# Proofs, Tables and Figures

In this electronic companion to the paper, we provide proofs of the theorems and lemmas. This companion also contains several accompanying tables and figures to the paper.

**Table EC.1** Range of critical fractile values where Assumption 1 holds.

| Distribution | When is Assumption 1 satisfied? | Notes |
|---|---|---|
| Normal$(\mu,\sigma)$ | $\frac{b}{b+h} \geq \frac{1}{2}$ | |
| Exponential$(\lambda)$ | $\frac{b}{b+h} \geq 0$ | |
| Lognormal$(\mu,\sigma)$ | $\frac{b}{b+h} \geq \frac{1}{2} + \frac{1}{2}\text{erf}\left(-\frac{\sigma}{2}\right)$ | erf: error function |
| Pareto$(x_m,\alpha)$ | $\frac{b}{b+h} \geq 0$ | |
| Uniform$(A,B)$ | $\frac{b}{b+h} \geq 0$ | |
| Gamma$(\alpha,\beta)$ | $\frac{b}{b+h} \geq \frac{1}{\Gamma(\alpha)}\gamma(\alpha,\alpha-1)$ | $\Gamma$: gamma function, $\gamma$: incomplete gamma function |
| Beta$(\alpha,\beta)$ | $\frac{b}{b+h} \geq \frac{B\left(\frac{\alpha-1}{\alpha+\beta-2};\alpha,\beta\right)}{B(\alpha,\beta)}$ | $B$: beta function |
| Power Law$(\alpha)$ | $\frac{b}{b+h} \geq 0$ | |
| Logistic$(\mu,s)$ | $\frac{b}{b+h} \geq \frac{1}{2}$ | |
| GEV$(\mu,\sigma,\xi)$ | $\frac{b}{b+h} \geq e^{-1-\xi}$ | for $\xi \geq 0$ |
| Chi$(k)$ | $\frac{b}{b+h} \geq P\left(\frac{k}{2},\frac{k-1}{2}\right)$ | for $k \geq 1$; $P$: regularized gamma function |
| Chi-squared$(k)$ | $\frac{b}{b+h} \geq \begin{cases} \frac{1}{\Gamma\left(\frac{k}{2}\right)}\gamma\left(\frac{k}{2},\frac{k-2}{2}\right), & \text{if } k \geq 2 \\ 0, & \text{if } k < 2 \end{cases}$ | |
| Laplace$(\mu,\beta)$ | $\frac{b}{b+h} \geq \frac{1}{2}$ | |
| Weibull$(\lambda,k)$ | $\frac{b}{b+h} \geq \begin{cases} 1-e^{-\frac{k-1}{k}}, & \text{if } k \geq 1 \\ 0 & \text{if } k < 1 \end{cases}$ | |

**Table EC.2** Average errors (%) with samples from an exponential distribution.

(a) Sample average approximation

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 2.39 | 1.83 | 2.08 | 2.24 | 2.62 | 3.22 | 4.05 | 4.67 | 7.65 | 10.87 | 33.76 |
| 50 | 0.77 | 0.73 | 0.81 | 0.87 | 1.35 | 1.49 | 1.93 | 2.38 | 3.10 | 7.33 | 16.89 |
| 100 | 0.54 | 0.34 | 0.48 | 0.60 | 0.70 | 0.91 | 0.96 | 1.50 | 2.03 | 3.24 | 8.56 |
| 200 | 0.27 | 0.23 | 0.27 | 0.29 | 0.34 | 0.40 | 0.49 | 0.64 | 1.22 | 2.22 | 4.36 |

(b) Distribution fitting

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 1.88 | 1.54 | 1.54 | 1.69 | 2.03 | 2.60 | 3.37 | 4.26 | 5.81 | 9.64 | 40.06 |
| 50 | 0.65 | 0.64 | 0.69 | 0.80 | 0.99 | 1.23 | 1.53 | 1.90 | 2.72 | 4.88 | 22.93 |
| 100 | 0.36 | 0.34 | 0.39 | 0.46 | 0.57 | 0.73 | 0.96 | 1.33 | 1.91 | 2.62 | 9.03 |
| 200 | 0.21 | 0.20 | 0.21 | 0.24 | 0.28 | 0.34 | 0.43 | 0.59 | 0.94 | 1.64 | 7.25 |

