Market Selection

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Abstract

The hypothesis that financial markets punish traders who make relatively inaccurate forecasts and eventually eliminate the effect of their beliefs on prices is of fundamental importance to the standard modeling paradigm in asset pricing. We establish straightforward necessary and sufficient conditions for agents to survive and to affect prices in the long run in a general setting with minimal restrictions on endowments, beliefs, or utility functions. We describe a new mechanism for the distinction between survival and price impact in a broad class of economies. Our results cover economies with time-separable utility functions, including possibly state-dependent preferences.

Keywords: Financial Markets, Heterogeneous Beliefs, Price Impact, Survival, General Equilibrium.

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1 Introduction

It has long been suggested that evolutionary forces work in financial markets: agents who are inferior at forecasting the future will either improve through learning or perish as their wealth diminishes relative to those superior in forecasting (e.g. Friedman (1953)). If such an evolutionary mechanism works effectively, then in the long run only those agents with the best forecasts will survive the market selection process and determine asset prices. This "market selection hypothesis" (MSH) is one of the major arguments behind the assumption of rational expectations in neoclassical asset pricing theory. After all, if agents with more accurate knowledge of fundamentals do not determine the behavior of market prices, there is little reason to assume that prices are driven by fundamentals and not by behavioral biases. More generally, it would be comforting that markets select for those agents with more accurate forecasts, even if agents with less accurate forecasts are replenished over time (e.g. in overlapping generations economies). We show that in frictionless, complete-market exchange economies, both parts of the MSH – that traders with inferior forecasts do not survive and that extinction destroys their price impact – are false in general. With minimal restrictions on endowments, preferences, and beliefs, we develop necessary and sufficient conditions for market selection of beliefs.

Despite the appeal and importance of the market selection hypothesis, its validity has remained ambiguous. Existing studies use specialized models, mostly for tractability and convenience, making it difficult to understand the economic mechanism behind the MSH and the scope of its validity. For instance, relying on partial equilibrium analysis, De Long et al. (1991) argue that agents making inferior forecasts can survive in wealth terms despite market forces exerted by agents with objective beliefs. Using a general equilibrium setting, Sandroni (2000) and Blume and Easley (2006) show that only agents with beliefs closest to the objective probabilities (in a sense they make precise) will survive and have price impact. Their results are obtained in economies with bounded aggregate endowment. Kogan et al. (2006) demonstrate in a general equilibrium setting without intermediate consumption that if aggregate endowment is unbounded, agents with incorrect beliefs can survive.[1] In this paper, we perform a comprehensive analysis of the MSH and its pricing implications in a general complete-market setting with time-separable preferences (including state-dependent

[1]A significant body of work exists examining pricing implications of heterogeneous beliefs in specific parameterized models, including Dumas et al. (2009), Fedyk et al. (2013), Xiong and Yan (2010), Yan (2008), and Borovička (2015).

preferences, e.g., catching up with the Joneses), not limiting ourselves to commonly used parametric specifications.[2]

Kogan et al. (2006) draw the distinction between the two parts of the MSH. They show, in a stylized setting, that even when agents with inferior beliefs do not survive in the long run, their impact on prices can persist. In other words, survival and price impact are two distinct concepts. In particular, an agent with relatively low consumption level can affect the prices of low-aggregate consumption states because the change in prices relative to wealth spent on consumption in such states is of order $\frac{1}{C}$, where C denotes agent's consumption. As a result, by distorting the prices of primitive Arrow-Debeu claims over a set of states of diminishing probability, it is possible for the agent to persistently distort valuations of non-primitive assets, like stocks and bonds, while failing to survive in the long run. In addition to relying on a particular set of non-primitive financial assets for their definition of price impact, Kogan et al. (2006) assume the absence of intermediate consumption, CRRA preferences, and IID endowment growth, leaving it unclear how their results apply in more general settings.

In this paper, we demonstrate that the distinction between survival and price impact arises in standard infinite-horizon models with intermediate consumption and flexible specification of time-separable preferences. Our analysis offers a more general and robust intuition for the distinction between price impact and survival. In particular, we show that long-run price impact can occur at the level of the primitive Arrow-Debreu claims, and not only at the level of certain non-primitive long-lived assets.[3] Moreover, we show that price impact in general settings does not hinge on the distortion in agents' consumption over a set of states of diminishing probability (specifically, the low-aggregate endowment states). Instead, price impact of distorted beliefs has to do with the ability of an agent holding such beliefs to provide non-trivial risk sharing to other agents in equilibrium despite of failing to survive in the long run.

We examine the MSH in frictionless and complete-market economies because common arguments in favor of its validity rely on unrestricted competition, no limits to

[2]While the class of preferences we consider is broad, it excludes non-separable recursive preference specifications. Thus, our analysis inevitably blurs the distinction between individual aversion to risk and the desire to smooth consumption intertemporally. General theoretical analysis of economies with non-separable preference is beyond the scope of this paper. Borovička (2015) obtains promising results in this direction.

[3]We discuss these alternative notions of price impact in detail in Section 2.

arbitrage, etc. To isolate the impact of disagreement, we populate our economies with competitive agents who only differ in their beliefs. We then analyze how survival and price impact properties of the economy depend on the primitives, such as errors in forecasts, endowment growth, and preferences.

Much of the asset pricing literature assumes that agents have homothetic preferences (constant relative risk aversion). Yan (2008) obtains strong results in support of the MSH under this assumption. We allow for a much broader family of preferences and find that the case of homothetic preferences is somewhat special. Without this assumption, validity of the MSH needs to be qualified, and depends on other economic primitives, such as the dynamics of the endowment process. Specifically, we find that if the curvature of the agents' utilities declines sufficiently fast as a function of their consumption level, or if the aggregate endowment is bounded, then agents with more accurate forecasts eventually dominate the economy and determine price behavior. Thus, the market selection hypothesis does hold in this particular class of economies, and the rates of extinction in both consumption and price impact are proportional to the growth rate of accumulated forecast errors. Without the above restrictions, the survival of agents with less accurate forecasts and their impact on state prices are determined by the tradeoff between agents' preferences, the magnitude of their forecast errors, and the aggregate endowment growth rate: if forecast errors accumulate slowly enough over time, agents with less accurate forecasts can maintain a nontrivial consumption share and affect prices.

Agents with heterogeneous beliefs trade with each other to share consumption across states, but whether this disagreement leaves one of the agents with a vanishing consumption share depends on the agents' preferences. When two agents disagree in their probability assessment of a particular state, the more optimistic agent buys a disproportionate share of the state-contingent consumption. If two agents have diverging beliefs, they end up with extreme disagreement asymptotically over most states. Pareto optimality implies that the ratio of agents' marginal utilities in each state must be inversely proportional to the ratio of their belief densities, and therefore, asymptotically, divergence in beliefs leads to divergence in marginal utilities. Whether or not large differences in marginal utilities correspond to large differences in consumption depends on the sensitivity of marginal utility to consumption, $d\ln(U'(C))/d\ln(C) = CU''(C)/U'(C) = -CA(C)$, where $A(C) = -U''(C)/U'(C)$ is the absolute risk aversion coefficient, which characterizes the local curvature of the utility function. If the utility curvature of the two agents declines slowly in their consumption level, their marginal utility differences may not translate into large consumption differences. In fact, as we show below (in Proposition 5.6), the two agents may consume equal consumption shares asymptotically despite their growing disagreement.

In addition, the conditions for survival and price impact are different. In equilibrium, the level of belief differences determines the relative consumption levels of the agents (survival), while the stochastic discount factor is determined by time-variation in the marginal utility of consumption. An agent with a diminishing consumption share may maintain a persistent impact on state prices as long as his presence affects the fluctuations in the marginal utility of the dominant agent. In other words, a disappearing agent may affect the stochastic discount factor as long as he provides nontrivial risk-sharing opportunities for the dominant agent.

To flesh out this intuition further, consider an exchange economy with two agents. Let D_t be the aggregate endowment, and let the agents have preferences given by $U(C_t)$. Assume that the first agent has objective beliefs, while beliefs of the second agent are such that he consumes $C_{2,t}$, which is an asymptotically diminishing share of the aggregate endowment. Thus, the second agent does not survive in the long run. The first agent consumes $C_{1,t} = D_t - C_{2,t}$. Next, compare the stochastic discount factor in this economy to the one in an identical economy without the second agent, i.e., with $C_{1,t} = D_t$.

The volatility of the stochastic discount factor equals the volatility of growth of the marginal utility of the first agent. Assume that all quantities are driven by Ito processes. In the second economy, the instantaneous stochastic component of the stochastic discount factor is, by Ito's lemma, $(dU'(D_t)-E_t[dU'(D_t)]/U'(D_t) = -A(D_t)(dD_t - E_t[dD_t])$, where $A(D) = -U''(D)/U'(D)$. This compares to $-A(C_{1,t})(dC_{1,t}-E_t[dC_{1,t}])$ in the first economy. Suppose that $A(C_{1,t}) \approx A(D_t)$. Then, the instantaneous difference between the stochastic discount factors in the two economies is approximately $A(D_t)(dC_{2,t} - E_t[dC_{2,t}])$. If the consumption level of the second agent is sufficiently volatile so that he is engaged in nontrivial risk sharing with the first agent, the discount factors in the two economies can be quite different. Whether such dynamics is observed in equilibrium depends on the agents' preferences as well as their beliefs and the aggregate endowment process.

Our results cover general state-dependent preferences, such as external habit formation or catching-up-with-the-Joneses. Such models of preferences are emerging as increasingly important in recent asset pricing research. State-dependent preferences change the risk attitudes of the agents in the economy, but they do not change how those risk attitudes affect survival or price impact. We are therefore able to apply the necessary and sufficient conditions for the validity of the MSH to models with state-dependent preferences that are commonly used in the literature. We conduct this analysis in the on-line Appendix.

2 The Model

We consider an infinite-horizon exchange (endowment) economy. Time is indexed by t, which takes values in $t \in [0, \infty)$. Time can either be continuous or discrete. While all of our general results can be stated either in discrete or continuous time, some of the examples are simpler in continuous time. We will use integrals to denote aggregation over time. When time is taken as discrete, time-integration will be interpreted as summation. We further assume that there is a single, perishable consumption good, which is also used as the numeraire.

Uncertainty and the Securities Market

The environment of the economy is described by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each element $\omega \in \Omega$ denotes a state of the economy. The information structure of the economy is given by a filtration on $\mathcal{F}$, $\{\mathcal{F}_t\}$, with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. The probability measure $\mathbb{P}$ is referred to as the objective probability measure. The endowment flow is given by an adapted process D_t. We assume that the aggregate endowment is strictly positive: $D_t > 0$, *a.s.* Here and in the rest of the paper we apply the concept of almost-sure convergence under measure $\mathbb{P}$.

In addition to the objective probability measure $\mathbb{P}$, we also consider other probability measures, referred to as subjective probability measures. Let $\mathbb{A}$ and $\mathbb{B}$ denote such measures. We assume that $\mathbb{A}$ and $\mathbb{B}$ share zero-probability events with $\mathbb{P}$ when restricted to any finite-time information set $\mathcal{F}_t$. Denote the Radon-Nikodym derivative of the probability measure $\mathbb{A}$ with respect to $\mathbb{P}$ by $\xi_t^{\mathbb{A}}$. Then

$$\mathrm{E}_t^{\mathbb{A}}\left[Z_s\right] = \mathrm{E}_t^{\mathbb{P}}\left[\frac{\xi_s^{\mathbb{A}}}{\xi_t^{\mathbb{A}}} Z_s\right] \tag{1}$$

for any $\mathcal{F}_s$-measurable random variable Z_s and $s \geq t$, where $\mathrm{E}_t[Z]$ denotes $\mathrm{E}[Z|\mathcal{F}_t]$. In addition, $\xi_0^{\mathbb{A}} \equiv 1$. The probability measure $\mathbb{B}$ has a corresponding Radon-Nikodym derivative $\xi_t^{\mathbb{B}}$. The random variable $\xi_t^{\mathbb{A}}$ can be interpreted as the density of the probability measure $\mathbb{A}$ with respect to the probability measure $\mathbb{P}$ conditional on the time-t information set.

We use $\mathbb{A}$ and $\mathbb{B}$ to model heterogeneous beliefs. We define the ratio of subjective belief densities

$$\xi_t = \frac{\xi_t^{\mathbb{B}}}{\xi_t^{\mathbb{A}}}. \tag{2}$$

Since both $\xi^{\mathbb{A}}$ and $\xi^{\mathbb{B}}$ are nonnegative martingales, they converge almost surely as time tends to infinity (e.g., Shiryaev (1996, §7.4, Th. 1)). Our results are most relevant for models in which the limit of ξ_t is either zero or infinity. We examine the asymptotic condition on subjective beliefs in more detail in Section 3.

We assume that there exists a complete set of Arrow-Debreu securities in the economy, so that the securities market is complete.

