Appendix A: M-Matrices

A square matrix whose diagonal elements are positive and off-diagonal elements are non-positive is called a Z-matrix. One definition of an M-matrix is that it is a Z-matrix with the additional property that all leading principal minors are positive. It suffices to note that column (or row) diagonally dominant Z-matrices are M-matrices. A symmetric Z-matrix is an M-matrix if and only if it is positive definite.

M-matrices enjoy a number of structural properties. We refer the reader to Horn and Johnson (1991) for a detailed treatment. The following two properties in particular are used extensively in our proofs. Let $X$ be an M-matrix and $Y$ be a Z-matrix such that $X \leq Y$. Then:

1. $X^{-1}$ exists and $X^{-1} \geq 0$;
2. $Y$ is an M-matrix and $Y^{-1} \leq X^{-1}$.

Appendix B: Proofs of Statements

Proof of Theorem 1:

PART 1 (Proof of the inequality):

Define the following functions of a real variable $\gamma$:

$$\overline{p_i}(\gamma) = \frac{1}{1 + x + \gamma} \left[ \tilde{d}_i + \frac{x (e' \tilde{d})}{1 - (n-1)x + \gamma} \right] + c_i,$$
\[ \tilde{\mathbf{p}}(\gamma) = \gamma [\mathbf{w}(\gamma) - c_i], \]
\[ \pi(\gamma) = [\mathbf{w}(\gamma) - c_i] \tilde{\mathbf{p}}(\gamma). \]

Bertrand profit for firm \( i \) is given by \( \pi_i(\gamma_b) \), where \( \gamma_b = 1 \), and Cournot profit for firm \( i \) is given by \( \pi_i(\gamma_c) \), where \( \gamma_c = 1 - \frac{r^2}{(n-1)(1-r)\tau} \). Substituting \( \mathbf{w}(\gamma) \) and \( \mathbf{q}(\gamma) \) into \( \pi_i(\gamma) \) yields:
\[ \pi_i(\gamma) = \frac{\gamma (e' \tilde{d})^2}{(1+x+\gamma)^2} \left[ \lambda_i + \frac{x}{1-(n-1)x+\gamma} \right]^2. \]

It is sufficient to consider the square root of \( \pi_i(\gamma) \):
\[ \sqrt{\pi_i(\gamma)} = \frac{\sqrt{\gamma} e' \tilde{d}}{1+x+\gamma} \left[ \lambda_i + \frac{x}{1-(n-1)x+\gamma} - \frac{1}{n} + \frac{1}{n} \right], \]
\[ = \frac{e' \tilde{d}}{n} \left[ \left( \lambda_i n - 1 \right) \sqrt{\gamma} + \frac{\sqrt{\gamma}}{1+x+\gamma} \right], \]
\[ = \frac{e' \tilde{d}}{n \sqrt{1+x}} \left[ \left( \lambda_i n - 1 \right) \frac{1}{\tau + v} + \frac{\sqrt{v}}{\tau + v} \right], \]

where \( v := \gamma/(1+x) \). Therefore, setting \( v_b := \gamma_b/(1+x) \) and \( v_c := \gamma_c/(1+x) \), it suffices to establish sufficient conditions for:
\[ \frac{\sqrt{v_b}}{\tau + v_b} - \frac{\sqrt{v_c}}{\tau + v_c} \leq \left( \lambda_i n - 1 \right) \left[ \frac{\sqrt{v_c}}{1+v_c} - \frac{\sqrt{v_b}}{1+v_b} \right]. \tag{B.1} \]

Some algebraic manipulation establishes the following:
\[ v_c \equiv \frac{1}{1+\frac{1}{n\tau}}, \]
\[ v_b \equiv \frac{n-1+\tau}{n}, \]
\[ v_b v_c \equiv \theta^2 := \frac{\tau^2 + (n-1)\tau}{1+(n-1)\tau}. \]

Note that \( \theta > \tau \). Re-arranging inequality (B.1),
\[ \left( \lambda_i n - 1 \right) \frac{1-\theta}{(1+v_b)(1+v_c)} \leq \frac{\theta - \tau}{(\tau + v_b)(\tau + v_c)}. \]

