbf{Y}^{-1} \leq \mathbf{X}^{-1}$.

Appendix B: Proofs of Statements

Proof of Theorem 1:

PART 1 (Proof of the inequality):

Define the following functions of a real variable γ:

$$\overline{p_i}(\gamma) = \frac{1}{1+x+\gamma}\left[\widetilde{d}_i + \frac{x\,(\mathbf{e}'\widetilde{\mathbf{d}})}{1-(n-1)x+\gamma}\right] + c_i,$$

$$\overline{q_i}(\gamma) = \gamma[\overline{p_i}(\gamma) - c_i],$$

$$\overline{\pi}(\gamma) = [\overline{p_i}(\gamma) - c_i]\, \overline{q_i}(\gamma).$$

Bertrand profit for firm i is given by $\overline{\pi_i}(\gamma^b)$, where $\gamma^b = 1$, and Cournot profit for firm i is given by $\overline{\pi_i}(\gamma^c)$, where $\gamma^c = 1 - \frac{r^2}{(n-1)(1-r)+r}$. Substituting $\overline{p_i}(\gamma)$ and $\overline{q_i}(\gamma)$ into $\overline{\pi_i}(\gamma)$ yields:

$$\overline{\pi_i}(\gamma) = \frac{\gamma\,(\mathbf{e}'\widetilde{\mathbf{d}})^2}{(1+x+\gamma)^2}\left[\widetilde{\lambda}_i + \frac{x}{1-(n-1)x+\gamma}\right]^2.$$

It is sufficient to consider the square root of $\overline{\pi_i}(\gamma)$:

$$\begin{aligned}
\sqrt{\overline{\pi_i}(\gamma)} &= \frac{\sqrt{\gamma}\,\mathbf{e}'\widetilde{\mathbf{d}}}{1+x+\gamma}\left[\widetilde{\lambda}_i + \frac{x}{1-(n-1)x+\gamma} - \frac{1}{n} + \frac{1}{n}\right],\\
&= \frac{\mathbf{e}'\widetilde{\mathbf{d}}}{n}\left[\frac{(\widetilde{\lambda}_i n - 1)\sqrt{\gamma}}{1+x+\gamma} + \frac{\sqrt{\gamma}}{1-(n-1)x+\gamma}\right],\\
&= \frac{\mathbf{e}'\widetilde{\mathbf{d}}}{n\sqrt{(1+x)}}\left[\frac{(\widetilde{\lambda}_i n - 1)\sqrt{v}}{1+v} + \frac{\sqrt{v}}{\tau+v}\right],
\end{aligned}$$

where $v := \gamma/(1+x)$. Therefore, setting $v_b := \gamma_b/(1+x)$ and $v_c := \gamma_c/(1+x)$, it suffices to establish sufficient conditions for:

$$\frac{\sqrt{v_b}}{\tau+v_b} - \frac{\sqrt{v_c}}{\tau+v_c} \le (\widetilde{\lambda}_i n - 1)\left[\frac{\sqrt{v_c}}{1+v_c} - \frac{\sqrt{v_b}}{1+v_b}\right]. \tag{B.1}$$

Some algebraic manipulation establishes the following:

$$\begin{aligned}
v_c &\equiv \frac{1}{1+\frac{1-\tau}{n\tau}},\\
v_b &\equiv \frac{n-1+\tau}{n},\\
v_b\, v_c &\equiv \theta^2 := \frac{\tau^2+(n-1)\tau}{1+(n-1)\tau}.
\end{aligned}$$

Note that $\theta > \tau$. Re-arranging inequality (B.1),

$$(\widetilde{\lambda}_i n - 1)\frac{1-\theta}{(1+v_b)(1+v_c)} \le \frac{\theta-\tau}{(\tau+v_b)(\tau+v_c)}.$$

Since $(1+v_b)/(\tau+v_b) \ge 2/(1+\tau)$, a sufficient condition for the above inequality to hold is:

$$(\widetilde{\lambda}_i n - 1)\frac{1-\theta}{\theta-\tau} \le \left(\frac{2}{1+\tau}\right)\left(\frac{1+v_c}{\tau+v_c}\right),$$

$$(\widetilde{\lambda}_i n-1)\frac{1-\theta^2}{\theta^2-\tau^2}\,\frac{\theta+\tau}{\theta+1}\leq\left(\frac{2}{1+\tau}\right)\frac{1}{\tau}\left(1-\frac{(\frac{1}{\tau}-1)v_c}{1+\frac{v_c}{\tau}}\right).$$