**Figure EC.1** Upper bound for a log-concave distribution with $\frac{b}{b+h}$ quantile $q^*$.

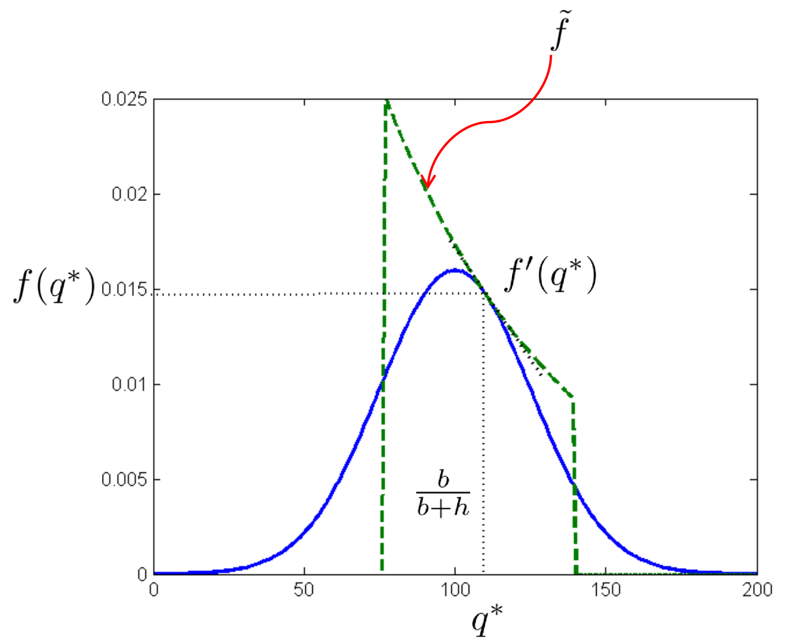


**Table EC.3** Average errors (%) with samples from a normal distribution.

(a) Sample average approximation

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 6.03 | 3.84 | 3.81 | 3.11 | 2.60 | 2.95 | 3.50 | 4.91 | 6.23 | 8.71 | 42.85 |
| 50 | 2.31 | 1.69 | 1.62 | 1.58 | 1.41 | 1.60 | 1.59 | 2.06 | 3.26 | 4.57 | 13.76 |
| 100 | 1.63 | 1.15 | 0.92 | 0.86 | 0.83 | 0.75 | 0.92 | 1.08 | 1.56 | 2.18 | 5.94 |
| 200 | 0.81 | 0.45 | 0.38 | 0.36 | 0.30 | 0.29 | 0.38 | 0.47 | 0.81 | 1.41 | 3.65 |

(b) Distribution fitting

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 4.65 | 3.53 | 3.07 | 2.73 | 2.62 | 2.77 | 3.16 | 3.74 | 5.48 | 12.83 | 75.12 |
| 50 | 1.91 | 1.43 | 1.27 | 1.20 | 1.24 | 1.38 | 1.60 | 1.87 | 2.53 | 4.41 | 18.77 |
| 100 | 1.13 | 0.90 | 0.78 | 0.71 | 0.68 | 0.69 | 0.76 | 0.89 | 1.17 | 1.75 | 6.59 |
| 200 | 0.47 | 0.36 | 0.28 | 0.25 | 0.25 | 0.27 | 0.33 | 0.42 | 0.63 | 1.03 | 3.92 |

**Table EC.4** Average errors (%) with samples from a Pareto distribution.

(a) Sample average approximation

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 0.88 | 0.69 | 0.83 | 0.93 | 1.14 | 1.59 | 2.18 | 3.12 | 6.70 | 28.33 | 34.35 |
| 50 | 0.28 | 0.28 | 0.31 | 0.37 | 0.60 | 0.73 | 1.02 | 1.39 | 2.28 | 6.12 | 33.39 |
| 100 | 0.19 | 0.13 | 0.19 | 0.24 | 0.29 | 0.39 | 0.45 | 0.83 | 1.54 | 3.28 | 39.74 |
| 200 | 0.09 | 0.08 | 0.10 | 0.11 | 0.14 | 0.17 | 0.24 | 0.35 | 0.80 | 1.97 | 6.86 |

(b) Distribution fitting

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 0.70 | 0.61 | 0.69 | 0.79 | 0.96 | 1.24 | 1.68 | 2.47 | 4.83 | 9.69 | 40.31 |
| 50 | 0.25 | 0.25 | 0.28 | 0.32 | 0.39 | 0.53 | 0.76 | 1.15 | 2.25 | 4.52 | 18.92 |
| 100 | 0.15 | 0.14 | 0.16 | 0.20 | 0.24 | 0.31 | 0.41 | 0.62 | 1.34 | 2.85 | 11.71 |
| 200 | 0.08 | 0.08 | 0.09 | 0.10 | 0.12 | 0.15 | 0.19 | 0.30 | 0.72 | 1.65 | 6.97 |

**Table EC.5** Average errors (%) with samples from a Beta distribution.

(a) Sample average approximation

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 5.15 | 4.80 | 4.07 | 3.07 | 3.06 | 2.63 | 2.92 | 2.90 | 4.30 | 4.27 | 14.99 |
| 50 | 2.69 | 2.28 | 2.15 | 1.99 | 1.63 | 1.41 | 1.26 | 1.25 | 1.34 | 1.88 | 2.47 |
| 100 | 1.86 | 1.17 | 0.94 | 0.82 | 0.88 | 0.85 | 0.73 | 0.77 | 0.79 | 0.89 | 0.78 |
| 200 | 1.11 | 0.59 | 0.40 | 0.35 | 0.36 | 0.35 | 0.32 | 0.31 | 0.32 | 0.30 | 0.41 |