Agents

There are two competitive agents in the economy. They have the same utility function, but differ in their beliefs. The first agent has $\mathbb{A}$ as his probability measure while the second agent has $\mathbb{B}$ as his probability measure. We refer to the agent who uses $\mathbb{A}$ as agent $\mathbb{A}$ and the agent who uses $\mathbb{B}$ as agent $\mathbb{B}$. It is clear from the context when we refer to an agent as opposed to a probability measure.

Until stated otherwise, we assume that the agents' preferences are time-additive and state-independent with the canonical form

$$\int_0^\infty e^{-\rho t} u(C_t) dt, \tag{3}$$

$u(\cdot)$ is the utility function, C_t is an agent's consumption at time t and ρ is the time-preference parameter. The common utility function $u(\cdot)$ is assumed to be increasing, weakly-concave, and twice continuously differentiable. We assume that $u(\cdot)$ satisfies the standard Inada condition at zero:

$$\lim_{x \to 0} u'(x) = \infty. \tag{4}$$

We make use of two standard measures of local utility curvature, $A(x) \equiv -u''(x)/u'(x)$ and $\gamma(x) \equiv -xu''(x)/u'(x) = xA(x)$ which are, respectively, an agent's absolute and relative risk aversion at the consumption level x.

Let $C_{\mathbb{A},t}$ and $C_{\mathbb{B},t}$ denote consumption of the two agents. Each agent maximizes his expected utility using his subjective beliefs. Agent i's objective is

$$\mathrm{E}_0^i\left[\int_0^\infty e^{-\rho t} u(C_{i,t})\,dt\right] = \mathrm{E}_0^{\mathbb{P}}\left[\int_0^\infty e^{-\rho t}\xi_t^i u(C_{i,t})\,dt\right], \quad i \in \{\mathbb{A},\mathbb{B}\}, \tag{5}$$

where the equality follows from (1). This implies that the two agents are observationally equivalent to the two agents with objective beliefs $\mathbb{P}$ but state-dependent utility functions $\xi_t^{\mathbb{A}} u(\cdot)$ and $\xi_t^{\mathbb{B}} u(\cdot)$ respectively.

The two agents are collectively endowed with a flow of the consumption good, with possibly different shares of the total endowment.

Equilibrium

Because the market is complete, if an equilibrium exists, it must be Pareto-optimal. In such situations, consumption allocations can be determined by maximizing a weighted sum of the utility functions of the two agents. The equilibrium is given at each time t by

$$\begin{aligned} \max_{C_{\mathbb{A},t},\, C_{\mathbb{B},t}} \quad & (1-\alpha)\,\xi_t^{\mathbb{A}} u(C_{\mathbb{A},t}) + \alpha\,\xi_t^{\mathbb{B}} u(C_{\mathbb{B},t}) \\ \text{s.t.} \quad & C_{\mathbb{A},t} + C_{\mathbb{B},t} = D_t \end{aligned} \tag{6}$$

where $\alpha \in [0,1]$.

Concavity of the utility function, together with the Inada condition, imply that the equilibrium consumption allocations satisfy the first-order condition

$$\frac{u'(C_{\mathbb{A},t})}{u'(C_{\mathbb{B},t})} = \lambda\,\xi_t, \tag{7}$$

where we denote $\alpha/(1-\alpha)$ by λ.

We define $w_t = \frac{C_{\mathbb{B},t}}{D_t}$ as the share of the aggregate endowment consumed by agent $\mathbb{B}$. The first-order condition for Pareto optimality (7) implies that w_t satisfies[4]

$$-\ln(\lambda\xi_t) = -\ln(u'((1-w_t)D_t)) + \ln(u'(w_t D_t)) = \int_{w_t D_t}^{(1-w_t)D_t} A(x)\,dx, \tag{8}$$

since $A(x) = -\frac{d}{dx}\ln(u'(x))$. This equation relates belief differences (ξ_t) to individual utility curvature ($A(x)$) and the equilibrium consumption allocation (w_t and D_t), and

[4]Throughout the paper we use the convention $\int_b^a = -\int_a^b$.

will be our primary analytical tool.

Definitions of Survival and Price Impact

Without loss of generality, we focus on the survival of agent $\mathbb{B}$ and that agent's impact on security prices in the long run. If one replaces $\lambda\xi_t$ with $\frac{1}{\lambda\xi_t}$ in our analysis, our results instead describe the survival and price impact of agent $\mathbb{A}$.

We first define formally the concepts of survival and price impact to be used in this paper and examine their properties.

Definition 1 **[Extinction and Survival]** *Agent $\mathbb{B}$ becomes extinct if*

$$\lim_{t\to\infty} \frac{C_{\mathbb{B},t}}{D_t} = 0, \quad \text{a.s.} \tag{9}$$

Agent $\mathbb{B}$ survives if he does not become extinct.

The above definition provides a weak condition for survival: an agent has to consume a positive fraction of the endowment with a positive probability in order to survive.

We opt for a relatively conservative definition of price impact, and define it in terms of distortions in the prices of primitive state-contingent claims (or the stochastic discount factor). This choice is natural for a frictionless complete-market economy. An alternative would be to define price impact over a set of non-primitive long-lived assets, in analogy with Kogan et al. (2006). The set of economies in which agent $\mathbb{B}$'s beliefs affect prices of *some* long-lived assets is larger than the set of economies with persistent distortions in the state-price density. In fact, it is typically easy to construct a particular long-lived asset that is persistently affected by the belief distortions, posing a question of which long-lived assets are more or less economically relevant, and therefore should or should not be considered when checking for price impact.[5] Our definition avoids such questions, which are ill-posed in the context of complete frictionless markets.

[5]See the on-line Appendix, Section B.5, for an example of a standard economy in which distorted beliefs have no long-run impact on the Arrow-Debreu prices, and yet prices of some of the long-lived non-primitive assets are affected by belief distortions. The reason for price impact on the non-primitive state-contingent claim we consider in our example is that its payoff is concentrated on a set of states in which agent $\mathbb{B}$ has a nontrivial consumption share. This set of states has an asymptotically vanishing probability but is relevant for the pricing of long-lived non-primitive state-contingent claims.

Let m_t denote the equilibrium state-price density. Pareto optimality and the individual optimality conditions imply that

$$m_t = e^{-\rho t} \frac{\xi_t^{\mathbb{A}} u'((1-w_t)D_t)}{u'((1-w_0)D_0)} = e^{-\rho t} \frac{\xi_t^{\mathbb{B}} u'(w_t D_t)}{u'(w_0 D_0)}. \tag{10}$$

In general, m_t depends on λ, the relative weight of the two agents in the economy. Thus, we write $m_t = m_t(\lambda)$. We denote by $m_t^\star(\lambda)$ the state-price density in the economy in which both agents have beliefs described by the measure $\mathbb{A}$ and hence $\xi_t = 1$. We define $m_t^\star(0)$ to be the state-price density in an economy in which all wealth is initially allocated to agent $\mathbb{A}$. We identify the price impact exerted by agent $\mathbb{B}$ by comparing m_t to $m^\star$.

Definition 2 [Price Impact] *Agent $\mathbb{B}$ has no price impact if there exists $\lambda^\star \geq 0$, such that for any $s > 0$,*

$$\lim_{t\to\infty} \frac{m_{t+s}(\lambda)/m_t(\lambda)}{m_{t+s}^\star(\lambda^\star)/m_t^\star(\lambda^\star)} = 1, \quad \text{a.s.} \tag{11}$$

Otherwise, he has price impact.

Our definition formalizes the notion that agent $\mathbb{B}$ has no price impact if the equilibrium stochastic discount factor is asymptotically indistinguishable from the one in a reference economy with the same preferences but with both agents using the beliefs of agent $\mathbb{A}$. We allow for the initial wealth distribution in the reference economy to differ from that in the original economy. Our motivation for this is straightforward, as we illustrate with the following argument.

Consider an economy with an equal initial wealth distribution between $\mathbb{A}$ and $\mathbb{B}$. Suppose that $\mathbb{A}$ maintains objective beliefs, while the beliefs of $\mathbb{B}$ are inaccurate. Suppose further that the primitives of the model are such that agent $\mathbb{B}$ becomes extinct asymptotically and therefore agent $\mathbb{A}$ dominates the economy in the long run. Assume that the stochastic discount factor in this economy converges asymptotically to the one in an economy in which only agent $\mathbb{A}$ is present. Intuitively, one would like to conclude that, because the stochastic discount factor converges to that of an economy without $\mathbb{B}$, the latter has no long-run impact on prices. If we insisted that the reference economy must be obtained from the original model by simply setting beliefs of both agents to those of agent $\mathbb{A}$, we would generally be forced to conclude that agent $\mathbb{B}$ maintains long-run impact on prices. The reason is that, in general, the stochastic discount factor in the model with homogeneous beliefs depends on the initial wealth distribution between the two agents.[6] Thus, to reflect our basic intuition, the definition for price impact must allow the reference economy to start with a wealth distribution different from that in the original model.

In contrast to the notion of long-run survival, equations (10) and (11) show that price impact is determined by ***changes*** in consumption over finite time intervals relative to a benchmark economy. In particular, we compare the stochastic discount factor in the original economy, $m_{t+s}(\lambda)/m_t(\lambda)$, to the one in a reference economy where both agents maintain the same beliefs, but, possibly, have a different initial wealth distribution, $m^\star_{t+s}(\lambda^\star)/m^\star_t(\lambda^\star)$.

The above definition may seem difficult to apply because condition (11) must be verified for all values of $\lambda^\star$. However, to demonstrate the absence of long-run price impact for economies in which agent $\mathbb{B}$ does not survive it is often sufficient to verify the definition for $\lambda^\star = 0$. As a measure of the magnitude of price impact, we use in that case

$$PI(t, s; 0) \equiv \ln\left(\frac{m_{t+s}(\lambda)/m_t(\lambda)}{m^\star_{t+s}(0)/m^\star_t(0)}\right) = \int_{D_{t+s}(1-w_{t+s})}^{D_{t+s}} A(x)\,dx - \int_{D_t(1-w_t)}^{D_t} A(x)\,dx, \quad (12)$$

where the final equality follows from

$$\frac{u'(D(1-w))}{u'(D)} = \exp\left(\int_{D(1-w)}^{D} A(x)\,dx\right). \quad (13)$$

Price impact can be similarly defined for any λ^*.

Discussion of the Assumptions

Our analysis focuses on a specific question: "Do markets select for relatively accurate forecasts?" Thus, we abstract away from differences in utility functions across agents. Understanding the behavior of economies with heterogeneous preferences is an important topic, but it is distinct from the market selection hypothesis, which postulates an evolutionary rationale for long-run market rationality. Thus, we isolate the effect of belief heterogeneity on long-run survival and price impact.

We consider the setting without constraints on trading to evaluate the economic

[6] As shown in Rubinstein (1974), the stochastic discount factor does not depend on the initial wealth distribution in the special case when the agents' utility function exhibits hyperbolic absolute risk aversion.

mechanism behind market selection. It is clear that market incompleteness and constraints may have a direct impact on the long-run dynamics of prices. As a stark illustration, consider an economy in which none of the agents are allowed to transact in financial markets. In this case, survival does not depend on beliefs, and prices are arbitrary. Similarly, in an economy with nontrivial labor income, liquidity constraints may guarantee that agents cannot lose their future labor income by making poor trading decisions, and thus survive. More generally, specific forms of market incompleteness may impede transactions among agents, thus reducing the long-run advantage of agents with more accurate forecasts (see, e.g., Blume and Easley (2006) and Beker and Chattopadhyay (2010)). Our frictionless framework helps evaluate the logic of the original argument for the validity of the MSH, which is based on unrestricted competition among agents, and we show under what assumptions the argument is or is not valid.

Our description of the long-run market dynamics is qualitative. We study the general conditions under which prices eventually reflect superior forecasts, and we characterize how the rate of convergence of consumption and prices depends on the economic primitives. Yan (2008) calibrates the empirically plausible speed of selection in a particular parameterized economy in which agents with CRRA preferences disagree about the growth rate of the aggregate endowment, and argues that market selection in such an economy is likely to be slow. In contrast, Fedyk et al. (2013) show that market selection speed is drastically higher when agents disagree about multiple sources of randomness than in analogous economies with a single source of disagreement. Thus, the market selection mechanism may work very slowly or very quickly, depending on the assumed economic primitives.[7] Our analysis makes it clear how these primitives drive the selection mechanism, and under what circumstances market selection is eventually successful.