The last inequality can be re-arranged to yield:

$$\left(\frac{\widetilde{\lambda}_i n-1}{n-1}\right)\left(\frac{\theta+\tau}{\theta+1}\right)\leq\left(\frac{2}{1+\tau}\right)\left(1-\frac{(1-\tau)}{(1+\tau)+\frac{(1-\tau)}{n}}\right).$$

PART 2 (Proof of the threshold level $n<8$):

It suffices to consider a firm with $\widetilde{\lambda}_i=1$. The firm's Cournot profit is at least as high as its Bertrand profit if the following inequality holds:

$$\frac{\theta+\tau}{\theta+1}\leq\left(\frac{2}{1+\tau}\right)\left(1-\frac{(1-\tau)}{(1+\tau)+\frac{(1-\tau)}{n}}\right).$$

Holding τ fixed, note that the right-hand side of this inequality is non-increasing in n. Also, note that θ is increasing in n and, therefore, the left-hand side of the inequality is increasing in n. Therefore, it suffices to show that the inequality holds for any value of $\tau\in(0,1]$ and $n=7$. Substituting $n=7$,

$$\theta=\sqrt{\frac{\tau^2+6\tau}{1+6\tau}}.$$

and the inequality reduces to:

$$\begin{aligned}
&\frac{\theta+\tau}{\theta+1}\leq\left(\frac{2}{1+\tau}\right)\left(\frac{13\tau+1}{6\tau+8}\right),\\
&(\sqrt{\tau^2+6\tau}+\tau\sqrt{1+6\tau})(1+\tau)(6\tau+8)\leq 2(13\tau+1)(\sqrt{\tau^2+6\tau}+\sqrt{1+6\tau}),\\
&\sqrt{\tau^2+6\tau}\,[(1+\tau)(6\tau+8)-2(13\tau+1)]\leq\sqrt{1+6\tau}\,[2(13\tau+1)-\tau(1+\tau)(6\tau+8)],\\
&6\sqrt{\tau^2+6\tau}(1-\tau)^2\leq\sqrt{1+6\tau}(2+18\tau-14\tau^2-6\tau^3).
\end{aligned}$$

The above inequality clearly holds for $\tau=1$. For $0<\tau<1$,

$$6\sqrt{\tau^2+6\tau}(1-\tau)\leq\sqrt{1+6\tau}(6\tau^2+20\tau+2).$$

Squaring both sides and simplifying, the inequality finally reduces to:

$$0\leq 54\tau^5+360\tau^4+660\tau^3+325\tau^2-28\tau+1. \qquad \text{(B.2)}$$

It suffices to show that $0 \leq 325\tau^2 - 28\tau + 1$ because the remaining polynomial, $54\tau^5 + 360\tau^4 + 660\tau^3$, has positive coefficients and is therefore non-negative for $\tau \in [0,1]$. Note, however, that the discriminant of $325\tau^2 - 28\tau + 1$ is negative. Therefore, the inequality (B.2) is satisfied for $\tau \in [0,1]$.

PART 3 (Proof of the threshold level $r < 0.739$):

It suffices to consider a firm with $\widetilde{\lambda}_i = 1$. The firm's Cournot profit is at least as high as its Bertrand profit if the following inequality holds:

$$\frac{\theta + \tau}{\theta + 1} \leq \left(\frac{2}{1+\tau}\right)\left(1 - \frac{(1-\tau)}{(1+\tau) + \frac{(1-\tau)}{n}}\right).$$

Note that the right-hand side of this inequality is non-increasing in n. Taking the limits as $n \to \infty$,

$$\frac{\theta + \tau}{\theta + 1} \leq \left(\frac{2}{1+\tau}\right)\left(1 - \frac{(1-\tau)}{(1+\tau)}\right).$$

Note that $(\theta + \tau)/(\theta + 1) \leq (1+\tau)/2$. Re-arranging, we get:

$$\tau - 2\sqrt[3]{\tau} + 1 \leq 0.$$

The roots of the left-hand side of the above inequality are $(-1-\sqrt{5})^3/8$, $(-1+\sqrt{5})^3/8$, and 1. Therefore, the firm's Cournot profit is at least as high as Bertrand profit if $\tau \geq (-1+\sqrt{5})^3/8$. Recall that $\tau := \frac{(n-1)(1-r)}{n-(1-r)}$ which can be re-arranged to get $r := \frac{(n-1)(1-\tau)}{n-(1-\tau)}$. For a fixed value of τ, r is non-decreasing in n. Given the result of PART 2, we can replace $n = 8$ and $\tau = (-1+\sqrt{5})^3/8$ in the expression for r to get $r \leq 0.739$ as a sufficient condition for a firm's Cournot profit to be at least as high as its Bertrand profit.