(b) Distribution fitting

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 5.62 | 4.42 | 3.41 | 2.70 | 2.38 | 2.32 | 2.34 | 2.39 | 3.40 | 7.13 | 35.94 |
| 50 | 2.90 | 2.24 | 1.77 | 1.50 | 1.43 | 1.49 | 1.61 | 1.65 | 1.68 | 2.88 | 9.40 |
| 100 | 1.43 | 1.06 | 0.83 | 0.69 | 0.62 | 0.61 | 0.62 | 0.63 | 0.64 | 0.99 | 4.45 |
| 200 | 0.70 | 0.43 | 0.30 | 0.26 | 0.25 | 0.25 | 0.26 | 0.26 | 0.24 | 0.37 | 2.29 |

**Table EC.6** Average errors (%) with samples from a mixed normal distribution.

(a) Sample average approximation

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 3.99 | 2.64 | 1.89 | 2.14 | 6.17 | 3.98 | 5.52 | 11.72 | 5.78 | 3.84 | 4.41 |
| 50 | 1.29 | 0.86 | 0.61 | 0.39 | 0.38 | 0.35 | 0.53 | 0.79 | 1.81 | 1.62 | 4.26 |
| 100 | 0.74 | 0.43 | 0.35 | 0.27 | 0.39 | 0.45 | 0.33 | 0.39 | 0.55 | 0.71 | 2.51 |
| 200 | 0.37 | 0.21 | 0.16 | 0.13 | 0.08 | 0.08 | 0.12 | 0.19 | 0.22 | 0.59 | 1.47 |

(b) Distribution fitting

| | Critical quantile | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample size | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 | 0.99 |
| 25 | 2.96 | 1.67 | 1.86 | 3.32 | 3.09 | 2.36 | 10.46 | 16.65 | 6.55 | 13.74 | 32.75 |
| 50 | 1.56 | 1.14 | 0.52 | 0.47 | 0.33 | 0.54 | 0.49 | 2.18 | 0.24 | 1.84 | 25.18 |
| 100 | 0.90 | 0.40 | 0.35 | 1.08 | 0.81 | 0.18 | 0.16 | 1.85 | 0.33 | 1.19 | 7.76 |
| 200 | 0.69 | 0.38 | 0.15 | 0.59 | 0.42 | 0.48 | 0.57 | 0.37 | 0.94 | 4.61 | 2.97 |

## EC.1. Proof of Theorem 2

As a preliminary for the proof, let us first state a version of Bernstein's inequality (Bernstein 1927):

THEOREM EC.1 **(Bernstein's inequality)**. *Let $X^1, X^2, \ldots, X^N$ be i.i.d. random variables such that $|X^1| \le c$ almost surely, and $Var(X^1) = \sigma^2$. Then, for any $t > 0$,*

$$\Pr\left(\frac{1}{N}\sum_{i=1}^{N} X^i - E[X^1] \ge t\right) \le \exp\left(\frac{-Nt^2}{2\sigma^2 + 2tc/3}\right).$$

For the proof of Theorem 2, we will require the following proposition.

PROPOSITION EC.1. *Suppose $\hat{Q}_N$ is the $\frac{b}{b+h}$ quantile of a random sample from $D$ with size $N$. Then, for any $\gamma > 0$,*

$$\Pr\left(\partial_- C(\hat{Q}_N) \le \gamma \text{ and } \partial_+ C(\hat{Q}_N) \ge -\gamma\right) \ge 1 - 2\exp\left(\frac{-3N\gamma^2}{6bh + 8\gamma(b+h)}\right).$$

*Proof.* *Let $\bar{F}$ be the complementary cdf of $D$, i.e., $\bar{F}(q) = \Pr(D \ge q) = 1 - F(q) + \Pr(D = q)$. For a random sample $\{D^1, \ldots, D^N\}$ drawn from $D$, let $\hat{Q}_N$ be the $\frac{b}{b+h}$ sample quantile. Define*

$$\hat{F}_N(q) \triangleq \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{[D^i \le q]},$$
$$\hat{\bar{F}}_N(q) \triangleq \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{[D^i \ge q]}.$$

*For simplicity, define $\alpha \triangleq \frac{\gamma}{b+h}$ and $\beta \triangleq \frac{b}{b+h}$. Define the events $B \triangleq [\partial_+ C(\hat{Q}_N) < -\gamma] = [F(\hat{Q}_N) < \beta - \alpha]$ and $L \triangleq [\partial_- C(\hat{Q}_N) > \gamma] = [\bar{F}(\hat{Q}_N) < 1 - \beta - \alpha]$. To prove Proposition EC.1, we need to find an upper bound for $\Pr(B)$ and for $\Pr(L)$.*