In our analysis we allow for general utility functions and do not restrict ourselves to CRRA preferences. In addition to helping us understand the role of preferences in the MSH, such a general setting also addresses the issue of market selection in many models used in the asset pricing literature. For example, models investigating the link between market dynamics and heterogenous beliefs/asymmetric information, wealth

[7]As we show, the rate at which ξ_t converges to zero is critical in determining the survival and price impact of agents with relatively inaccurate forecasts. So, for example, if there is a constant disagreement of δ over the drift of N different independent Brownian Motions Z_t^i – there are N sources of disagreement – then $\ln(\xi_t) = -\frac{1}{2}N\delta^2 t + \delta\sum_{i=1}^N Z_t^i$. Thus, the selection speed in this example is proportional to the number of sources of disagreement.

accumulation under uncertainty, savings in the presence of taxes, trading costs, and market liquidity often use preferences with constant absolute risk aversion.[8] As we show below, such preferences have very different implications for market selection compared to the utilities with constant, or, more generally, bounded relative risk aversion. Without taking a stand on which utility specification is most convenient or realistic in any particular setting, we instead cover a broad spectrum of possible individual preferences. Importantly, our results apply to state-dependent preferences. These increasingly common specifications allow risk aversion to be a stochastic function of consumption, departing from pure power utility.

We define survival and extinction in terms of consumption ***share***, as opposed to consumption ***level***. In economies in which the aggregate dividend is bounded, extinction in level and extinction in share are equivalent, but this is generally not true in economies with a growing aggregate endowment. If we define extinction in level instead of share ($w_t D_t \to 0$ rather than $w_t \to 0$), then the result that extinction implies no price impact is immediate for utility functions with decreasing absolute risk aversion. Since $w_t D_t A(D_t - w_t D_t) \geq \int_{D_t - w_t D_t}^{D_t} A(x)dx \geq w_t D_t A(D_t)$, and both bounds go to zero as $w_t D_t \to 0$, equation (12) shows that there is no price impact. The reverse – that survival implies price impact – is not always true, as we show later. However, extinction in level is a very strong condition, which is not met in many economies of interest. For example, in an economy in which aggregate endowment follows a geometric Brownian motion with a 5% expected growth and unit volatility, investors have log utility, and one investor knows the true growth rate while the second agent persistently believes the true growth rate is 25%, there is no extinction in consumption level. The second agent does, however, face extinction in terms of his consumption share and has no long-run price impact.

Finally, we have assumed that both agents have the same time discount factor. This allows us to isolate the effect of heterogeneous beliefs, but this assumption can be relaxed without loss in tractability. With different time discount factors, we would have that the Pareto optimal allocation is $\frac{u'(C_{\mathbb{A},t})}{u'(C_{\mathbb{B},t})} = \lambda e^{(\rho_{\mathbb{A}} - \rho_{\mathbb{B}})t} \xi_t$ instead of (7). This is equivalent to replacing the belief process ξ_t in our setting with the process $\hat{\xi}_t = e^{(\rho_{\mathbb{A}} - \rho_{\mathbb{B}})t} \xi_t$.

[8] See for example Shefrin (2005) on heterogeneous beliefs; Caballero (1991) on aggregate wealth accumulation; Kimball and Mankiw (1989) on savings; Vayanos (1999) and Lo et al. (2004) on volume and microstructure; Gromb and Vayanos (2002), Huang and Wang (2009), Yuan (2005) on liquidity and crashes; Garleanu (2009) on search.

3 Examples

In this section we use a series of examples to illustrate how survival and price impact properties depend on the interplay of the model primitives and to provide basic intuition for the more general results in the next section. Our examples are organized in three sets. Each set compares economies differing from each other with respect to only one of the primitives: endowments, beliefs, or preferences. Formal derivations are presented in the on-line Appendix.

3.1 Endowments

The following two examples illustrate the dependence of survival and price impact results on the endowment process. We show that a change in the properties of the economic environment (the endowment), holding the agents' characteristics (beliefs and preferences) fixed, may affect the validity of the MSH, altering both the survival and price impact results. In addition, we demonstrate an example in which survival and price impact are not equivalent.

Our first example has no aggregate uncertainty: investors differ in their beliefs only over an extraneous source of randomness:

Example 3.1 *Consider a continuous-time endowment economy. The aggregate endowment process is given by*

$$D_t = (1 + \mu t), \tag{14}$$

where $\mu > 0$*. There is an additional state variable* X_t*, which evolves according to*

$$dX_t = -\theta X_t \, dt + \sigma \, dZ_t, \tag{15}$$

where $\theta > 0$ *and* Z *is a Brownian motion under the objective probability measure.*

Both agents have the same utility function with $A(x) = 1$ *for* $x \geq 1$*. We also assume* $A(x) = \frac{1}{x}$ *for* $x < 1$ *to preserve the Inada condition at zero. The utility function thus exhibits decreasing absolute risk aversion for low levels of consumption and constant absolute risk aversion for high levels of consumption.*

Assume that agent $\mathbb{A}$ *knows the true distribution, while agent* $\mathbb{B}$ *disagrees with* $\mathbb{A}$ *and believes that* Z *has a drift. Assume that the disagreement is constant,* $\delta \neq 0$*, and* *therefore the di erence in agents' beliefs is described by the density process*

$$\xi_t = \exp\left(-\frac{1}{2}\delta^2 t + \delta Z_t\right). \tag{16}$$

Agents' beliefs thus diverge asymptotically, with $\lim_{t\to\infty} \xi_t = 0$ $a.s.$ *Then,*

i. If $\mu > \delta^2/2$*, agent* $\mathbb{B}$ *survives in the long run and exerts long-run impact on prices;*

ii. If $\mu < \delta^2/2$*, agent* $\mathbb{B}$ *fails to survive in the long run and does not exert long-run impact on prices;*

iii. If $\mu = \delta^2/2$*, agent* $\mathbb{B}$ *fails to survive in the long run but exerts long-run impact on prices.*

Since there is no aggregate uncertainty and investors have the same preferences and equal initial endowments, differences in consumption between the two agents are driven by their beliefs. Investors act to smooth their marginal utilities, as modified by their beliefs, granting more consumption to an agent in states that agent believes are relatively more likely. This happens even in the case in which disagreement is over events that are irrelevant to the aggregate endowment.

Our example cover cases in which the aggregate endowment grows faster, slower, or at the same rate as the agent's belief differences accumulate. The key comparison is between the rate of accumulation of belief differences – and hence the rate of growth of consumption differences – with the level of aggregate consumption.

In addition, this example illustrates the importance of endowment growth overall. The two cases in which agent $\mathbb{B}$ survives or has price impact, (i) and (iii) are impossible without a growing endowment. We show in Sections 4 and 5 that this result can be made general: a necessary condition for either price impact or survival is that the endowment is not bounded from above and below by positive constants.

In the first case, (i), aggregate consumption grows fast enough so that the growth in local utility curvature, evaluated at the aggregate consumption level, outpaces the rate of accumulation of differences in beliefs (growth rate of ξ) and agent $\mathbb{B}$ both survives and has price impact. Here, belief differences drive a wedge between agents' consumptions, but that wedge is small relative to the size of the economy, and so agent $\mathbb{B}$ can survive and exert price impact. In the second case, (ii), aggregate consumption grows slowly so that growth in the local utility curvature is slower than the rate of accumulation of belief differences, and so the wedge that belief difference creates between the two consumption levels is large relative to aggregate consumption. Thus, agent $\mathbb{B}$ does not survive and cannot exert price impact.

In the third case, (iii), aggregate consumption, and hence local utility curvature, grow at an intermediate rate. Here, belief differences create consumption differences that are large enough relative to the aggregate endowment that agent $\mathbb{B}$ cannot survive. However, over time, belief differences can still drive changes in the consumption allocation: consumption levels are volatile enough that marginal utilities are affected. In particular, as we show in the appendix, $\lim_{t\to\infty} \frac{C_{\mathbb{B},t}}{D_t} = 0$, while $\limsup_{t\to\infty} C_{\mathbb{B},t} = \infty$. This means that for large enough t, $A(C_{\mathbb{A},t}) = 1$, while, whenever $C_{\mathbb{B},t} > 1$, Pareto optimality implies $C_{\mathbb{A},t} = \frac{1}{2}D_t - \frac{1}{2}\ln(\lambda\xi_t) = \frac{1}{2}(1 - \ln\lambda - \delta Z_t)$. As we discuss in the introduction, agent $\mathbb{B}$ can exert nontrivial impact on prices if he is able to provide agent $\mathbb{A}$ with nontrivial risk sharing in the long run. This is the case in our example. The conditional volatility of the marginal utility of agent $\mathbb{A}$ is equal to $\mathrm{vol}(dm_t) = A(C_{\mathbb{A},t})\mathrm{vol}(dC_{\mathbb{A},t}) = \delta/2$. In contrast, if agent $\mathbb{B}$ had the same beliefs as $\mathbb{A}$, the volatility of the stochastic discount factor would be zero (since the aggregate endowment is deterministic). Because agent $\mathbb{B}$ maintains non-vanishing volatility of his consumption changes in equilibrium, his beliefs distort equilibrium prices asymptotically.

Figure 1 illustrates the tradeoff between endowment growth and accumulation of belief differences in the economy described in Example 3.1. Specifically, Figure 1 shows the ***median path*** of the economy in each of the three cases considered in Example 3.1, plotted against level curves for the consumption share of agent $\mathbb{B}$ (solid lines). The median path of the economy is obtained by setting the driving Brownian motion Z_t to zero. Each level curve represents pairs $(D, \ln(\xi))$ that give rise to a particular consumption share w. These lines depend only on preferences, and they can be found by fixing w and plotting the value of $\ln(\xi)$ as a function of D, with the function given by the Pareto optimality condition (8). Note that, because of constant absolute risk aversion at high consumption levels, a given difference in consumption shares requires a larger difference in beliefs at higher endowment levels. This explains why the level curves tend to be spaced wider at higher endowment levels.

The economy illustrated in Figure 1 is growing, since $\mu > 0$, and so the median path is traced from left to right as time passes. In case (i), the median path (marked with circles) does not cross the level curves for large t, which shows that as the economy grows over time, the consumption share of agent $\mathbb{B}$ along the median path

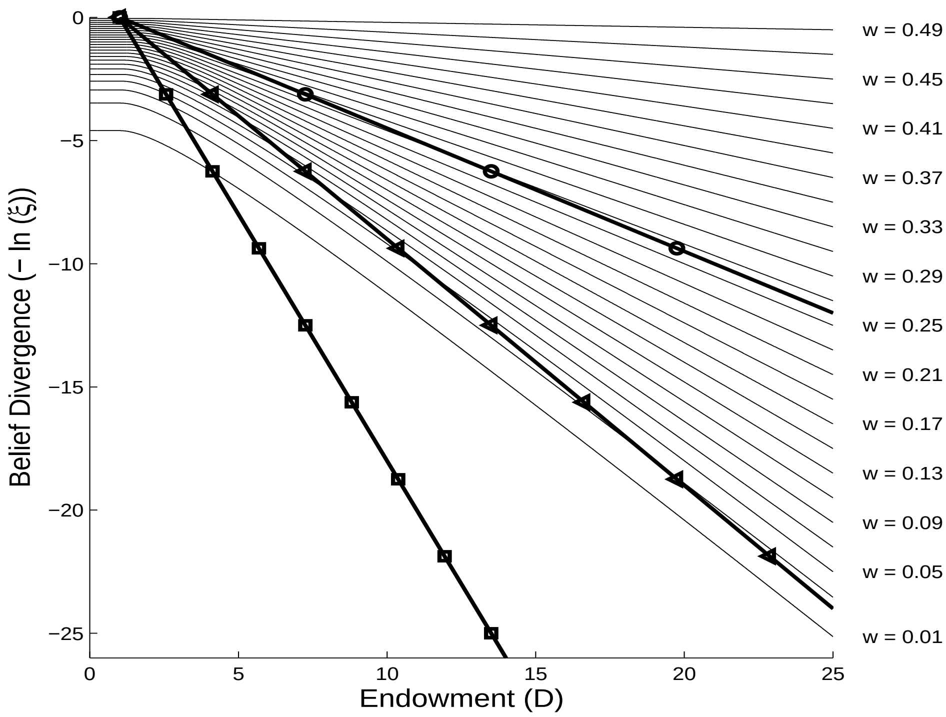

Figure 1: Survival. This figure illustrates survival results in Examples 3.1 and 3.2. We set $\lambda = 1$. We plot the aggregate endowment D on the horizontal axis, and the log relative belief density, $\ln(\xi)$, on the vertical axis. Solid lines are the level curves for the consumption share of agent $\mathbb{B}$, so that each solid line plots pairs $(D, \ln(\xi))$ that give rise to a given consumption share w. These pairs can be found by fixing w and plotting the value of $\ln(\xi)$ as a function of D, with the function given by the Pareto optimality condition (8). Labels for agent $\mathbb{B}$'s consumption share are shown alone the right margin. The marked lines show the median path ($Z_t = 0$) of $(D_t, \ln(\xi_t))$ for each case in Example 3.1. We choose $\delta = 0.5$ and set μ in cases (i), (ii), and (iii) to δ^2, $\delta^2/4$, and $\delta^2/2$ respectively. We mark the corresponding median paths with circles, squares, and triangles.

converges to a constant, and hence agent $\mathbb{B}$ survives. In case (ii), the median path (marked with squares) crosses consumption-share level curves from above, showing that as the economy grows, agent $\mathbb{B}$'s consumption share vanishes. Since the rate of accumulation of belief differences is identical in all three cases, the only reason why the three median paths have different slopes is because of the different growth rates of the endowment process. Slow endowment growth (and hence slow growth of local utility curvature) generates a steep median path, leading to agent $\mathbb{B}$'s extinction. The median path for case (iii) (marked with triangles) also crosses consumption-share level curves from above, showing that agent $\mathbb{B}$'s consumption share vanishes as the economy grows. The difference between cases (ii) and (iii) is in the rate at which agent $\mathbb{B}$'s consumption share vanishes. The rate of extinction is lower in case (iii), allowing agent $\mathbb{B}$ to retain impact on prices in the long run.