Since \( (1+v_b)/(\tau + v_b) \geq 2/(1+\tau) \), a sufficient condition for the above inequality to hold is:
\[ \left( \lambda_i n - 1 \right) \frac{1-\theta}{\theta - \tau} \leq \left( \frac{2}{1+\tau} \right) \left( \frac{1+v_c}{\tau + v_c} \right), \]
\[ \left( \tilde{\lambda}_i, n - 1 \right) \frac{1 - \theta^2}{\theta^2 - \tau^2} \frac{\theta + \tau}{\theta + 1} \leq \left( \frac{2}{1 + \tau} \right) \frac{1}{\tau} \left( 1 - \frac{(1 - \tau) v_c}{1 + \frac{\tau}{\theta}} \right). \]

The last inequality can be re-arranged to yield:

\[ \left( \frac{\tilde{\lambda}_i}{n - 1} \right) \left( \frac{\theta + \tau}{\theta + 1} \right) \leq \left( \frac{2}{1 + \tau} \right) \left( 1 - \frac{(1 - \tau)}{(1 + \tau) + \frac{(1 - \tau)}{n}} \right). \]

**PART 2 (Proof of the threshold level \( n < 8 \)):**

It suffices to consider a firm with \( \tilde{\lambda}_i = 1 \). The firm’s Cournot profit is at least as high as its Bertrand profit if the following inequality holds:

\[ \frac{\theta + \tau}{\theta + 1} \leq \left( \frac{2}{1 + \tau} \right) \left( 1 - \frac{(1 - \tau)}{(1 + \tau) + \frac{(1 - \tau)}{n}} \right). \]

Holding \( \tau \) fixed, note that the right-hand side of this inequality is non-increasing in \( n \). Also, note that \( \theta \) is increasing in \( n \) and, therefore, the left-hand side of the inequality is increasing in \( n \). Therefore, it suffices to show that the inequality holds for any value of \( \tau \in (0, 1] \) and \( n = 7 \).

Substituting \( n = 7 \),

\[ \theta = \sqrt{\frac{\tau^2 + 6\tau}{1 + 6\tau}}. \]

and the inequality reduces to:

\[ \frac{\theta + \tau}{\theta + 1} \leq \left( \frac{2}{1 + \tau} \right) \left( \frac{13\tau + 1}{6\tau + 8} \right), \]

\[ (\sqrt{\tau^2 + 6\tau + \tau(1 + 6\tau)}(1 + \tau))(6\tau + 8) \leq 2(13\tau + 1)(\sqrt{\tau^2 + 6\tau} + \sqrt{1 + 6\tau}), \]

\[ \sqrt{\tau^2 + 6\tau}[(1 + \tau)(6\tau + 8) - 2(13\tau + 1)] \leq \sqrt{1 + 6\tau} [2(13\tau + 1) - \tau(1 + \tau)(6\tau + 8)], \]

\[ 6\sqrt{\tau^2 + 6\tau}(1 - \tau)^2 \leq \sqrt{1 + 6\tau}(2 + 18\tau - 14\tau^2 - 6\tau^3). \]

The above inequality clearly holds for \( \tau = 1 \). For \( 0 < \tau < 1 \),

\[ 6\sqrt{\tau^2 + 6\tau}(1 - \tau) \leq \sqrt{1 + 6\tau}(6\tau^2 + 20\tau + 2). \]

Squaring both sides and simplifying, the inequality finally reduces to:

\[ 0 \leq 54\tau^5 + 360\tau^4 + 660\tau^3 + 325\tau^2 - 28\tau + 1. \quad (B.2) \]
It suffices to show that $0 \leq 325\tau^2 - 28\tau + 1$ because the remaining polynomial, $54\tau^5 + 360\tau^4 + 660\tau^3$, has positive coefficients and is therefore non-negative for $\tau \in [0, 1]$. Note, however, that the discriminant of $325\tau^2 - 28\tau + 1$ is negative. Therefore, the inequality (B.2) is satisfied for $\tau \in [0, 1]$.