*Define the quantile $q_1 \triangleq \inf\{q : F(q) \ge \beta - \alpha\}$. Since $F$ is nondecreasing, we have that $B = [\hat{Q}_N < q_1]$. Consider a monotonically decreasing, nonnegative sequence $\{\tau^k\}_{k=1}^{\infty}$, where $\tau^k \downarrow 0$. Define the sequence of events $\{B_k\}_{k=1}^{\infty}$, where*

$$B_k \triangleq [\hat{Q}_N \le q_1 - \tau^k] = [\hat{F}_N(q_1 - \tau^k) \ge \beta].$$

*Note that since $\hat{F}_N(q_1 - \tau^k) \le \hat{F}_N(q_1 - \tau^{k+1})$, then it follows that $B_k \subseteq B_{k+1}$. Thus, we have that $B_k \uparrow \lim_{k\to\infty} B_k \triangleq \bar{B}$, which implies $\Pr(B_k) \uparrow \Pr(\bar{B})$. Note also that $B \subseteq \bar{B}$, thus $\Pr(B) \le \Pr(\bar{B})$.*

*From the definition of $q_1$, observe that for every $k \ge 1$, there exists $\varepsilon_k > \alpha$ such that $F(q_1 - \tau^k) = \beta - \varepsilon_k < \beta - \alpha$. Note that*

$$F(q_1 - \tau^k)\left(1 - F(q_1 - \tau^k)\right) < (\beta - \alpha)(1 - \beta + \varepsilon_k). \tag{EC.1}$$

*Thus, we have that*

$$\begin{aligned}\Pr(B_k) &= \Pr(\hat{F}_N(q_1-\tau^k)\geq\beta),\\ &= \Pr(\hat{F}_N(q_1-\tau^k)-F(q_1-\tau^k)\geq\varepsilon_k),\\ &\leq \exp\left(\frac{-N\varepsilon_k^2/2}{F(q_1-\tau^k)(1-F(q_1-\tau^k))+\frac{\varepsilon_k}{3}}\right), \qquad \text{(EC.2)}\\ &\leq \exp\left(\frac{-N\varepsilon_k/2}{\frac{1}{\varepsilon_k}(\beta-\alpha)(1-\beta)+\beta-\alpha+\frac{1}{3}}\right), \qquad \text{(EC.3)}\end{aligned}$$

*where (EC.2) follows from Bernstein's inequality and (EC.3) follows from inequality (EC.1). Now, since* $\varepsilon_k>\alpha$, *for all* $k\geq 1$, *we have that*

$$\begin{aligned}\Pr(B_k) &\leq \exp\left(\frac{-N\alpha/2}{\frac{1}{\alpha}\beta(1-\beta)-\frac{2}{3}+2\beta-\alpha}\right),\\ &\leq \exp\left(\frac{-N\alpha/2}{\frac{1}{\alpha}\beta(1-\beta)+\frac{4}{3}-2\min(\beta,1-\beta)-\alpha}\right),\\ &\leq \exp\left(\frac{-N\alpha/2}{\frac{1}{\alpha}\beta(1-\beta)+\frac{4}{3}}\right)=\exp\left(\frac{-3N\gamma^2}{6bh+8\gamma(b+h)}\right)\triangleq\delta.\end{aligned}$$

*Thus,* $\Pr(B)\leq\Pr(\bar{B})\leq\delta$. *In fact, by going through a similar argument, we can show that* $\Pr(L)\leq\delta$. *Thus, by the union bound, we have that*

$$\Pr\left(\partial_-C(\hat{Q}_N)>\gamma \ \textit{or}\ \partial_+C(\hat{Q}_N)<-\gamma\right)=\Pr(B\cup L)\leq\Pr(B)+\Pr(L)\leq 2\delta,$$

*proving Proposition EC.1. Q.E.D.*

We can now proceed with the proof of Theorem 2. Note that $S_\epsilon^{LRS}$ consists of all $q$ for which $\partial_-C(q)\leq\gamma$ and $\partial_+C(q)\geq-\gamma$, with $\gamma=\frac{\epsilon}{3}\min(b,h)$. From Proposition EC.1, the SAA solution from a random sample with size $N$ lies in $S_\epsilon^{LRS}$ with probability at least

$$\begin{aligned}1-2\exp\left(\frac{-N\epsilon^2\min\{b,h\}^2}{18bh+8\epsilon(b+h)\min\{b,h\}}\right) &= 1-2\exp\left(\frac{-N\epsilon^2\min\{b,h\}}{18\max\{b,h\}+8\epsilon(b+h)}\right).\\ &\geq 1-2\exp\left(\frac{-N\epsilon^2}{18+8\epsilon}\cdot\frac{\min\{b,h\}}{b+h}\right).\end{aligned}$$