Proof of Theorem 2:

PART 1 (Proof of the inequality):

Define the following functions of a real variable γ:

$$\begin{aligned}
\overline{\mathbf{p}}(\gamma) &= \frac{1}{1+x+\gamma}\left[\widetilde{\mathbf{d}} + \frac{x\,(\mathbf{e}'\widetilde{\mathbf{d}})}{1-(n-1)x+\gamma}\mathbf{e}\right] + \mathbf{c}, \\
\overline{\mathbf{q}}(\gamma) &= \gamma[\overline{\mathbf{p}}(\gamma) - \mathbf{c}], \\
\overline{\pi}(\gamma) &= [\overline{\mathbf{p}}(\gamma) - \mathbf{c}]'\,\overline{\mathbf{q}}(\gamma).
\end{aligned}$$

Bertrand total profit is given by $\overline{\pi}(\gamma^b)$, where $\gamma^b = 1$, and Cournot total profit is given by $\overline{\pi}(\gamma^c)$, where $\gamma^c = 1 - \frac{r^2}{(n-1)(1-r)+r}$. Substituting $\overline{\mathbf{p}}(\gamma)$ and $\overline{\mathbf{q}}(\gamma)$ into $\overline{\pi}(\gamma)$ yields:

$$\overline{\pi}(\gamma) = \frac{\gamma}{(1+x+\gamma)^2}\left[\widetilde{\mathbf{d}}'\widetilde{\mathbf{d}} + \frac{2x(1+x+\gamma)-nx^2}{(1-(n-1)x+\gamma)^2}(\mathbf{e}'\widetilde{\mathbf{d}})^2\right].$$

Letting $\alpha = \widetilde{\mathbf{d}}'\widetilde{\mathbf{d}}$ and $\beta = (\mathbf{e}'\widetilde{\mathbf{d}})^2$,

$$\begin{aligned}\overline{\pi}(\gamma) &= \frac{\gamma}{(1+\gamma+x)^2}\left[\alpha + \frac{2x(1+x+\gamma)-nx^2}{[1-(n-1)x+\gamma]^2}\beta - \frac{\beta}{n} + \frac{\beta}{n}\right], \\ &= \frac{\beta}{n}\left[\left(\frac{n\alpha}{\beta}-1\right)\frac{\gamma}{(1+x+\gamma)^2} + \frac{\gamma}{(1-(n-1)x+\gamma)^2}\right], \\ &= \frac{\beta}{n(1+x)}f\left(\frac{\gamma}{1+x}\right),\end{aligned}$$

where

$$\begin{aligned}f(v) &= \left(\frac{n\alpha}{\beta}-1\right)\frac{v}{(1+v)^2} + \frac{v}{(\tau+v)^2}, \\ &= (n-1)s^2\frac{v}{(1+v)^2} + \frac{v}{(\tau+v)^2}.\end{aligned}$$

The last equality follows from the fact that $\frac{n\alpha}{\beta} - 1 = (c.v.)^2 = (n-1)s^2$.

We need to derive a sufficient condition for $\overline{\pi}(\gamma^b) \leq \overline{\pi}(\gamma^c)$. This is equivalent to showing that $f(v_b) \leq f(v_c)$ where $v_b = \frac{\gamma_b}{1+x}$ and $v_c = \frac{\gamma_c}{1+x}$. That is, we need to establish when the following inequality holds:

$$(n-1)s^2\frac{v_b}{(1+v_b)^2} + \frac{v_b}{(\tau+v_b)^2} \leq (n-1)s^2\frac{v_c}{(1+v_c)^2} + \frac{v_c}{(\tau+v_c)^2}.$$

Rearranging yields:

$$\begin{aligned}\frac{v_b}{(\tau+v_b)^2} - \frac{v_c}{(\tau+v_c)^2} &\leq (n-1)s^2\left[\frac{v_c}{(1+v_c)^2} - \frac{v_b}{(1+v_b)^2}\right], \\ \frac{v_b(\tau+v_c)^2 - v_c(\tau+v_b)^2}{(\tau+v_b)^2(\tau+v_c)^2} &\leq (n-1)s^2\left[\frac{v_c(1+v_b)^2 - v_b(1+v_c)^2}{(1+v_c)^2(1+v_b)^2}\right], \\ \frac{(v_c-v_b)(v_bv_c-\tau^2)}{(\tau+v_b)^2(\tau+v_c)^2} &\leq (n-1)s^2\left[\frac{(v_c-v_b)(1-v_bv_c)}{(1+v_c)^2(1+v_b)^2}\right].\end{aligned}$$