**PART 3 (Proof of the threshold level $r < 0.739$):**

It suffices to consider a firm with $\tilde{\lambda}_i = 1$. The firm’s Cournot profit is at least as high as its Bertrand profit if the following inequality holds:

$$\frac{\theta + \tau}{\theta + 1} \leq \left( \frac{2}{1 + \tau} \right) \left( 1 - \frac{(1 - \tau)}{(1 + \tau) + \frac{(1-\tau)}{n}} \right).$$

Note that the right-hand side of this inequality is non-increasing in $n$. Taking the limits as $n \to \infty$,

$$\frac{\theta + \tau}{\theta + 1} \leq \left( \frac{2}{1 + \tau} \right) \left( 1 - \frac{(1 - \tau)}{(1 + \tau)} \right).$$

Note that $(\theta + \tau)/(\theta + 1) \leq (1 + \tau)/2$. Re-arranging, we get:

$$\tau - 2\sqrt{\tau} + 1 \leq 0.$$

The roots of the left-hand side of the above inequality are $(-1 - \sqrt{5})^3/8$, $(-1 + \sqrt{5})^3/8$, and 1. Therefore, the firm’s Cournot profit is at least as high as Bertrand profit if $\tau \geq (-1 + \sqrt{5})^3/8$.

Recall that $\tau := \frac{(n-1)(1-r)}{n-(1-r)}$ which can be re-arranged to get $r := \frac{(n-1)(1-\tau)}{n-(1-\tau)}$. For a fixed value of $\tau$, $r$ is non-decreasing in $n$. Given the result of PART 2, we can replace $n = 8$ and $\tau = (-1 + \sqrt{5})^3/8$ in the expression for $r$ to get $r \leq 0.739$ as a sufficient condition for a firm’s Cournot profit to be at least as high as its Bertrand profit.

**Proof of Theorem 2:**

**PART 1 (Proof of the inequality):**

Define the following functions of a real variable $\gamma$:

$$\mathbf{p}(\gamma) = \frac{1}{1 + x + \gamma} \left[ \tilde{d} + \frac{x(e\tilde{d})}{1 - (n-1)x + \gamma} \right] + c,$$

$$\mathbf{q}(\gamma) = \gamma[\mathbf{p}(\gamma) - c],$$

$$\pi(\gamma) = [\mathbf{p}(\gamma) - c]'\mathbf{q}(\gamma).$$
Bertrand total profit is given by \( \pi(\gamma^b) \), where \( \gamma^b = 1 \), and Cournot total profit is given by \( \pi(\gamma^c) \), where \( \gamma^c = 1 - \frac{r^2}{n - 1} \). Substituting \( \bar{p}(\gamma) \) and \( \bar{q}(\gamma) \) into \( \pi(\gamma) \) yields:

\[
\pi(\gamma) = \frac{\gamma}{(1 + x + \gamma)^2} \left[ d'e' + \frac{2x(1 + x + \gamma) - nx^2}{(1 - (n - 1)x + \gamma)^2} (e'd')^2 \right].
\]

Letting \( \alpha = d'e' \) and \( \beta = (e'd')^2 \),

\[
\pi(\gamma) = \frac{\gamma}{(1 + x + \gamma)^2} \left[ \alpha + \frac{2x(1 + x + \gamma) - nx^2}{(1 - (n - 1)x + \gamma)^2} \beta - \frac{\beta}{n} \frac{\gamma}{n} \right],
\]

\[
= \frac{\beta}{n} \left[ (\frac{n\alpha}{\beta} - 1) \frac{\gamma}{(1 + x + \gamma)^2} \right] \left[ (1 - (n - 1)x + \gamma)^2 \right],
\]

\[
= \frac{\beta}{n(1 + x)^2} f(\frac{\gamma}{1 + x}),
\]

where

\[
f(v) = \left( \frac{n\alpha}{\beta} - 1 \right) \frac{v}{(1 + v)^2} + \frac{v}{(\gamma + v)^2},
\]

\[
= (n - 1)s^2 \frac{v}{(1 + v)^2} + \frac{v}{(\gamma + v)^2}.
\]

The last equality follows from the fact that \( \frac{n\alpha}{\beta} - 1 = (c.v.)^2 = (n - 1)s^2 \).