## EC.2. Proof of Theorem 3

Since $C$ is convex, $S_\epsilon^f\cap[q^*,\infty)$ can be equivalently expressed as $\{q: C'(q)\leq C'(\overline{q})$ and $q\geq q^*\}$. Note that,

$$\begin{aligned}C'(\overline{q}) &= (b+h)(F(\overline{q})-F(q^*))=(b+h)\left[(\overline{q}-q^*)f(q^*)+O(\overline{q}-q^*)^2\right]\\ &= \sqrt{2\epsilon bh\Delta(q^*)f(q^*)}+O(\epsilon), \qquad \text{(EC.4)}\end{aligned}$$

which follows from Taylor series approximation and from the definition of $\overline{q}$ in (7).

To prove Theorem 3, note that the event that $\tilde{Q}_N^\alpha \in S_\epsilon^f \cap [q^*, \infty)$, where $\alpha = C'(\overline{q})$, is equivalent to the intersection of events $[\tilde{Q}_N^\alpha \geq q^*]$ and $[C'(\tilde{Q}_N^\alpha) \leq \alpha]$. We will prove an upper bound on the probability of $[\tilde{Q}_N^\alpha < q^*]$ and on the probability of $[C'(\tilde{Q}_N^\alpha) > \alpha]$. It follows similar lines to the proof of Lemma 3.5 in Levi et al. (2007), except we will use Bernstein's inequality instead of Hoeffding's inequality.

Define $\beta \triangleq \frac{b}{b+h}$ and $\gamma \triangleq \frac{1}{2}\frac{\alpha}{b+h}$. First, let us bound the probability of $B \triangleq [\tilde{Q}_N^\alpha < q^*]$. For a real-valued sequence $\{\tau^k\}_{k=1}^\infty$ where $\tau^k \downarrow 0$, define

$$B_k \triangleq [\tilde{Q}_N^\alpha \leq q^* - \tau^k] = \left[-b + (b+h)\hat{F}_N(q^* - \tau^k) \geq \frac{\alpha}{2}\right] = [\hat{F}_N(q^* - \tau^k) \geq \beta + \gamma].$$

Note that since $\hat{F}_N$ is monotonically increasing, it follows that $B_k \subseteq B_{k+1}$. Thus, if $\bar{B}$ is the limiting event of the sequence of events $\{B_k\}_{k=1}^\infty$, then $B_k \uparrow \bar{B}$, implying that $\Pr(B_k) \uparrow \Pr(\bar{B})$. Note also that $B \subseteq \bar{B}$, thus $\Pr(B) \leq \Pr(\bar{B})$. Therefore, to bound $\Pr(B)$, we only need to find a uniform upper bound for $\Pr(B_k)$.

Note that for any $k \geq 1$, there exists $\varepsilon^k > 0$ such that $F(q^* - \tau^k) = \beta - \varepsilon^k$. Thus,

$$F(q^* - \tau^k)\left(1 - F(q^* - \tau^k)\right) = (\beta - \varepsilon^k)(1 - \beta + \varepsilon^k) < \beta(1 - \beta + \varepsilon^k).$$

From Bernstein's inequality, we have that

$$\begin{aligned}
\Pr(B_k) &= \Pr\left(\hat{F}_N(q^* - \tau^k) \geq \beta + \gamma\right) = \Pr\left(\hat{F}_N(q^* - \tau^k) - F(q^* - \tau^k) \geq \gamma + \varepsilon^k\right) \\
&\leq \exp\left(\frac{-N(\gamma + \varepsilon^k)^2}{2F(q^* - \tau^k)(1 - F(q^* - \tau^k)) + \frac{2}{3}(\gamma + \varepsilon^k)}\right) \\
&= \exp\left(\frac{-N(\gamma + \varepsilon^k)}{\frac{2}{(\gamma+\varepsilon^k)}(\beta - \varepsilon^k)(1 - \beta - \gamma) + 2(\beta - \varepsilon^k) + \frac{2}{3}}\right) \\
&\leq \exp\left(\frac{-N(\gamma + \varepsilon^k)}{\frac{2}{(\gamma+\varepsilon^k)}\beta(1 - \beta - \gamma) + 2\beta + \frac{2}{3}}\right) \leq \exp\left(\frac{-N\gamma}{\frac{2}{\gamma}\beta(1 - \beta - \gamma) + 2\beta + \frac{2}{3}}\right)
\end{aligned}$$

where the inequality follows when $1 - \beta - \gamma \geq 0$. Hence, for all $k \geq 1$,

$$\Pr(B_k) \leq \exp\left(\frac{-3N\alpha^2}{24bh + 4\alpha(b+h)}\right).$$

Since $\alpha = C'(\overline{q})$, from (EC.4) we have that

$$\Pr(B_k) \leq \exp\left(-\frac{6N\epsilon bh\Delta(q^*)f(q^*) + O(\epsilon^{3/2})}{24bh + O(\epsilon^{1/2})}\right) \triangleq U(\epsilon). \tag{EC.5}$$