In Example 3.1, the aggregate endowment process is deterministic. The same conclusions carry over to a setting in which the two agents disagree about the distribution of the aggregate endowment process, as we show in the Example 3.2 below. In this example, the stochastic component of the endowment is stationary and thus does not affect the relation between the asymptotic growth rate of endowment and the rate of accumulation of belief differences. Different conclusions could result under alternative assumptions of endowment growth.

Example 3.2 *We modify Example 3.1 so that there is aggregate risk and agents disagree over the evolution of the aggregate endowment process. The aggregate endowment process is now given by*

$$D_t = (1 + \mu t)e^{X_t}, \tag{17}$$

$$dX_t = -\theta X_t \, dt + \sigma \, dZ_t. \tag{18}$$

Then,

i. If $\mu > \delta^2/2$*, agent* $\mathbb{B}$ *survives in the long run and exerts long-run impact on prices;*

ii. If $\mu < \delta^2/2$*, agent* $\mathbb{B}$ *fails to survive in the long run and does not exert long-run impact on prices;*

iii. If $\mu = \delta^2/2$*, agent* $\mathbb{B}$ *fails to survive in the long run but exerts long-run impact on prices.*

3.2 Beliefs

Our second set of examples illustrates how extinction depends on the assumptions on agents' beliefs.

Example 3.3 *Consider a continuous-time economy with the aggregate endowment given by a geometric Brownian motion:*

$$\frac{dD_t}{D_t} = \mu \, dt + dZ_t, \quad D_0 > 0. \tag{19}$$

Assume that the two agents have logarithmic preferences: $U(c) = \ln(c)$ *and that they do not know the growth rate of the endowment process. The agents start with Gaussian* *prior beliefs about μ, $\mathcal{N}(\hat{\mu}^i, (\nu^i)^2)$, $i \in \{\mathbb{A}, \mathbb{B}\}$, and update their beliefs based on the observed history of the endowment process according to the Bayes rule. Then, if both agents have non-degenerate priors ($\min(\nu^{\mathbb{A}}, \nu^{\mathbb{B}}) > 0$), then both agents survive in the long run. If agent $\mathbb{A}$ knows the exact value of the endowment growth rate but agent $\mathbb{B}$ does not, i.e., $\nu^{\mathbb{B}} > \nu^{\mathbb{A}} = 0$, then agent $\mathbb{B}$ fails to survive.*

In the above example, both agents' beliefs tend to the true value of the unknown parameter μ asymptotically and survival depends on the relative rate of learning. If both agents start not knowing the true value of μ, then, regardless of the bias or precision of their prior, they both learn at comparable rates. Formally, the ratio of the agents' belief densities converges to a positive finite constant. However, if one agent starts with perfect knowledge of the true parameter value, eventual convergence of the learning process by the other agent is not sufficient to guarantee that agent's survival.[9]

Our second example is motivated by Dumas et al. (2009), who study an economy with an irrational ("overconfident") agent who fails to account for noise in his signal during the learning process. We do not model the learning process of the overconfident agent explicitly, as Dumas et al. (2009) do, but instead postulate a qualitatively similar belief process exogenously. In our example, agent $\mathbb{B}$ is the analog of the overconfident agent in Dumas et al. (2009).

Example 3.4 *Consider a discrete-time economy with uncertainty described by the i.i.d sequence of independent normal variables $(\varepsilon_t, u_t) \sim N(0, I)$. Aggregate endowment is given by*

$$D_t = D_{t-1} \exp\left(\mu_{t-1} + \epsilon_t\right), \quad D_0 > 0, \tag{20}$$

where the conditional growth rate of the endowment, μ_{t-1} is a stationary moving average of the shocks $\epsilon_{t-1}, \epsilon_{t-2}, \ldots$. Assume that agent $\mathbb{A}$ knows the true value of μ_t, but agent $\mathbb{B}$. Specifically, agent $\mathbb{B}$'s estimate of the current growth rate of the endowment is given by $\mu_{t-1} + \delta_{t-1}$, where δ_t follows a finite-order moving average process driven by u_t. Assume that both agents have logarithmic preferences. Then, agent $\mathbb{B}$ fails to survive in the long run.

In the above example, agent $\mathbb{B}$'s errors follow a stationary process and thus do not diminish over time. Agent $\mathbb{B}$'s mistakes accumulate asymptotically so that

[9] Our example builds on the models of Basak (2005) and Detemple and Murthy (1994). See also Blume and Easley (2006) for further discussion of Bayesian learning and its implications for survival.

$\lim_{t\to\infty} \xi_t = 0$, and therefore he fails to survive.

The above examples emphasize the importance of the requirement that $\xi_t \to 0$, which is our usual condition on belief dispersion. In Example 3.3, both agents are learning about the true data-generating process at the same rate, and thus their relative errors do not accumulate fast enough over time. In contrast, in Example 3.4, one agent knows the true probability law while errors of the other agent accumulate fast enough so he fails to survive.

In our general analysis we do not take a stand on whether one set of incorrect beliefs is more or less "wrong" than another. We simply observe that the condition $\xi \to 0$ defines a particular criterion by which errors of agent $\mathbb{B}$ accumulate faster than errors of agent $\mathbb{A}$. In particular, market selection is ***not*** selection for "better" learning. For instance, in Example 3.3 $\ln \xi_t^{\mathbb{B}}$ grows at rate $\frac{1}{2}\ln(t)$. Thus, if agent $\mathbb{A}$ has any set of beliefs, not necessarily "rational", for which the differences from the true measure accumulate slower than $\ln(t)$, then agent $\mathbb{A}$ will survive and agent $\mathbb{B}$ will not. For example, agent $\mathbb{A}$ may be correct most of the time but make infrequent large errors so that $\ln \xi_t^{\mathbb{A}}$ asymptotically grows at rate $\frac{1}{3}\ln(t)$.[10] In this case $\mathbb{B}$'s mistakes vanish in magnitude over time, $\mathbb{A}$'s do not, and yet $\mathbb{A}$ survives and $\mathbb{B}$ does not because $\mathbb{B}$'s accumulated errors grow faster than $\mathbb{A}$'s accumulated errors.

3.3 Preferences

We now illustrate how survival depends on preferences. We consider a family of economies that differ only with respect to the agents' utility function. Suppose that agent $\mathbb{B}$ accumulates forecast errors at a higher rate than agent $\mathbb{A}$. For agent $\mathbb{B}$ to become extinct, agent $\mathbb{A}$ must bet sufficiently aggressively on his beliefs. If these agents have sufficiently high utility curvature at high consumption levels, $\mathbb{A}$ does not bet on his beliefs aggressively enough, which allows $\mathbb{B}$ to survive and have price impact. This result bears similarity to the finding that inferior forecasters can survive in incomplete market economies, since the agents are limited in their ability to bet on their beliefs by the available menu of financial assets. Whether the agents lack willingness to bet on their beliefs because to their preferences or face constraints, the end result is that agents with inferior forecasts are able to survive.

Example 3.5 *Consider a continuous-time economy with the aggregate endowment*

[10] Beker and Espino (2011), example 4, contains a similar construction.

given by a geometric Brownian motion:

$$D_t = \exp\left(\mu\, t + \sigma\, Z_t\right), \quad D_0 > 0, \quad \mu, \sigma > 0. \tag{21}$$

Assume that agent $\mathbb{A}$ uses the correct probability measure, $\mathbb{A} = \mathbb{P}$, but agent $\mathbb{B}$ has a constant bias, $\delta\sigma \neq 0$, in his forecasts of the growth rate of the endowment. Therefore,

$$\xi_t = \exp\left(-\frac{1}{2}\delta^2 t + \delta Z_t\right). \tag{22}$$

Let the absolute risk aversion function be $A(x) = \frac{1}{x}$ for $x < 1$ to preserve the Inada condition at zero, and

$$A(x) = x^{\alpha}, \quad \alpha \leq 0. \tag{23}$$

for $x \geq 1$. Then, if local utility curvature is declining rapidly enough, $\alpha \leq -1$, agent $\mathbb{B}$ does not survive and does not a ect prices asymptotically. If local utility curvature declines only slowly, $\alpha \in (-1, 0]$, then agent $\mathbb{B}$ survives and has price impact in the long run.

Decreasing absolute risk aversion (DARA) is a weak a priori restriction on utility functions, and within this family of preferences we can see how the properties of local utility curvature affect survival. When high levels of consumption generate a high propensity to accept gambles – local utility curvature declines rapidly in consumption – agent $\mathbb{B}$ does not survive and has no price impact.

4 Survival

In this section we present general necessary and sufficient conditions for survival. The following theorem shows formally how survival depends on the tradeoff between endowments, beliefs, and preferences.

Theorem 4.1 *A necessary condition for agent $\mathbb{B}$ to become extinct is that for all $\epsilon \in (0, \frac{1}{2})$,*

$$\limsup_{t\to\infty} \frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\, dx}{-\ln(\lambda \xi_t)} \leq 1, \quad \text{a.s.} \tag{24}$$

A su cient condition for his extinction is that the inequality is strict, i.e., for all $\epsilon \in (0, \frac{1}{2})$,

$$\limsup_{t\to\infty} \frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} < 1, \quad \text{a.s.} \tag{25}$$

From the conditions in Theorem 4.1, it is clear that survival depends on the joint properties of aggregate endowments (D_t), preferences (in particular, risk aversion $A(x)$), and beliefs (ξ_t). Survival is determined by the relation between the rate of accumulation of belief differences and the rate of decline of local utility curvature, evaluated at the aggregate endowment. Theorem 4.1 formalizes the informal discussion in the introduction. If utility curvature declines sufficiently slowly, the numerator in (24) dominates and agent $\mathbb{B}$ survives. Intuitively, the numerator captures the relation between differences in consumption and differences in marginal utilities between the two agents. The Pareto optimality condition (7) implies that if belief differences of the two agents accumulate sufficiently rapidly, their marginal utilities evaluated at their equilibrium consumption must diverge. But if utility curvature declines too slowly, increasing differences in marginal utilities fail to generate large differences in consumption. Equation (24) provides the precise restriction on the rate of decline in utility curvature necessary for agent $\mathbb{B}$'s extinction.

Next, we place strict assumptions on some of the primitives, thus simplifying the interplay between the endowment, beliefs, and preferences. The following straightforward applications of Theorem 4.1 identify a broad class of models in which agent $\mathbb{B}$ does not survive, under easily verifiable conditions on the utility function (in Corollary 4.2) or on the endowment process (in Corollary 4.3).

Corollary 4.2 *If local utility curvature is bounded so that* $A(x) \leq \overline{C}x^{-1}$*, and* $\lim_{t\to\infty} \xi_t = 0$*, a.s., then agent* $\mathbb{B}$ *never survives.*

If local utility curvature is bounded as in Corollary 4.2, then large differences in marginal utilities imply large differences in consumption. Therefore, as belief differences accumulate, agent $\mathbb{B}$ fails to survive. The class of models with bounded utility curvature in the manner of Corollary 4.2 (i.e., utilities with a bounded relative risk aversion coefficient) is quite large.

If the endowment process is bounded, then large differences in marginal utilities require one agent's consumption to approach zero. Sandroni (2000) and Blume and Easley (2006) study models with bounded endowment and heterogeneous beliefs and find that agent $\mathbb{B}$ fails to survive regardless of the exact form of preferences. We replicate this result as a consequence of Theorem 4.1.

Corollary 4.3 *If the aggregate endowment process is bounded away from zero and infinity,* $0 < \underline{D} \leq D_t \leq \overline{D} < \infty$*, and* $\lim_{t\to\infty} \xi_t = 0$*, a.s., then agent* $\mathbb{B}$ *never survives.*

In models covered by Corollaries 4.2 and 4.3, we sharpen the survival results further by establishing the rate of extinction of agents with inferior beliefs. In particular, in models satisfying restrictions on the local utility curvature as in Corollary 4.2, the rate of extinction is directly related to the rate of accumulation of differences in beliefs.