Note that $v_c < v_b$ because, from (3), $\gamma_c < \gamma_b$. Therefore, the above inequality reduces to:

$$(n-1)s^2(\tau+v_b)^2(\tau+v_c)^2(1-v_bv_c) \leq (1+v_b)^2(1+v_c)^2(v_bv_c-\tau^2).$$

Dividing both sides by $(v_c)^3$,

$$(n-1)s^2(\tau+v_b)^2(\frac{\tau}{v_c}+1)^2(\frac{1}{v_c}-v_b) \leq (1+v_b)^2(\frac{1}{v_c}+1)^2(v_b-\frac{\tau^2}{v_c}). \tag{B.3}$$

Some algebraic manipulation establishes the following:

$$\begin{aligned} v_c &\equiv \frac{1}{1+\frac{1-\tau}{n\tau}}, \\ v_b &\equiv \frac{n-1+\tau}{n}, \\ \frac{\tau}{v_c}+1 &\equiv \frac{1}{n}\left[(n+1)+(n-1)\tau\right], \\ \frac{1}{v_c}-v_b &\equiv \frac{1-\tau^2}{n\tau}, \\ \frac{1}{v_c}+1 &\equiv \frac{1}{n\tau}\left[1+(2n-1)\tau\right], \\ v_b-\frac{\tau^2}{v_c} &\equiv \frac{(n-1)(1-\tau^2)}{n}. \end{aligned}$$

Substituting into inequality (B.3), we get:

$$s\sqrt{\tau}\left(\frac{\tau+v_b}{1+v_b}\right)\left[(n+1)+(n-1)\tau\right] \leq 1+(2n-1)\tau. \tag{B.4}$$

Since $0 \leq v_b$ and $0 \leq \tau \leq 1$, therefore:

$$\frac{\tau+v_b}{1+v_b} \leq \frac{\tau+1}{2}.$$

The above relaxation when substituted into inequality (B.4) yields:

$$s\sqrt{\tau}(1+\tau)\left[(n+1)+(n-1)\tau\right] \leq 2\left[1+(2n-1)\tau\right].$$

PART 2 (Proof of the threshold level $n < 28$):

It suffices to consider the case where $s = 1$. Total profit under Cournot competition is at least as high as total profit under Bertrand competition if the following inequality holds:

$$\sqrt{\tau}(1+\tau)\left[(n+1)+(n-1)\tau\right] \leq 2\left[1+(2n-1)\tau\right].$$

It is clear that this inequality holds for $\tau = 1$. Therefore, we restrict attention to $0 < \tau < 1$. The inequality can be expressed as:

$$\frac{1}{n} \geq h(\tau);$$

where $h(\tau) := \frac{2}{2-\sqrt{\tau}(1+\tau)} - \frac{1+\tau}{1-\tau}$. It is easy to establish that $h(\tau) \geq 0$ if and only if:

$$1 + \tau - 2\tau^{1/4} \geq 0.$$

The above fourth order polynomial can be expressed as:

$$1 + \tau - 2\tau^{1/4} = (\tau^{1/4} - 1)(\tau^{3/4} + \tau^{1/2} + \tau^{1/4} - 1);$$

Since $\tau < 1$, it is necessary and sufficient to examine the sign of polynomial $(\tau^{3/4}+\tau^{1/2}+\tau^{1/4}-1)$. This polynomial is strictly increasing in τ. Therefore, $\tau^{3/4} + \tau^{1/2} + \tau^{1/4} - 1 = 0$ has a unique root. It can be verified that $h(\tau)$ is positive for $\tau \in (0, 0.087]$ and negative for $\tau \in (0.088, 1)$. Therefore, in the latter range of τ values, $\tau \in (0.088, 1)$, Cournot total profit is at least as large as Bertrand total profit, regardless of n. In the former range of τ values, $\tau \in (0, 0.087]$, we need to establish an upper bound on $h(\tau)$ over the interval $(0, 0.087]$. h is concave over that interval as can be verified from its second derivative. It can also be verified that $h'(0.022) > 0$ and $h'(0.023) < 0$. This implies that the maximizer τ_{max} of function $h(\tau)$ lies in the interval $(0.022, 0.023)$. Therefore,

$$\begin{aligned} h(\tau) &\leq h(\tau_{max}), \\ &\leq h(0.022) + [\tau_{max} - 0.022]\, h'(0.022), \\ &\leq h(0.022) + 0.001\, h'(0.022), \\ &< 1/27. \end{aligned}$$