Now, let us bound the probability of $L \triangleq [C'(\tilde{Q}_N^\alpha) > \alpha] = \left[\bar{F}(\tilde{Q}_N^\alpha) < \frac{h}{b+h} - \frac{\alpha}{b+h}\right]$. Define $q_0 \triangleq \sup\left\{q : \bar{F}(q) \geq \frac{h}{b+h} - \frac{\alpha}{b+h}\right\}$. Thus, $L = [\tilde{Q}_N^\alpha > q_0]$. Note that $\tilde{Q}_N^\alpha = \sup\{q : h - (b+h)\hat{\bar{F}}_N(q) \leq \frac{\alpha}{2}\}$. For a real-valued sequence $\{\tau^k\}_{k=1}^\infty$ where $\tau^k \downarrow 0$, define

$$\begin{aligned}
L_k &\triangleq [\tilde{Q}_N^\alpha \geq q_0 + \tau^k] = \left[h - (b+h)\hat{\bar{F}}_N(q_0 + \tau^k) \leq \frac{\alpha}{2}\right] \\
&= \left[\hat{\bar{F}}_N(q_0 + \tau^k) \geq \frac{h}{b+h} - \frac{1}{2}\frac{\alpha}{b+h}\right] = \left[\hat{\bar{F}}_N(q_0 + \tau^k) \geq 1 - \beta - \gamma\right].
\end{aligned}$$

Since $\hat{\bar{F}}_N$ is nonincreasing, then it follows that $L_k \subseteq L_{k+1}$. Thus, if $\bar{L}$ is the limiting event of the sequence $\{L_k\}_{k=1}^{\infty}$, then $L_k \uparrow \bar{L}$, implying that $\Pr(L_k) \uparrow \Pr(\bar{L})$. Note also that $L \subseteq \bar{L}$, implying that $\Pr(L) \leq \Pr(\bar{L})$. Therefore, to prove a bound on $\Pr(L)$, it is sufficient to prove a uniform upper bound on $\Pr(L_k)$.

Note that for some $\epsilon^k > 0$, we have that $\bar{F}(q_0 + \tau^k) = 1 - \beta - 2\gamma - \epsilon^k$. Thus, $L_k = [\hat{\bar{F}}_N(q_0 + \tau^k) - \bar{F}(q_0 + \tau^k) \geq \gamma + \epsilon^k]$. Finally, from Bernstein's inequality, we have that

$$\begin{aligned}
\Pr(L_k) &\leq \exp\left(\frac{-N(\gamma+\epsilon^k)^2}{2\bar{F}(q_0+\tau^k)(1-\bar{F}(q_0+\tau^k)) + \frac{2}{3}(\gamma+\epsilon^k)}\right) \\
&= \exp\left(\frac{-N(\gamma+\epsilon^k)}{\frac{2}{\gamma+\epsilon^k}(1-\beta-2\gamma-\epsilon^k)(\beta+2\gamma+\epsilon^k) + \frac{2}{3}}\right) \\
&= \exp\left(\frac{-N(\gamma+\epsilon^k)}{\frac{2}{\gamma+\epsilon^k}(1-\beta-2\gamma-\epsilon^k)(\beta+\gamma) + 2(1-\beta-2\gamma-\epsilon^k) + \frac{2}{3}}\right) \\
&\leq \exp\left(\frac{-N(\gamma+\epsilon^k)}{\frac{2}{\gamma+\epsilon^k}(1-\beta-2\gamma)(\beta+\gamma) + 2(1-\beta-2\gamma) + \frac{2}{3}}\right) \\
&\leq \exp\left(\frac{-N\gamma}{\frac{2}{\gamma}(1-\beta-2\gamma)(\beta+\gamma) + 2(1-\beta-2\gamma) + \frac{2}{3}}\right) \\
&= \exp\left(\frac{-N\gamma}{\frac{2}{\gamma}\beta(1-\beta-2\gamma) + 4(1-\beta-2\gamma) + \frac{2}{3}}\right).
\end{aligned}$$

Therefore, we have that for all $k \geq 1$,

$$\Pr(L_k) \leq \exp\left(\frac{-3N\alpha^2}{24bh + 4\alpha(7h - 5b - 6\alpha)}\right).$$

Since $\alpha = C'(\overline{q})$, we have from (EC.4) that

$$\Pr(L_k) \leq \exp\left(-\frac{6N\epsilon bh\Delta(q^*)f(q^*) + O(\epsilon^{3/2})}{24bh + O(\epsilon^{1/2})}\right) \triangleq U(\epsilon). \tag{EC.6}$$