We need to derive a sufficient condition for \( \pi(\gamma^b) \leq \pi(\gamma^c) \). This is equivalent to showing that \( f(v_b) \leq f(v_c) \) where \( v_b = \frac{2\alpha}{1 + x} \) and \( v_c = \frac{\gamma}{1 + x} \). That is, we need to establish when the following inequality holds:

\[
(n - 1)s^2 \frac{v_b}{(1 + v_b)^2} + \frac{v_b}{(\gamma + v_b)^2} \leq (n - 1)s^2 \frac{v_c}{(1 + v_c)^2} + \frac{v_c}{(\gamma + v_c)^2}.
\]

Rearranging yields:

\[
\frac{v_b}{(\gamma + v_b)^2} - \frac{v_c}{(\gamma + v_c)^2} \leq (n - 1)s^2 \left[ \frac{v_c}{(1 + v_c)^2} - \frac{v_b}{(1 + v_b)^2} \right],
\]

\[
\frac{v_b(\gamma + v_b)^2 - v_c(\gamma + v_c)^2}{(\gamma + v_b)^2(\gamma + v_c)^2} \leq (n - 1)s^2 \left[ \frac{v_c(1 + v_c)^2 - v_b(1 + v_b)^2}{(1 + v_c)^2(1 + v_b)^2} \right],
\]

\[
\frac{(v_c - v_b)(v_b v_c - \tau^2)}{(\gamma + v_b)^2(\gamma + v_c)^2} \leq (n - 1)s^2 \left[ \frac{(v_c - v_b)(1 + v_c)}{(1 + v_c)^2(1 + v_b)^2} \right].
\]

Note that \( v_c < v_b \) because, from (3), \( \gamma_c < \gamma_b \). Therefore, the above inequality reduces to:

\[
(n - 1)s^2(\gamma + v_b)^2(\gamma + v_c)^2(1 - v_b v_c) \leq (1 + v_b)^2(1 + v_c)^2(v_b v_c - \tau^2).
\]
Dividing both sides by \((v_c)^3\),

\[
(n-1)s^2(\tau + v_b)^2(\frac{\tau}{v_c} + 1)^2(\frac{1}{v_c} - v_b) \leq (1 + v_b)^2(\frac{1}{v_c} + 1)^2(v_b - \frac{\tau^2}{v_c}).
\]  \(\text{(B.3)}\)

Some algebraic manipulation establishes the following:

\[
\begin{align*}
v_c &\equiv \frac{1}{1 + \frac{\tau}{n\tau}}, \\
v_b &\equiv \frac{n - 1 + \tau}{n}, \\
\frac{\tau}{v_c} + 1 &\equiv \frac{1}{n}[(n+1) + (n-1)\tau], \\
\frac{1}{v_c} - v_b &\equiv \frac{1 - \tau^2}{n\tau}, \\
\frac{1}{v_c} + 1 &\equiv \frac{1}{n\tau}[1 + (2n - 1)\tau], \\
v_b - \frac{\tau^2}{v_c} &\equiv \frac{(n-1)(1-\tau^2)}{n}.
\end{align*}
\]

Substituting into inequality (B.3), we get:

\[
s\sqrt{\tau}\left(\frac{\tau + v_b}{1 + v_b}\right)[(n+1) + (n-1)\tau] \leq 1 + (2n - 1)\tau. \tag{B.4}
\]

Since \(0 \leq v_b\) and \(0 \leq \tau \leq 1\), therefore:

\[
\frac{\tau + v_b}{1 + v_b} \leq \frac{\tau + 1}{2}.
\]

The above relaxation when substituted into inequality (B.4) yields:

\[
s\sqrt{\tau}(1 + \tau)[(n+1) + (n-1)\tau] \leq 2[1 + (2n - 1)\tau].
\]

PART 2 (Proof of the threshold level \(n < 28\)):

It suffices to consider the case where \(s = 1\). Total profit under Cournot competition is at least as high as total profit under Bertrand competition if the following inequality holds:

\[
\sqrt{\tau}(1 + \tau)[(n+1) + (n-1)\tau] \leq 2[1 + (2n - 1)\tau].
\]

It is clear that this inequality holds for \(\tau = 1\). Therefore, we restrict attention to \(0 < \tau < 1\). The inequality can be expressed as:

\[
\frac{1}{n} \geq h(\tau);
\]
where \( h(\tau) := \frac{2}{2 - \sqrt{\tau(1+\tau)}} - \frac{1+\tau}{1-\tau}. \) It is easy to establish that \( h(\tau) \geq 0 \) if and only if:

\[
1 + \tau - 2\tau^{1/4} \geq 0.
\]

The above fourth order polynomial can be expressed as:

\[
1 + \tau - 2\tau^{1/4} = (\tau^{1/4} - 1)(\tau^{3/4} + \tau^{1/2} + \tau^{1/4} - 1);
\]

Since \( \tau < 1 \), it is necessary and sufficient to examine the sign of polynomial \((\tau^{3/4} + \tau^{1/2} + \tau^{1/4} - 1)\). This polynomial is strictly increasing in \( \tau \). Therefore, \( \tau^{3/4} + \tau^{1/2} + \tau^{1/4} - 1 = 0 \) has a unique root.