Proposition 4.4 *Assume that utility curvature is bounded so that* $\underline{C}x^{-1} \leq A(x) \leq \overline{C}x^{-1}$*, and* $\lim_{t\to\infty} \xi_t = 0$*, a.s. For each element of the probability space where* $\lim_{t\to\infty} \xi_t = 0$*, let* t *be large enough that* $\lambda\xi_t < 1$*. Then agent* $\mathbb{B}$*'s consumption share satisfies*

$$\frac{(\lambda\xi_t)^{1/\underline{C}}}{1 + (\lambda\xi_t)^{1/\underline{C}}} \leq w_t \leq \frac{(\lambda\xi_t)^{1/\overline{C}}}{1 + (\lambda\xi_t)^{1/\overline{C}}}. \tag{26}$$

The rate of extinction is proportional to the rate of accumulation of belief differences. For instance, consider the specification of endowment and beliefs as in Example 3.1. Under the preference restrictions in Proposition 4.4, agent $\mathbb{B}$'s consumption share approaches zero exponentially at the rate bounded between $\frac{1}{2}\underline{C}\delta^2$ and $\frac{1}{2}\overline{C}\delta^2$. As Fedyk et al. (2013) make clear, belief differences in realistic settings can accumulate arbitrarily quickly if agents disagree on the distribution of multiple sources of randomness. Thus, our convergence rate result shows that extinction happens at high rates in economies with quantitatively large disagreement between agents, or at low rates in economies with mild degrees of disagreement.

We obtain a similar characterization of extinction rates in economies with bounded aggregate endowment.

Proposition 4.5 *Assume that the aggregate endowment process is bounded away from zero and infinity and* $\lim_{t\to\infty} \xi_t = 0$*, a.s. For each element of the probability space where* $\lim_{t\to\infty} \xi_t = 0$*, let* t *be large enough that* $\lambda\xi_t < 1$*. Then, agent* $\mathbb{B}$*'s consumption share satisfies*

$$\frac{1}{\overline{D}}(u')^{-1}\left(\frac{u'(\underline{D}/2)}{\lambda\xi_t}\right) \leq w_t \leq \frac{1}{\underline{D}}(u')^{-1}\left(\frac{u'(\overline{D})}{\lambda\xi_t}\right). \tag{27}$$

As in Proposition 4.4, we find that the rate of extinction of agent $\mathbb{B}$ is directly related to the rate of accumulation of belief differences. However, the relationship between the two is modulated by the assumed utility function. Qualitatively, the sharper the rise in marginal utility as consumption level approaches zero, the lower the rate of extinction. For instance, with CRRA preferences, the rate of extinction is proportional to the rate of accumulation of belief differences. It is clear, however, that without restricting preferences it is impossible to place tighter bounds on the rate of extinction.

If the endowment process and the utility curvature are not bounded, as in Propositions 4.4 and 4.5, then the precise relation between the primitives is important in determining agent $\mathbb{B}$'s survival. We simplify the conditions of Theorem 4.1 for the class of utilities with (weakly) decreasing absolute risk aversion (DARA), which is generally considered to be the weakest a priori restriction on utility functions.

Proposition 4.6 *Suppose that the utility function exhibits DARA and* $\lim_{t\to\infty} \xi_t = 0$*, a.s. Then, for agent* $\mathbb{B}$ *to become extinct asymptotically, it is su cient that there exists a sequence of numbers* $\epsilon_n \in (0, \frac{1}{2})$ *converging to zero such that for any* n

$$\lim_{t\to\infty} \frac{A(\epsilon_n D_t) D_t}{-\ln(\lambda \xi_t)} = 0, \quad \text{a.s.} \tag{28}$$

For agent $\mathbb{B}$ *to survive, it is su cient that for some* $\epsilon \in (0, \frac{1}{2})$

$$\text{Prob}\left[\limsup_{t\to\infty} \frac{A(\epsilon D_t) D_t}{-\ln(\lambda \xi_t)} = \infty\right] > 0. \tag{29}$$

If, in addition,

$$\lim_{t\to\infty} \frac{A(D_t) D_t}{-\ln(\lambda \xi_t)} = \infty, \quad \text{a.s.} \tag{30}$$

then $\lim_{t\to\infty} w_t = \frac{1}{2}$, a.s.

The above proposition clarifies the trade-off between beliefs, endowments, and preferences that determines survival. Intuitively, given the endowment growth rate in the economy, agent $\mathbb{B}$ fails to survive if local utility curvature declines sufficiently fast with the consumption level, or if differences in beliefs accumulate at a sufficiently high rate.

We finally consider a generalization of the setting analyzed in Kogan et al. (2006) and Yan (2008), where endowment follows a Geometric Brownian motion and agent

$\mathbb{B}$ is persistently optimistic about the growth rate of the endowment. We make a weaker assumption that the endowment and belief differences grow at proportional asymptotic rates, i.e. $\lim_{t\to\infty} \frac{\ln(D_t)}{-\ln(\xi_t)} = b < \infty$. Such models, with geometric Brownian motion specifications for D and ξ in particular, are common in the literature.

Corollary 4.7 *Consider an economy with* $0 < \lim_{t\to\infty} \frac{\ln(D_t)}{-\ln(\xi_t)} = b < \infty$ *and* $\lim_{t\to\infty} \xi_t = 0$*, a.s. Assume that the utility function is of DARA type. Then, if*

$$\lim_{x\to\infty} \frac{A(x)}{\frac{1}{x}\ln(x)} = 0, \tag{31}$$

agent $\mathbb{B}$ *becomes extinct. If*

$$\lim_{x\to\infty} \frac{A(x)}{\frac{1}{x}\ln(x)} = \infty, \tag{32}$$

agent $\mathbb{B}$ *survives.*

We thus identify two broad classes of preferences for which survival does and does not take place under the above assumption on the endowment and beliefs. Agent $\mathbb{B}$ becomes extinct if local utility curvature at high consumption levels declines rapidly enough, and he survives if local utility curvature declines sufficiently slowly. Models with CRRA preferences and geometric Brownian motions for D and ξ satisfy condition (31): with relative risk aversion equal to γ, $\frac{A(x)}{\frac{1}{x}\ln(x)} = \frac{\gamma}{\ln(x)} \to 0$, and therefore agent $\mathbb{B}$ becomes extinct in such economies.

5 Price Impact

We now consider the influence agent $\mathbb{B}$ has on the long-run behavior of prices and how this influence is related to his survival. Below we identify broad classes of economies in which the agent making relatively inaccurate forecast does or does not have impact on prices in the long run.

As we have shown in Corollaries 4.2 and 4.3, if local utility curvature declines sufficiently rapidly with consumption level, or if the aggregate endowment is bounded, agent $\mathbb{B}$ does not survive. In these cases, agent $\mathbb{B}$ also has no price impact in the long run:

Proposition 5.1 *If local utility curvature is bounded such that* $A(x) \le \overline{C}x^{-1}$*, and* $\lim_{t\to\infty} \xi_t = 0$*, a.s., then agent* $\mathbb{B}$ *has no asymptotic price impact.*

Proposition 5.2 *If the aggregate endowment process is bounded away from zero and infinity,* $0 < \underline{D} \le D_t \le \overline{D} < \infty$*, and* $\lim_{t\to\infty} \xi_t = 0$*, a.s., then agent* $\mathbb{B}$ *has no price impact asymptotically.*

The relation between Arrow-Debreu prices and marginal utilities means that agent $\mathbb{B}$ can affect prices if he has nontrivial impact on the marginal utility of agent $\mathbb{A}$. When local utility curvature declines at a high enough rate, this requires him to have significant impact on consumption growth of agent $\mathbb{A}$. This in turn is impossible since agent $\mathbb{B}$ does not survive. Similarly, in economies with bounded aggregate endowment, agent $\mathbb{B}$ cannot have significant impact on consumption growth for agent $\mathbb{A}$, and so agent $\mathbb{B}$ cannot have price impact.

In addition to showing that agent $\mathbb{B}$ has no long-run impact on prices in economies covered by Propositions 5.1 and 5.2, we characterize how rapidly price impact of agent $\mathbb{B}$ disappears. We measure the magnitude of price impact according to (12) relative to the reference economy with utility weight $\lambda^\star = 0$ because price impact of agent $\mathbb{B}$ vanishes asymptotically with respect to this reference economy under either bounded coefficient of relative risk aversion or with a bounded aggregate endowment.

Proposition 5.3 *Assume that local utility curvature is bounded so that* $A(x) \le \overline{C} x^{-1}$*, and* $\lim_{t\to\infty} \xi_t = 0$*, a.s.. For each element of the probability space where* $\lim_{t\to\infty} \xi_t = 0$*, let* t *be large enough that* $\lambda \xi_t < 1$*. Then, the measure of price impact* $PI(t, s; 0)$ *satisfies*

$$|PI(t, s; 0)| \le \overline{C} \left((\lambda \xi_t)^{1/\overline{C}} + (\lambda \xi_{t+s})^{1/\overline{C}} \right). \tag{33}$$

In economies covered by Proposition 5.3, the distortion of the stochastic discount factor created by $\mathbb{B}$'s beliefs disappears at a rate at least proportional to the rate of accumulation of belief differences. As shown in Proposition 4.5, $\mathbb{B}$'s consumption share in such economies also vanishes at the rate of accumulation of belief differences.

The price impact of agent $\mathbb{B}$'s beliefs in economies with bounded aggregate endowment vanishes at least as quickly as $\mathbb{B}$'s consumption share:

Proposition 5.4 *Assume that the aggregate endowment process is bounded away from zero and infinity, and* $\lim_{t\to\infty} \xi_t = 0$*, a.s. For each element of the probability space where* $\lim_{t\to\infty} \xi_t = 0$*, let* t *be large enough that* $\lambda \xi_t < 1$*. Then, the measure of price impact* $PI(t, s; 0)$ *satisfies*

$$|PI(t, s; 0)| \le \left(\max_{x \in [\underline{D}/2, \overline{D}]} A(x) \right) (w_t + w_{t+s}) \overline{D}, \tag{34}$$

where w_t is the equilibrium consumption share of agent $\mathbb{B}$.

As we show in Proposition 4.5, in economies with bounded aggregate endowment, $\mathbb{B}$'s consumption share vanishes at the rate directly related to the rate of accumulation of belief differences. Since the exact extinction rate generally depends on preferences in a nonlinear manner, we do not derive an explicit bound on price impact in Proposition 5.4 in terms of the primitives, and instead express an upper bound on price impact in terms of $\mathbb{B}$'s consumption share.

When conditions on the primitives are relaxed, consumption and price dynamics may have more complex properties. It is possible for agent $\mathbb{B}$ to affect the marginal utility of agent $\mathbb{A}$ without having non-vanishing asymptotic effect on agent $\mathbb{A}$'s consumption growth, which means that agent $\mathbb{B}$'s price impact may be consistent with his extinction. We have described such an economy in Example 3.1. In the context of Example 3.1, price impact without survival occurs in a "knife-edge" case. We next explore more generally the conditions under which such a phenomenon occurs.

Proposition 5.5 *Consider an economy with a DARA utility function, assume that* $\lim_{t\to\infty} D_t = \infty$*, a.s., and* $\lim_{t\to\infty} \xi_t = 0$*, a.s. Then, a necessary condition for price impact is*

$$\limsup_{t\to\infty} \frac{\int_1^{D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} \geq 1, \text{ a.s.} \tag{35}$$

A necessary condition for price impact without survival is

$$\limsup_{t\to\infty} \frac{\max_{z\in[1,D_t]} \gamma(z)\ln(D_t)}{-\ln(\lambda\xi_t)} \geq 1 \geq \limsup_{t\to\infty} \frac{\gamma(D_t)}{-\ln(\lambda\xi_t)}, \text{ a.s.} \tag{36}$$

The above result hinges on the difference between extinction in consumption share ($w_t \to 0$) and in consumption level ($w_t D_t \to 0$). As we have shown in Section 2 (and show again formally in the Appendix), there is no price impact when $\mathbb{B}$ experiences extinction in consumption level. Thus, a necessary condition for price impact is $\limsup_{t\to\infty} w_t D_t > 0$, a.s., which we use to derive (35). We combine equation (35) with Theorem 4.1 to find DARA-utility economies that may exhibit price impact with extinction (or, without survival). The result is (36).

The necessary conditions for asymptotic price impact without survival give us an idea of how special this phenomenon is. Consider, for instance, Examples 3.1 and 3.2. In these examples, $A(x) = 1$ for $x \geq 1$, and $A(x) = \frac{1}{x}$ for $x < 1$, and hence for $D > 1$,

$\max_{z\in[1,D]}\gamma(z) = \gamma(D) = D$. Then, Proposition 5.5 implies that in the context of these examples, necessary conditions for price impact without survival are satisfied only in case (iii), for which

$$\lim_{t\to\infty}\frac{D_t}{-\ln(\lambda\xi_t)} = 1, \text{ a.s.} \tag{37}$$

While one can design more general specifications of $A(x)$ that allow for price impact without survival, informally, Proposition 5.5 shows that the set of such economies is rather restricted.