Summarizing from (EC.5) and (EC.6), we have that $\Pr(B) \leq \Pr(\bar{B}) \leq U(\epsilon)$ and that $\Pr(L) \leq \Pr(\bar{L}) \leq U(\epsilon)$. Thus,

$$\begin{aligned}
\Pr\left\{\tilde{Q}_N^\alpha < q^* \text{ or } C'(\tilde{Q}_N^\alpha) > C'(\overline{q})\right\} &= \Pr(B \cup L) \leq \Pr(B) + \Pr(L) \\
&\leq 2U(\epsilon) \sim 2\exp\left(-\frac{1}{4}N\epsilon\Delta(q^*)f(q^*)\right), \text{ as } \epsilon \to 0.
\end{aligned}$$

## EC.3. Proof of Lemma 1

Denote by $\partial_- g(x)$ (or $\partial_+ g(x)$) the left-side (or right-side) derivative of a function $g$ at $x$. The failure rate and reverse hazard rate is given by $\bar{r}(x) = \frac{f(x)}{1-F(x)}$ and $r(x) = \frac{f(x)}{F(x)}$. Since $f$ is a log-concave distribution, it has an increasing failure rate. This implies that $\log \bar{r}(x) = \log f(x) - \log(1-F(x))$ is increasing, and $\partial_- \log \bar{r}(x) \geq 0$ for all $x$. Thus,

$$\gamma_1 + \gamma_0 \frac{b+h}{h} = \gamma_1 + \frac{f(q^*)}{1-F(q^*)} \geq \partial_- \log f(q^*) + \frac{f(q^*)}{1-F(q^*)} = \partial_- \log \bar{r}(q^*) \geq 0. \tag{EC.7}$$

A log-concave distribution also has a decreasing reversed hazard rate. This implies that $\log r(x) = \log f(x) - \log F(x)$ is decreasing and $\partial_+ \log r(x) \leq 0$ for all $x$. Thus,

$$\gamma_1 - \gamma_0 \frac{b+h}{b} = \gamma_1 - \frac{f(q^*)}{F(q^*)} \leq \partial_+ \log f(q^*) - \frac{f(q^*)}{F(q^*)} = \partial_+ \log \bar{r}(q^*) \leq 0. \tag{EC.8}$$

Combining (EC.7) and (EC.8), we have that $-\frac{b+h}{h} \leq \frac{\gamma_1}{\gamma_0} \leq \frac{b+h}{b}$.

## EC.4. Proof of Lemma 2

Note that since $\log f$ is concave, then $\log f(x) \leq \log \gamma_0 + \gamma_1(x-t)$, for all $x$ such that $f(x) > 0$. Taking the exponent on both sides proves our result.

## EC.5. Proof of Lemma 3

Note that $\frac{d}{dx}F_1(x) \leq \frac{d}{dx}F_2(x)$ by our assumption that $f_1(x) \leq f_2(x)$. Moreover, since $F_1(t) = F_2(t)$, then $F_1(x) \geq F_2(x)$ for all $x \leq t$ and $F_1(x) \leq F_2(x)$ for all $x \geq t$. Note that

$$\begin{aligned} E(D_1 - t | D_1 > t) &= \int_0^\infty \Pr(D_1 > t + s | D_1 > t) ds, \\ &= \tfrac{1}{1-F_1(t)} \int_0^\infty (1 - F_1(t+s)) ds, \\ &\geq \tfrac{1}{1-F_2(t)} \int_0^\infty (1 - F_2(t+s)) ds, \\ &= E(D_2 - t | D_2 > t) \end{aligned}$$

With the same technique, we can also prove that $E(t - D_1 | D_1 \leq t) \geq E(t - D_2 | D_2 \leq t)$. Combining these results proves the lemma.

## EC.6. Proof of Lemma 4

We first introduce the following notation:

$$\begin{aligned} G(\alpha) &\triangleq \left(\frac{1}{1-\beta} + \alpha\right) \log\left(1 + \alpha(1-\beta)\right) + \left(\frac{1}{\beta} - \alpha\right) \log\left(1 - \alpha\beta\right) - \min\{\beta, 1-\beta\}\alpha^2, \\ U(\beta) &\triangleq \frac{\beta}{1-\beta} \log\left(\frac{1}{\beta}\right) - \beta, \\ L(\beta) &\triangleq \frac{1-\beta}{\beta} \log\left(\frac{1}{1-\beta}\right) - (1-\beta). \end{aligned}$$

We need to prove that each of the three functions are nonnegative.