It can be verified that \( h(\tau) \) is positive for \( \tau \in (0, 0.087] \) and negative for \( \tau \in (0.088, 1) \). Therefore, in the latter range of \( \tau \) values, \( \tau \in (0.088, 1) \), Cournot total profit is at least as large as Bertrand total profit, regardless of \( n \). In the former range of \( \tau \) values, \( \tau \in (0, 0.087] \), we need to establish an upper bound on \( h(\tau) \) over the interval \((0, 0.087]\). \( h \) is concave over that interval as can be verified from its second derivative. It can also be verified that \( h'(0.022) > 0 \) and \( h'(0.023) < 0 \). This implies that the maximizer \( \tau_{\text{max}} \) of function \( h(\tau) \) lies in the interval \((0.022, 0.023)\). Therefore,

\[
h(\tau) \leq h(\tau_{\text{max}}),
\]

\[
\leq h(0.022) + [\tau_{\text{max}} - 0.022] h'(0.022),
\]

\[
\leq h(0.022) + 0.001 h'(0.022),
\]

\[
< 1/27.
\]

PART 3 (Proof of the threshold level \( r < 0.90 \)):

The proof of PART 2 established that Cournot total profit is at least as high as Bertrand total profit for \( \tau \in (0.088, 1) \) regardless of \( n \). Recall that \( \tau := \frac{(n-1)(1-r)}{n(1-r)} \) which can be re-arranged to get \( r := \frac{(n-1)(1-\tau)}{n(1-\tau)} \). Note that \( r \) is decreasing in \( \tau \) and non-decreasing in \( n \). We have established in PART 2 that Cournot total profit is at least as high as Bertrand total profit for \( n < 28 \). Therefore, substituting \( n = 28 \) and \( \tau = 0.088 \), Cournot total profit is at least as high as Bertrand total profit for

\[
r \leq \frac{(28-1)(1-0.088)}{28 - (1-0.088)} = 0.909.
\]
Proof of Theorem 3:

Using the notation introduced in section 3.2.1, recall that \( Q^b := (\Gamma^b)^{1/2} (B + \Gamma^b)^{-1} \) and \( Q^c := (\Gamma^c)^{1/2} (B + \Gamma^c)^{-1} \). Define \( G := \Gamma^c (\Gamma^b)^{-1} \) and let \( g_i \) denote the \( i \)th diagonal element of \( G \). Let \( K := G^{-1/2} (I + G)/2 \). First, we show that \( K \) is positive definite. Therefore, its inverse is non-negative. The rea...
is the submatrix obtained by deleting the $i$th row and the $i$th column of $B$. Therefore, we need to show that:

$$\tilde{d}_i \det(B) \geq |b_{ii}| \det(B_{ii}) \tilde{d}_i - \sum_{j \neq i} |b_{ij}| \det(B_{jj}) \tilde{d}_j,$$

for all $i$. In the above inequality we have used the fact that the determinant of a diagonally dominant M-matrix is positive. Using the Laplace expansion:

$$\det(B) = \sum_j (-1)^{i+j} b_{ij} \det(B_{ij}),$$

$$\geq |b_{ii}| \det(B_{ii}) - \sum_{j \neq i} |b_{ij}| |\det(B_{ij})|.$$ 

Therefore, it suffices to show that

$$\sum_{j \neq i} |b_{ij}| \left[ \det(B_{jj}) \tilde{d}_j - |\det(B_{ij})| \tilde{d}_i \right] \geq 0,$$

for all $i$. It follows from a result by Ostrowski (1952) that $|\det(B_{ij})| \leq r_i \det(B_{jj})$. Therefore,

$$\det(B_{jj}) \tilde{d}_j - |\det(B_{ij})| \tilde{d}_i \geq \det(B_{jj}) \left[ \tilde{d}_j - r_i \tilde{d}_i \right] \geq \det(B_{jj}) \left[ \tilde{d}_{\min} - r_i \tilde{d}_i \right] \geq 0.$$