PART 3 (Proof of the threshold level $r < 0.90$):

The proof of PART 2 established that Cournot total profit is at least as high as Bertrand total profit for $\tau \in (0.088, 1)$ regardless of n. Recall that $\tau := \frac{(n-1)(1-r)}{n-(1-r)}$ which can be re-arranged to get $r := \frac{(n-1)(1-\tau)}{n-(1-\tau)}$. Note that r is decreasing in τ and non-decreasing in n. We have established in PART 2 that Cournot total profit is at least as high as Bertrand total profit for $n < 28$. Therefore, substituting $n = 28$ and $\tau = 0.088$, Cournot total profit is at least as high as Bertrand total profit for

$$r \leq \frac{(28-1)(1-0.088)}{28-(1-0.088)} = 0.909.$$

Proof of Theorem 3:

Using the notation introduced in section 3.2.1, recall that $\mathbf{Q}^b := (\mathbf{\Gamma}^b)^{1/2}(\mathbf{B}+\mathbf{\Gamma}^b)^{-1}$ and $\mathbf{Q}^c := (\mathbf{\Gamma}^c)^{1/2}(\mathbf{B}+\mathbf{\Gamma}^c)^{-1}$. Define $\mathbf{G} := \mathbf{\Gamma}^c(\mathbf{\Gamma}^b)^{-1}$ and let g_i denote the ith diagonal element of $\mathbf{G}$. Let $\mathbf{K} := \mathbf{G}^{-1/2}(\mathbf{I}+\mathbf{G})/2$. First, we show that $\mathbf{K}\mathbf{Q}^c \geq \mathbf{Q}^b$. It suffices to show that $(\mathbf{Q}^b)^{-1}\mathbf{K} = (\mathbf{B}+\mathbf{\Gamma}^b)(\mathbf{\Gamma}^b)^{-1/2}\mathbf{K} \geq (\mathbf{B}+\mathbf{\Gamma}^c)(\mathbf{\Gamma}^c)^{-1/2} = (\mathbf{Q}^c)^{-1}$ because $\mathbf{Q}^b$ and $\mathbf{Q}^c$ are nonnegative by the property of M-matrices. This inequality can be verified by checking the (positive) diagonal and the (non-negative) off-diagonal elements separately and by noting that $\mathbf{0} \leq \mathbf{G} \leq \mathbf{I}$. Let $[\mathbf{Q}^b]_i$ and $[\mathbf{Q}^c]_i$ denote, respectively, the ith rows of $\mathbf{Q}^b_i$ and $\mathbf{Q}^c_i$. As argued in section 3.2.1, $\overline{\pi}^b_i = ([\mathbf{Q}^b]_i\widetilde{\mathbf{d}})^2$ and $\overline{\pi}^c_i = ([\mathbf{Q}^c]_i\widetilde{\mathbf{d}})^2$. Therefore,

$$\frac{\overline{\pi}^c_i}{\overline{\pi}^b_i} = \frac{([\mathbf{Q}^c]_i\widetilde{\mathbf{d}})^2}{([\mathbf{Q}^b]_i\widetilde{\mathbf{d}})^2} \geq \frac{(\mathbf{K}^{-1}[\mathbf{Q}^b]_i\widetilde{\mathbf{d}})^2}{([\mathbf{Q}^b]_i\widetilde{\mathbf{d}})^2} = \left(\frac{2\sqrt{g_i}}{1+g_i}\right)^2 = \frac{4g_i}{(1+g_i)^2}.$$

The above lower bound is increasing in g_i. Note from (3), that $g_i \geq 1-r_i^2$. This concludes the proof.