1. Let us prove the result for $G$. First, we prove the result for the case when $\beta \geq \frac{1}{2}$. Note that

$$G'(\alpha) = \log\left(\frac{1 + \alpha(1-\beta)}{1 - \alpha\beta}\right) - 2(1-\beta)\alpha.$$

The derivative is nonnegative if and only if $G_1(\alpha) \triangleq (1+\alpha(1-\beta))e^{-(1-\beta)\alpha} - (1-\alpha\beta)e^{(1-\beta)\alpha} \geq 0$. Note that for $\alpha \geq 0$,

$$\begin{aligned} G_1'(\alpha) &= -\alpha(1-\beta)^2 e^{-(1-\beta)\alpha} + \beta e^{(1-\beta)\alpha} - (1-\beta)(1-\alpha\beta)e^{(1-\beta)\alpha}, \\ &\geq -\alpha(1-\beta)^2 e^{-(1-\beta)\alpha} + \beta(1-\beta)\alpha e^{(1-\beta)\alpha}, \\ &\geq \alpha(1-\beta)^2\left(e^{(1-\beta)\alpha} - e^{-(1-\beta)\alpha}\right) \geq 0 \end{aligned}$$

Note that $G_1(0)=0$, thus, $G_1(\alpha) \geq 0$ for all $\alpha \geq 0$. Now define $G_2(\alpha) \triangleq (1+\alpha(1-\beta))e^{-(1-\beta)\alpha} - (1-\alpha(1-\beta))e^{(1-\beta)\alpha}$. Note that $G_2(\alpha) \geq G_1(\alpha)$ if $\alpha \leq 0$. We have

$$G_2'(\alpha) = \alpha(1-\beta)^2\left(e^{(1-\beta)\alpha} - e^{-(1-\beta)\alpha}\right) \geq 0, \quad \text{for } \alpha \leq 0.$$

Note that $G_2(0)=0$, thus, $G_1(\alpha) \leq G_2(\alpha) \leq 0$ for all $\alpha \leq 0$. Thus, $G(\alpha)$ is nondecreasing in $\alpha \geq 0$, and non-increasing in $\alpha \leq 0$. Since at $\alpha = 0$, this function is zero, then $G(\alpha) \geq 0$ for all $\alpha$. Now we can also prove the result for $\beta \leq \frac{1}{2}$, if we define the function $\tilde{\beta} = 1-\beta \geq \frac{1}{2}$ and $\tilde{G}(\alpha) = G(-\alpha)$. Q.E.D.

2. Let us prove the result for $U$. The result is true if and only if $-\log\beta \geq 1-\beta$. Note that $-\log\beta$ is a convex function of $\beta$, thus the linear approximation at $\beta=1$ (i.e., the function $1-\beta$) bounds it from below. Q.E.D.

3. Let us prove the result for $L$. Defining $\tilde{\beta} = 1-\beta$, note that $L(\beta) = U(\tilde{\beta}) \geq 0$, which follows from (2).

## EC.7. Proof of Theorem 4

Recall that if $q \in S_\epsilon^f \cap [q^*, \infty)$, then $C(q) \leq (1+\epsilon)C(q^*)$. Also, $S_\epsilon^f \cap [q^*,\infty)$ can be equivalently expressed as $\{q : C'(q) \leq C'(\overline{q}) \text{ and } q \geq q^*\}$. Let $\tilde{Q}_N^\alpha$ be defined in (8), but with $\alpha = \sqrt{2\epsilon bh\frac{\min\{b,h\}}{b+h}} + O(\epsilon)$. Since,

$$\begin{aligned} C'(\overline{q}) &= (b+h)(F(\overline{q}) - F(q^*)) = (b+h)\left[(\overline{q}-q^*)f(q^*) + O(\overline{q}-q^*)^2\right] \\ &= \sqrt{2\epsilon bh\Delta(q^*)f(q^*)} + O(\epsilon), \end{aligned}$$

then it follows from Proposition 2 that $\alpha \leq C'(\overline{q})$ when the demand distribution is log-concave. This implies that

$$\left[\tilde{Q}_N^\alpha \geq q^*\right] \cap \left[C'(\tilde{Q}_N^\alpha) \leq \alpha\right] \subseteq \left[\tilde{Q}_N^\alpha \geq q^*\right] \cap \left[C'(\tilde{Q}_N^\alpha) \leq C'(\overline{q})\right].$$

Thus, we only need to derive a lower bound on the probability of the left-hand side event to prove Theorem 4. Modifying the proof of Theorem 3 by letting $\alpha = \sqrt{2\epsilon bh\frac{\min\{b,h\}}{b+h}} + O(\epsilon)$, we can prove that

$$\Pr\left(\tilde{Q}_N^\alpha < q^* \text{ or } C'(\tilde{Q}_N^\alpha) > \alpha\right) \leq 2U^*(\epsilon) \sim 2\exp\left(-\frac{1}{4}N\epsilon\frac{\min\{b,h\}}{b+h}\right), \text{ as } \epsilon \to 0.$$