Survival does not always imply asymptotic price impact. Below, we establish necessary and sufficient conditions for price impact in a broad class of economies in which agent $\mathbb{B}$ survives. We present the proof of Propositions 5.6, 5.7, and 5.8 in the on-line Appendix.

Proposition 5.6 *Consider a growing economy with* $\lim_{t\to\infty} D_t = \infty$*, a.s. Assume that the utility function is such that* $A(x)$ *is weakly decreasing and* $xA(x)$ *is weakly increasing in consumption level. Further, assume that the endowment and the beliefs satisfy*

$$\lim_{t\to\infty}\frac{A(\frac{1}{2}D_t)D_t}{(\ln(\lambda\xi_t))^2} = \infty, \quad \text{a.s.} \tag{38}$$

Then agent $\mathbb{B}$ *survives and asymptotically consumes half of the aggregate endowment.*

Proposition 5.7 *In the economy defined in Proposition 5.6, if the belief process* ξ_t *has non-vanishing growth rate asymptotically, i.e., there exist* $s > 0$ *and* $\epsilon > 0$ *such that*

$$\text{Prob}\left[\limsup_{t\to\infty}|\ln(\xi_{t+s}) - \ln(\xi_t)| > \epsilon\right] > 0, \tag{39}$$

and, in addition, there exist $s' > 0$ *and* $\epsilon' > 0$ *such that*

$$\text{Prob}\left[\limsup_{t\to\infty}\left|\int_{D_{t+s'}/2}^{D_{t+s'}} A(x)\,dx - \int_{D_t/2}^{D_t} A(x)\,dx - \frac{1}{2}\ln(\xi_{t+s'}) + \frac{1}{2}\ln(\xi_t)\right| > \epsilon'\right] > 0, \tag{40}$$

then agent $\mathbb{B}$ *exerts long-run price impact. Moreover, asymptotically the state price density does not depend on the initial wealth distribution, i.e., does not depend on* λ.

Condition (39) is necessary for asymptotic price impact to exist in this economy.

Proposition 5.7 shows that survival does not necessarily lead to price impact: there can be an additional condition on the variation in the belief process. Note that if condition (39) holds, (40) would be violated only in economies with a very particular combination of primitives. Such cases are covered by the additional condition (40).

The next proposition helps clarify why the survival and price impact properties of various economies are connected to the rate of decrease in local utility curvature. It states a general result for consumption sharing rules in growing economies with unbounded relative risk aversion.

Proposition 5.8 *Consider the economy defined in Proposition 5.6, and assume that both agents hold the same beliefs. Then, for any initial allocation of wealth between the agents, their consumption shares become asymptotically equal. Moreover, the state price density in this economy is asymptotically the same as in an economy in which the two agents start with equal endowments.*

As we know from Corollary 4.2 and Proposition 5.1, economies with agents having local utility curvature $A(x)$ rapidly declining in consumption exhibit simple behavior: agent $\mathbb{B}$ does not survive and has no asymptotic impact on the state-price density. When local utility curvature is only slowly declining, survival is also determined by belief differences. Proposition 5.8 shows that in a homogeneous-belief economy consumption shares of the agents tend to become equalized over time no matter how uneven the initial wealth distribution is. Similarly, the state-price density does not depend (asymptotically) on the initial wealth distribution.

The above convergence mechanism remains at work in economies with heterogeneous beliefs. However, there is another force present under belief heterogeneity: agent $\mathbb{B}$ tends to mis-allocate his consumption across states due to his distorted beliefs, which reduces his asymptotic consumption share. The tradeoff between the two competing forces is intuitive: distortions in agents' consumption shares caused by belief differences tend to disappear over time, unless the belief differences grow sufficiently rapidly. Condition (38) guarantees that differences in beliefs do not accumulate too quickly, and so under this condition agent $\mathbb{B}$ survives and consumes half of the aggregate endowment asymptotically. Agent $\mathbb{B}$ can exert asymptotic price impact as long as the differences between his beliefs and those of agent $\mathbb{A}$ do not vanish asymptotically in a sense made precise by the conditions in Proposition 5.7.

6 Conclusion

In this paper we examine the economic mechanism behind the Market Selection Hypothesis and establish necessary and sufficient conditions for its validity in a general setting with minimal restrictions on endowments, beliefs, or utility functions. We show that the MSH holds in economies with bounded endowments or bounded relative risk aversion. The commonly studied special case of constant relative risk aversion preferences belongs to this class of models. However, we show that the MSH cannot be generalized without additional qualifications to a broader class of models. Instead, survival is determined by a comparison of the forecast errors to risk attitudes. The price impact of inaccurate forecasts is distinct from survival because price impact is determined by the volatility of traders' consumption shares rather than by their level. We show a new mechanism by which an agent who fails to survive in the long run can exert persistent impact on prices. This phenomenon exists because the disappearing agent can provide a non-trivial degree of risk sharing in equilibrium.

Our results apply to economies with state-dependent time-separable preferences, such as external habit formation. In the on-line Appendix, we show how our approach extends to this broader class of preferences. One limitation of our approach is that we consider only time-additive preferences, and thus cannot separate risk aversion effects from inter-temporal substitution effects. Extending theoretical analysis of the Market Selection Hypothesis to a broader class of preferences remains an important open problem.

A Appendix

A.1 Proof of Theorem 4.1

Suppose that agent $\mathbb{B}$ becomes extinct, i.e., $w_t = \frac{C_{\mathbb{B},t}}{D_t}$ converges to zero almost surely. For each element of the probability space ω for which w_t vanishes asymptotically, one can find $T(\omega;\epsilon)$, such that $w_t(\omega) < \epsilon$ for any $t > T(\omega;\epsilon)$. Since $\int_{wD}^{(1-w)D} A(x)\,dx$ is a decreasing function of w, the first-order condition (8) implies that for all $t > T(\omega;\epsilon)$

$$1 = \frac{\int_{w_t D_t}^{(1-w_t)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} \geq \frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)}. \tag{A.1}$$

Thus, the desired result follows by applying $\limsup_{t\to\infty}$ to both sides of the inequality.

We now prove the sufficient condition. Consider the subset of the probability space over which $\limsup_{t\to\infty} \frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} < 1$ for any $\epsilon > 0$. For each element of the probability space w in such set, we can define $T(\omega;\epsilon)$ and $\delta(\omega) > 0$, such that

$$\frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} \leq 1 - \delta \tag{A.2}$$

for all $t > T(\omega;\epsilon)$. If $\limsup_{t\to\infty} w_t \neq 0$, then one can always find $\epsilon > 0$ and $t > T(\omega;\epsilon)$, such that $w_t > \epsilon$. But then

$$1 = \frac{\int_{w_t D_t}^{(1-w_t)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} \leq \frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)}. \tag{A.3}$$

Taking $\limsup_{t\to\infty}$ on both sides, implies $1 \leq 1 - \delta$, which is a contradiction.

A.2 Proof of Corollary 4.2

By assumption $xA(x) < \overline{C}$ for all x and $\overline{C} > 0$. For every path of ξ that converges to zero, let t be large enough so that $\lambda\xi_t < 1$. Then

$$\frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} \leq \overline{C}\frac{\ln(1-\epsilon) - \ln(\epsilon)}{-\ln(\lambda\xi_t)} \tag{A.4}$$

which converges to zero almost surely as $t \to \infty$.

A.3 Proof of Corollary 4.3

Let $\underline{D}$ and $\overline{D}$ denote the upper and lower bound of D_t, $0 < \underline{D} \leq \overline{D}$. Let $\overline{A}(\epsilon)$ denote the maximum of $A(x)$ on $[\epsilon\underline{D}, (1-\epsilon)\overline{D}]$. We then have $\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)dx \leq \left((1-\epsilon)\overline{D} - \epsilon\underline{D}\right)\overline{A}(\epsilon)$, which is finite. Given that $\xi_t \to 0$, *a.s.*, as $t \to \infty$ and hence $-\ln(\xi_t) \to \infty$, *a.s.*, we conclude that (25) holds, and agent $\mathbb{B}$ does not survive.

A.4 Proof of Proposition 4.4

We establish the upper bound. Derivation of the lower bound is analogous. For every path of ξ that converges to zero, let t be large enough so that $\lambda\xi_t < 1$. Using the Pareto optimality condition and the restriction $A(x) \leq \overline{C}x^{-1}$,

$$-\ln\lambda\xi_t = \int_{w_t D_t}^{(1-w_t)D_t} A(x)dx \leq \int_{w_t D_t}^{(1-w_t)D_t} \overline{C}x^{-1}dx = \overline{C}\ln\frac{1-w_t}{w_t}, \tag{A.5}$$

which implies

$$\frac{w_t}{1-w_t} \leq (\lambda\xi_t)^{1/\overline{C}}. \tag{A.6}$$

The upper bound then follows.

A.5 Proof of Proposition 4.5

Using Pareto optimality condition and concavity of the utility function,

$$\lambda\xi_t = \frac{u'((1-w_t)D_t)}{u'(w_t D_t)} \geq \frac{u'(\overline{D})}{u'(w_t D_t)}, \tag{A.7}$$

which implies

$$u'(w_t D_t) \geq \frac{u'(\overline{D})}{\lambda\xi_t} \tag{A.8}$$

and therefore

$$w_t \leq \frac{1}{D_t}(u')^{-1}\left(\frac{u'(\overline{D})}{\lambda\xi_t}\right) \leq \frac{1}{\underline{D}}(u')^{-1}\left(\frac{u'(\overline{D})}{\lambda\xi_t}\right). \tag{A.9}$$

Similarly, to derive the lower bound, use

$$\lambda\xi_t = \frac{u'((1-w_t)D_t)}{u'(w_tD_t)} \le \frac{u'(\underline{D}/2)}{u'(w_tD_t)}, \tag{A.10}$$

which implies

$$u'(w_tD_t) \le \frac{u'(\underline{D}/2)}{\lambda\xi_t} \tag{A.11}$$

and therefore

$$w_t \ge \frac{1}{D_t}(u')^{-1}\left(\frac{u'(\underline{D}/2)}{\lambda\xi_t}\right) \ge \frac{1}{\overline{D}}(u')^{-1}\left(\frac{u'(\underline{D}/2)}{\lambda\xi_t}\right). \tag{A.12}$$

A.6 Proof of Proposition 4.6

For every path of ξ that converges to zero, let t be large enough so that $\lambda\xi_t < 1$. Since $A(x)$ is a non-increasing function,

$$\int_{\epsilon' D}^{(1-\epsilon')D} A(x)\,dx \ge \int_{\epsilon' D}^{\epsilon D} A(x)\,dx \ge A(\epsilon D)D(\epsilon-\epsilon'), \tag{A.13}$$

where $0 < \epsilon' < \epsilon < 1-\epsilon'$. Condition (29) then implies that with positive probability, we have

$$\limsup_{t\to\infty} \frac{\int_{\epsilon' D_t}^{(1-\epsilon')D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} = \infty \tag{A.14}$$

and hence a necessary condition for extinction is violated (from Theorem 4.1). Thus, agent $\mathbb{B}$ survives.

Next, for any $\epsilon \in (0, 1/2)$, find $\epsilon_n < \epsilon$. Then, since $A(x)$ is a non-increasing function,

$$\int_{\epsilon D}^{(1-\epsilon)D} A(x)\,dx \le A(\epsilon D)D(1-2\epsilon) \le A(\epsilon_n D)D(1-2\epsilon). \tag{A.15}$$

Then,

$$\frac{\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} \le \frac{A(\epsilon_n D_t)D_t(1-2\epsilon)}{-\ln(\lambda\xi_t)}, \tag{A.16}$$

and the result follows from applying (28) to show that we have a sufficient condition for extinction (25).

Lastly, since the utility function is of DARA type, condition (8) implies that

$$-\ln(\lambda\xi_t) \geq A((1-w_t)D_t)D_t(1-2w_t) \tag{A.17}$$

and therefore, using condition (30), $\lim_{t\to\infty} w_t = 1/2$ $a.s.$

A.7 Proof of Corollary 4.7

Consider a set (of measure one) on which $\lim_{t\to\infty} \frac{\ln(D_t)}{-\ln(\xi_t)} = b$ and $\lim_{t\to\infty} \xi_t = 0$. On this set,

$$\lim_{t\to\infty} \frac{A(\epsilon D_t)\epsilon D_t}{-\ln(\lambda\xi_t)} = \lim_{t\to\infty} \frac{\frac{1}{D_t}\ln(D_t)}{-\ln(\lambda\xi_t)} \frac{A(\epsilon D_t)\epsilon D_t}{\frac{1}{D_t}\ln(D_t)} = \epsilon b \lim_{t\to\infty} \frac{A(\epsilon D_t)}{\frac{1}{D_t}\ln(D_t)} \tag{A.18}$$

for any positive ϵ. Thus, by Proposition 4.6, agent $\mathbb{B}$ becomes extinct as long as the risk aversion coefficient satisfies (31). According to the same proposition, if the risk aversion coefficient satisfies (32), then agent $\mathbb{B}$ survives.