Proof of Theorem 4:

Let $\pi(\mathbf{p}) = (\mathbf{d}-\mathbf{B}\mathbf{p})'(\mathbf{p}-\mathbf{c})$ denote the total profit for a given price vector $\mathbf{p}$. Assumptions A3 and A4 coupled with the symmetry of $\mathbf{B}$ imply that $\mathbf{B}$ is positive definite. Therefore, π is a concave function of $\mathbf{p}$ and

$$\pi(\overline{\mathbf{p}}^c) - \pi(\overline{\mathbf{p}}^b) \geq [\nabla\pi(\overline{\mathbf{p}}^c)]'(\overline{\mathbf{p}}^c - \overline{\mathbf{p}}^b).$$

Since $\overline{\mathbf{p}}^b \leq \overline{\mathbf{p}}^c$, therefore it suffices to show that $\nabla\pi(\overline{\mathbf{p}}^c) \geq \mathbf{0}$.

$$\begin{aligned}
\nabla\pi(\overline{\mathbf{p}}^c) &= \widetilde{\mathbf{d}} - 2\mathbf{B}\overline{\mathbf{p}}^c, \\
&= \widetilde{\mathbf{d}} - 2\mathbf{B}(\mathbf{B}+\mathbf{\Gamma}^c)^{-1}\widetilde{\mathbf{d}}, \\
&= \widetilde{\mathbf{d}} - 2(\mathbf{B}+\mathbf{\Gamma}^{\mathbf{c}}-\mathbf{\Gamma}^{\mathbf{c}})(\mathbf{B}+\mathbf{\Gamma}^c)^{-1}\widetilde{\mathbf{d}}, \\
&= 2(\mathbf{I}+\mathbf{B}(\mathbf{\Gamma}^c)^{-1})^{-1}\widetilde{\mathbf{d}} - \widetilde{\mathbf{d}}.
\end{aligned}$$

$\mathbf{I}+\mathbf{B}(\mathbf{\Gamma}^c)^{-1}$ is an M-matrix. Therefore, its inverse is non-negative. Therefore, the condition $2(\mathbf{I}+\mathbf{B}(\mathbf{\Gamma}^c)^{-1})^{-1}\widetilde{\mathbf{d}} \geq \widetilde{\mathbf{d}}$ is implied by the inequality $2\widetilde{\mathbf{d}} \geq (\mathbf{I}+\mathbf{B}(\mathbf{\Gamma}^c)^{-1})\widetilde{\mathbf{d}}$. Therefore, it suffices to show that $\widetilde{\mathbf{d}} \geq \mathbf{B}(\mathbf{\Gamma}^c)^{-1}\widetilde{\mathbf{d}}$. Note that the ith diagonal element of $\mathbf{\Gamma}^c$ is equal to $det(\mathbf{B})/det(\mathbf{B}_{ii})$ where $\mathbf{B}_{ii}$

is the submatrix obtained by deleting the ith row and the ith column of $\mathbf{B}$. Therefore, we need to show that:

$$\widetilde{\mathbf{d}}_i \det(\mathbf{B}) \geq |b_{ii}| \det(\mathbf{B}_{ii})\widetilde{\mathbf{d}}_i - \sum_{j \neq i} |b_{ij}| \det(\mathbf{B}_{jj})\widetilde{\mathbf{d}}_j,$$

for all i. In the above inequality we have used the fact that the determinant of a diagonally dominant M-matrix is positive. Using the Laplace expansion:

$$\begin{aligned} \det(\mathbf{B}) &= \sum_j (-1)^{i+j} b_{ij} \det(\mathbf{B}_{ij}), \\ &\geq |b_{ii}| \det(\mathbf{B}_{ii}) - \sum_{j \neq i} |b_{ij}| |\det(\mathbf{B}_{ij})|. \end{aligned}$$

Therefore, it suffices to show that

$$\sum_{j \neq i} |b_{ij}| \left[\det(\mathbf{B}_{jj})\widetilde{\mathbf{d}}_j - |\det(\mathbf{B}_{ij})| \widetilde{\mathbf{d}}_i \right] \geq 0,$$

for all i. It follows from a result by Ostrowski (1952) that $|\det(\mathbf{B}_{ij})| \leq r_i \det(\mathbf{B}_{jj})$. Therefore,

$$\det(\mathbf{B}_{jj})\widetilde{\mathbf{d}}_j - |\det(\mathbf{B}_{ij})| \widetilde{\mathbf{d}}_i \geq \det(\mathbf{B}_{jj}) \left[\widetilde{\mathbf{d}}_j - r_i \widetilde{\mathbf{d}}_i \right] \geq \det(\mathbf{B}_{jj}) \left[\widetilde{\mathbf{d}}_{min} - r_i \widetilde{\mathbf{d}}_i \right] \geq 0.$$