A.8 Proof of Propositions 5.1 and 5.2

As we show in corollary 4.2, there is no survival in models with bounded relative risk aversion. Thus, w_t converges to zero almost surely. Consider now the first term in (12). By the mean value theorem and the bound given in the proposition, this term equals

$$A(x^\star_{t+s})D_{t+s}w_{t+s} \leq \overline{C}\frac{D_{t+s}w_{t+s}}{x^\star_{t+s}}, \tag{A.19}$$

for some $x^\star_{t+s} \in [(1-w_{t+s})D_{t+s}, D_{t+s}]$. Since, almost surely, the ratio $\frac{D_{t+s}}{x_{t+s}*}$ converges to one and w_{t+s} converges to zero, we conclude that the first term in (12) converges to zero almost surely. The same argument implies that the second term converges to zero almost surely, and therefore there is no price impact. This proves Proposition 5.1. Proposition 5.2 follows from the fact that bounding the endowment implies bounding relative risk aversion on the interval $(D_t(1-w_t), D_t)$.

A.9 Proof of Proposition 5.3

We use the tighter of the two upper bounds on the consumption share w_t of agent $\mathbb{B}$ in Proposition 4.4:

$$\left|\int_{D_t(1-w_t)}^{D_t} A(x)\,dx\right| \leq \overline{C}\,|\ln(1-w_t)| \leq \overline{C}\left|\ln\left(1+(\lambda\xi_t)^{1/\overline{C}}\right)\right| \leq \overline{C}\,(\lambda\xi_t)^{1/\overline{C}}.$$

Then,

$$\begin{aligned}|PI(t,s;0)| &= \left|\int_{D_{t+s}(1-w_{t+s})}^{D_{t+s}} A(x)\,dx - \int_{D_t(1-w_t)}^{D_t} A(x)\,dx\right| \\ &\leq \left|\int_{D_{t+s}(1-w_{t+s})}^{D_{t+s}} A(x)\,dx\right| + \left|\int_{D_t(1-w_t)}^{D_t} A(x)\,dx\right| \\ &\leq \overline{C}\left((\lambda\xi_t)^{1/\overline{C}} + (\lambda\xi_{t+s})^{1/\overline{C}}\right) \qquad (A.20)\end{aligned}$$

A.10 Proof of Proposition 5.4

For every path of ξ that converges to zero, let t be large enough so that $\lambda\xi_t < 1$ and therefore $w_t \leq 1/2$. Because aggregate endowment is bounded, $\underline{D} \leq D_t \leq \overline{D}$,

$$\left|\int_{D_t(1-w_t)}^{D_t} A(x)\,dx\right| \leq \left(\sup_{x\in[\underline{D}/2,\overline{D}]} A(x)\right) w_t\overline{D} \qquad (A.21)$$

and therefore

$$\begin{aligned}|PI(t,s;0)| &= \left|\int_{D_{t+s}(1-w_{t+s})}^{D_{t+s}} A(x)\,dx - \int_{D_t(1-w_t)}^{D_t} A(x)\,dx\right| \\ &\leq \left|\int_{D_{t+s}(1-w_{t+s})}^{D_{t+s}} A(x)\,dx\right| + \left|\int_{D_t(1-w_t)}^{D_t} A(x)\,dx\right| \\ &\leq \left(\sup_{x\in[\underline{D}/2,\overline{D}]} A(x)\right)\overline{D}(w_t + w_{t+s}). \qquad (A.22)\end{aligned}$$

A.11 Proof of Proposition 5.5

For every path of ξ that converges to zero, let t be large enough so that $\lambda\xi_t < 1$. Consider the measure of price impact defined with respect to $\lambda^\star = 0$, as given in (12).

In an economy with DARA utility function,

$$0 \leq \int_{(1-w_t)D_t}^{D_t} A(x)dx \leq w_t D_t A((1-w_t)D_t), \tag{A.23}$$

and therefore, for price impact to exist asymptotically, it is necessary that

$$\limsup_{t\to\infty} w_t D_t > 0, \quad \text{a.s.} \tag{A.24}$$

The equilibrium consumption sharing rule in (8) implies that

$$-\ln(\lambda\xi_t) \leq \int_{w_t D_t}^{D_t} A(x)\,dx, \tag{A.25}$$

and therefore, to satisfy the condition (A.24), the model primitives must satisfy

$$\limsup_{t\to\infty} \frac{\int_1^{D_t} A(x)\,dx}{-\ln(\lambda\xi_t)} \geq 1, \quad \text{a.s.} \tag{A.26}$$

where, because $\lim_{t\to\infty}\xi_t = 0$, a.s., the lower limit in the integral can be set to any constant.

Next, we use

$$\int_1^{D_t} A(x)dx = \int_1^{D_t} A(x)\frac{1}{x}x\,dx \leq \max_{z\in[1,D_t]} \gamma(z)\ln(D_t), \tag{A.27}$$

to establish the first inequality in (36). Then, starting from (24), we have

$$\int_{\epsilon D_t}^{(1-\epsilon)D_t} A(x)dx \geq A(D_t)(1-2\epsilon)D_t = \gamma(D_t)(1-2\epsilon) \tag{A.28}$$

for any $\epsilon > 0$. Thus, a corollary of Theorem 4.1 is that a weaker necessary condition for extinction is

$$\limsup_{t\to\infty} \frac{\gamma(D_t)}{-\ln(\lambda\xi_t)} \leq 1, \text{ a.s..} \tag{A.29}$$

which yields the second inequality in (36).

References

Abel, A. B. (1990). Asset prices under habit formation and catching up with the joneses. *American Economic Review 80*, 38–42.

Basak, S. (2005). Asset pricing with heterogeneous beliefs. *Journal of Banking and Finance 29*, 2849–2881.

Beker, P. and S. Chattopadhyay (2010). Consumption dynamics in general equilibrium: A characterisation when markets are incomplete. *Journal of Economic Theory 145* (6), 2133–2185.

Beker, P. and E. Espino (2011). The dynamics of efficient asset trading with heterogeneous beliefs. *Journal of Economic Theory 146*, 189–229.

Blume, L. and D. Easley (2006). If you're so smart, why aren't you rich? belief selection in complete and incomplete markets. *Econometrica 74*, 929–966.

Borovička, J. (2015). Survival and long-run dynamics with heterogeneous beliefs under recursive preferences. Working Paper, University of Chicago.

Caballero, R. J. (1991). Earnings uncertainty and aggregate wealth accumulation. *American Economic Review 81* (4), 859–871.

Campbell, J. Y. and J. H. Cochrane (1999). By force of habit: a consumption based explanation of aggragate stock market behavior. *Journal of Political Economy 107*, 205–251.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldman (1991). The survival of noise traders in financial markets. *Journal of Business 64*, 1–19.

Detemple, J. and S. Murthy (1994). Intertemporal asset pricing with heterogeneous beliefs. *Journal of Economic Theory 62*, 294–320.

Dumas, B., A. Kurshev, and R. Uppal (2009). Equilibrium portfolio strategies in the presence of sentiment risk and excess volatility. *Journal of Finance 64* (2), 579–629.

Fedyk, Y., C. Heyerdahl-Larsen, and J. Walden (2013). Market selection and welfare in a multi-asset economy. *Review of Finance 17* (3), 1179–1237.

Friedman, M. (1953). The case for flexible exchange rates. In *Essays in Positive Economics*, pp. 157–203. Chicago: University of Chicago Press.

Garleanu, N. (2009). Portfolio choice and pricing in illiquid markets. *Journal of Economic Theory 144*(2), 532–564.

Gromb, D. and D. Vayanos (2002). Equilibrium and welfare in markets with financially constrained arbitrageurs. *Journal of Financial Economics 66*(2-3), 361–407.

Huang, J. and J. Wang (2009). Liquidity and market crashes. *Review of Financial Studies 22*(7), 2607–2643.

Kimball, M. S. and N. G. Mankiw (1989). Precautionary saving and the timing of taxes. *Journal of Political Economy 97*(4), 863–879.

Kogan, L., S. A. Ross, J. Wang, and M. M. Westerfield (2006). The price impact and survival of irrational traders. *Journal of Finance 61*, 195–229.

Lo, A. W., H. Mamaysky, and J. Wang (2004). Asset prices and trading volume under fixed transactions costs. *Journal of Political Economy 112*(5), 1054–1090.

Rubinstein, M. (1974). An aggregation theorem for securities markets. *Journal of Financial Economics 1*(3), 225–244.

Sandroni, A. (2000). Do markets favor agents able to make accurate predictions? *Econometrica 68*, 1303–134.

Shefrin, H. (2005). *A Behavioral Approach to Asset Pricing* (first ed.). Academic Press Advanced Finance Series. San Diego: Elsevier Academic Press.

Shiryaev, A. (1996). *Probability* (second ed.), Volume 95 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.

Vayanos, D. (1999). Strategic trading and welfare in a dynamic market. *Review of Economic Studies 66*(2), 219–254.

Xiong, W. and H. Yan (2010). Heterogeneous expectations and bond markets. *Review of Financial Studies 23*, 1433–1466.

Yan, H. (2008). Natural selection in financial markets: does it work? *Management Science 54*(11), 1935–1950.

Yuan, K. (2005). Asymmetric price movements and borrowing constraints: A rational expectations equilibrium model of crises, contagion, and confusion. *Journal of Finance 60*(1), 379–411.

On-Line Appendix (Appendices B and C) for **Market Selection**

In this on-line Appendix for "Market Selection" we include explicit calculations for our examples (Section B), including an example of the impact of distorted beliefs on the prices of non-primitive claims. We follow that with our analysis of the case of state dependent preferences (Section C).

B Examples

B.1 Examples 3.1 and 3.2

First, we establish survival results. For every path of ξ that converges to zero, let t be large enough so that $\lambda\xi_t < 1$. Consider the Pareto optimality condition (8). w_t is given by

$$w_t = \frac{1}{2}\left(1 + \frac{\ln(\lambda\xi_t)}{D_t}\right) \tag{B.1}$$

if it satisfies $w_t D_t > 1$, i.e., (B.1) describes the equilibrium consumption share of agent $\mathbb{B}$ if

$$\frac{1}{2}\left(\ln(\lambda\xi_t) + D_t\right) > 1. \tag{B.2}$$

Given the specification of the beliefs and the endowment process,

$$\lim_{t\to\infty} \frac{1}{2}\left(1 + \frac{\ln(\lambda\xi_t)}{D_t}\right) = \frac{1}{2}\left(1 - \frac{\delta^2/2}{\mu}\right) \quad \text{a.s.} \tag{B.3}$$

In case (i), $\mu > \delta^2/2$, and therefore (B.2) is satisfied for large enough t. Then, (B.1) holds and $\lim_{t\to\infty} w_t > 0$, thus agent $\mathbb{B}$ survives.

In case (ii), $\mu < \delta^2/2$, and therefore, for large enough t, (B.2) is violated. We conclude then that, for large enough t, $w_t \le D_t^{-1}$, and therefore agent $\mathbb{B}$ fails to survive.

In case (iii), either (B.2) holds, or $w_t D_t \le 1$, so that

$$w_t \le \min\left(D_t^{-1}, \frac{1}{2}\left(1 + \frac{\ln(\lambda\xi_t)}{D_t}\right)\right). \tag{B.4}$$

Since both terms on the right-hand-side converge to zero almost surely, we conclude that agent $\mathbb{B}$ fails to survive.

Next, we establish the price impact results. Consider a reference economy ($\xi_t = 1$) with the utility weight $\lambda^\star \geq 0$. The Pareto optimality condition (8) implies that

$$-\ln(\lambda^\star) = \int_{w_t^\star D_t}^{(1-w_t^\star)D_t} A(x)\, dx, \tag{B.5}$$

where $w_t^\star$ is the consumption share process of agent $\mathbb{B}$. Since $\lim_{t\to\infty} D_t = \infty$, this implies that for almost every path of the economy, for large enough t,

$$w_t^* = \begin{cases} \frac{1}{2}\left(1 + \frac{\ln(\lambda^\star)}{D_t}\right), & \lambda^\star > 0, \\ 0, & \lambda^\star = 0. \end{cases} \tag{B.6}$$

Then, the ratio of the pricing kernel in the heterogeneous-belief economy to the one in the reference economy is given by

$$\frac{m_t}{m_t^\star} = \begin{cases} e^{D_t w_t - \frac{1}{2}D_t - \frac{1}{2}\ln(\lambda^\star)}, & \lambda^\star > 0, \\ e^{D_t w_t}, & \lambda^\star = 0. \end{cases} \tag{B.7}$$

Consider case (i), $\mu > \delta^2/2$. In this case, $\lim_{t\to\infty} w_t = w_\infty > 0$ *a.s.* Thus, for almost every path of the economy, for large enough t,

$$w_t D_t = \frac{1}{2}\left(D_t + \ln(\lambda\xi_t)\right). \tag{B.8}$$

To show that there exists price impact, we must show that $\ln\left(\frac{m_{t+s}}{m_{t+s}^\star}\frac{m_t^\star}{m_t}\right)$ does not converge to zero. For large enough t,

$$\ln\left(\frac{m_{t+s}}{m_{t+s}^\star}\frac{m_t^\star}{m_t}\right) = \begin{cases} \frac{1}{2}\left(-\frac{\delta^2}{2}s + \delta(Z_{t+s} - Z_t)\right), & \lambda^\star > 0, \\ \frac{1}{2}\left(D_{t+s} - D_t - \frac{\delta^2}{2}s + \delta(Z_{t+s} - Z_t)\right), & \lambda^\star = 0, \end{cases} \tag{B.9}$$

The fact that $Z_{t+s} - Z_t \sim \mathcal{N}(0, s)$ is independent of filtration at t establishes that there is price impact in case (i) for $\lambda^\star > 0$. The case of $\lambda^\star = 0$ follows an analogous argument.

In case (ii), $\mu < \delta^2/2$ and thus $\lim_{t\to\infty} \frac{D_t}{-\ln(\xi_t)} = 0$ *a.s.* The Pareto optimality then states that for almost every path of the economy, for large enough t,

$$D_t(1 - w_t) - 1 - \ln\left(w_t D_t\right) = -\ln(\lambda\xi_t) \tag{B.10}$$

and therefore $\lim_{t\to\infty} w_t D_t = 0$ $a.s.$. This implies that there is no price impact relative to the benchmark economy with $\lambda^\star = 0$.

In case (iii), $\mu = \delta^2/2$ and therefore $\limsup_{t\to\infty} w_t D_t = \infty$ $a.s.$ Then, for almost every path, there exists an unbounded increasing sequence of times t_k such that $w_{t_k} D_{t_k} > 1$. For $t = t_k$, using (B.1), the ratio $\frac{m_t}{m_t^\star}$ is given by

$$\frac{m_t}{m_t^\star} = \begin{cases} e^{\frac{1}{2}\ln(\lambda\xi_t) - \frac{1}{2}\ln(\lambda^\star)}, & \lambda^\star > 0, \\ e^{D_t w_t}, & \lambda^\star = 0. \end{cases} \tag{B.11}$$

Since, along such a sequence, $Z_{t_k+s} - Z_{t_k} > |\delta| s$ infinitely often, we can select a further subsequence t'_k, such that $w_{t'_k} D_{t'_k} > 1$ and $Z_{t'_k+s} - Z_{t'_k} > |\delta| s$. Note that this implies that $w_{t'_k+s} D_{t'_k+s} > 1$ and therefore $\frac{m_{t'_k+s}}{m^\star_{t'_k+s}}$ is also given by (B.11). Then, for $\lambda^\star > 0$,

$$\ln\left(\frac{m_{t'_k+s}}{m^\star_{t'_k+s}} \frac{m^\star_{t'_k}}{m_{t'_k}}\right) = \frac{1}{2}\left(-\frac{\delta^2}{2}s + \delta(Z_{t'_k+s} - Z_{t'_k})\right) \tag{B.12}$$

and therefore $\ln\left(\frac{m_{t'_k+s}}{m^\star_{t'_k+s}} \frac{m^\star_{t'_k}}{m_{t'_k}}\right) > \delta^2 s/4$. Thus, $\ln\left(\frac{m_{t+s}}{m^\star_{t+s}} \frac{m^\star_t}{m_t}\right)$ fails to converge to zero on a set of measure one and we conclude that there exists price impact for $\lambda^\star > 0$. The case of $\lambda^\star = 0$ follows an analogous argument.

B.2 Example 3.3

Define $\delta^{\mathbb{A}}_t = \hat{\mu}^{\mathbb{A}}_t - \mu$. Then, using the Kalman Filter,

$$d\delta^{\mathbb{A}}_t = -\delta^{\mathbb{A}}_t \nu^{\mathbb{A}}_t dt + \nu^{\mathbb{A}}_t dZ_t \quad \text{and} \quad d\nu^{\mathbb{A}}_t = -\left(\nu^{\mathbb{A}}_t\right)^2 dt \tag{B.13}$$

and therefore

$$\delta^{\mathbb{A}}_t = \frac{\delta^{\mathbb{A}}_0}{\nu^{\mathbb{A}}_0 t + 1} + \frac{\nu^{\mathbb{A}}_0}{\nu^{\mathbb{A}}_0 t + 1} Z_t. \tag{B.14}$$

Next, from the definition of $\xi_t^{\mathbb{A}}$, we have

$$\begin{aligned}\ln(\xi_t^{\mathbb{A}}) &= -\frac{1}{2}\int_0^t \left(\delta_s^{\mathbb{A}}\right)^2 ds + \int_0^t \delta_s^{\mathbb{A}} dZ_s \\ &= -\int_0^t \left(\frac{1}{2}\left(\delta_0^{\mathbb{A}}\right)^2 \frac{1}{(\nu_0^{\mathbb{A}}s+1)^2} + \frac{1}{2}\left(\nu_0^{\mathbb{A}}\right)^2 \frac{1}{(\nu_0^{\mathbb{A}}s+1)^2} Z_s^2 + \delta_0^{\mathbb{A}}\nu_0^{\mathbb{A}} \frac{1}{(\nu_0^{\mathbb{A}}s+1)^2} Z_s\right) ds \\ &+ \int_0^t \delta_s^{\mathbb{A}} dZ_s. \end{aligned} \tag{B.15}$$

Integration by parts yields

$$\begin{aligned}\int_0^t \delta_s^{\mathbb{A}} dZ_s &= \int_0^t \left(\frac{\delta_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}s+1} + \frac{\nu_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}s+1} Z_s\right) dZ_s \\ &= \frac{1}{2}\frac{\nu_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}t+1} Z_t^2 + \frac{\delta_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}t+1} Z_t \\ &+ \int_0^t \left(-\frac{1}{2}\frac{\nu_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}s+1} + \frac{1}{(\nu_0^{\mathbb{A}}s+1)^2}\left(\frac{1}{2}\left(\nu_0^{\mathbb{A}}\right)^2 Z_s^2 + \delta_0^{\mathbb{A}}\nu_0^{\mathbb{A}} Z_s\right)\right) ds. \end{aligned} \tag{B.16}$$

Plugging the last equation into the expression for $\ln(\xi_t^{\mathbb{A}})$ results in

$$\ln(\xi_t^{\mathbb{A}}) = \frac{1}{2}\frac{\nu_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}t+1} Z_t^2 + \frac{\delta_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}t+1} Z_t - \int_0^t \frac{1}{2}\left(\delta_0^{\mathbb{A}}\right)^2 \frac{1}{(\nu_0^{\mathbb{A}}s+1)^2} ds - \int_0^t \frac{1}{2}\frac{\nu_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}s+1} ds. \tag{B.17}$$

Combining a similar expression for $\mathbb{B}$, and assuming $\min(\nu^{\mathbb{A}}, \nu^{\mathbb{B}}) > 0$, we have

$$\ln(\xi_t^{\mathbb{B}}) - \ln(\xi_t^{\mathbb{A}}) = \left(\frac{1}{2}\frac{\nu_0^{\mathbb{B}}}{\nu_0^{\mathbb{B}}t+1} - \frac{1}{2}\frac{\nu_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}t+1}\right) Z_t^2 + \left(\frac{\delta_0^{\mathbb{B}}}{\nu_0^{\mathbb{B}}t+1} - \frac{\delta_0^{\mathbb{A}}}{\nu_0^{\mathbb{A}}t+1}\right) Z_t \tag{B.18}$$

$$+ \left(-\frac{1}{2}\frac{\left(\delta_0^{\mathbb{B}}\right)^2}{\nu_0^{\mathbb{B}}} + \frac{1}{2}\frac{\left(\delta_0^{\mathbb{A}}\right)^2}{\nu_0^{\mathbb{A}}}\right) - \frac{1}{2}\ln\left(\frac{1+\nu_0^{\mathbb{B}}t}{1+\nu_0^{\mathbb{A}}t}\right). \tag{B.19}$$

The first two terms converge to a constant since $Z_t^2/t \to 1$, a.s., and $Z_t/t \to 0$, a.s. The last term converges to a constant as well, $-\frac{1}{2}\ln\frac{\nu_0^{\mathbb{B}}}{\nu_0^{\mathbb{A}}}$. Thus $\ln\xi_t = \ln(\xi_t^{\mathbb{B}}) - \ln(\xi_t^{\mathbb{A}})$ converges to a constant, and so (8) with $A(x) = x^{-1}$ implies that both agents survive.

If $\nu^{\mathbb{B}} > \nu^{\mathbb{A}} = 0$, then $\xi_t^{\mathbb{B}} = \xi_t \to 0$, a.s., and so Proposition 4.6 implies that $\mathbb{B}$ does not survive.

B.3 Example 3.4

Agent $\mathbb{B}$'s beliefs are characterized by the density process

$$\xi_t = \exp\left(\sum_{s=1}^{t}\left(-\frac{\delta_{s-1}^2}{2} + \delta_{s-1}\epsilon_s\right)\right), \tag{B.20}$$

The process $M_t = \sum_{s=1}^{t} \delta_{s-1}\epsilon_s$ is a martingale.

Since $\lim_{t\to\infty}\left(\frac{1}{t}\sum_{s=1}^{t}\delta_{s-1}^2\right) = \mathrm{E}[\delta_t^2]$, the quadratic variation process of M_t converges to infinity almost surely under $\mathbb{P}$, and therefore $\lim_{t\to\infty} M_t/\left(\sum_{s=1}^{t}\delta_{s-1}^2\right) = 0$, a.s. (see Shiryaev 1996, §7.5, Th. 4). This implies that $\lim_{t\to\infty}\xi_t = 0$ *a.s.* and hence the condition (28) in Proposition 4.6 is satisfied. We conclude that agent $\mathbb{B}$ does not survive in the long run.

B.4 Example 3.5

Survival results follow from Proposition 4.6, since the logarithm of the belief density ratio $\ln(\xi_t)$ exhibits linear growth, while the aggregate endowment D_t grows exponentially. Price impact results follow from Proposition 5.7.

B.5 Example of the Impact of Distorted Beliefs on the Prices of Non-Primitive Claims

Consider a discrete-time economy with log-utility preferences, time-preference parameter ρ, and the aggregate endowment given by D_t. Assume that agent $\mathbb{A}$ has correct beliefs, and agent $\mathbb{B}$'s beliefs are distorted, with the density equal to ξ_t. Assume that $\xi_t \to 0$ *a.s.* as $t \to \infty$ (see Section 3 for the discussion of this condition on beliefs). Let the utility weight of agent $\mathbb{B}$ be one.

According to Corollary 4.2, agent $\mathbb{B}$ fails to survive in the long run. Moreover, according to Proposition 5.1, agent $\mathbb{B}$ has no long-run price impact on the Arrow-Debreu prices in this economy.

Consider a long-lived, non-primitive state-contingent claim H, with the cash flow stream given by $D_t^H = 1_{[\xi_t \geq 1]}$. Note that the payoff of H is a function of the underlying state space, and hence is defined without an explicit connection to agent $\mathbb{B}$.

In an economy without belief distortions, the state-price density is $\pi_t = \exp(-\rho t)D_t^{-1}$, and hence the price of asset H is

$$P_t^{H,\text{hom}} = D_t \, \mathrm{E}_t \left[\sum_{t+1}^{\infty} e^{-\rho(s-t)} D_s^{-1} \, 1_{[\xi_s \geq 1]} \right]. \tag{B.21}$$

In the economy with heterogeneous beliefs,

$$\pi_t = e^{-\rho t} \frac{1+\xi_t}{D_t}, \tag{B.22}$$

and the price of H is

$$P_t^{H,\text{het}} = D_t \left(1+\xi_t\right)^{-1} \mathrm{E}_t \left[\sum_{t+1}^{\infty} e^{-\rho(s-t)} D_s^{-1} \left(1+\xi_s\right) 1_{[\xi_s \geq 1]} \right]. \tag{B.23}$$

Since

$$\frac{P_t^{H,\text{het}}}{P_t^{H,\text{hom}}} \geq \frac{2}{1+\xi_s}, \tag{B.24}$$

and $\xi_t \to 0$, *a.s.*, we conclude that the price of asset H is affected, in the long run, by the distorted beliefs of agent $\mathbb{B}$. The reason for price impact of $\mathbb{B}$'s beliefs on asset H is that the payoff of H is concentrated on a set of states in which agent $\mathbb{B}$ has a nontrivial consumption share. This set of states has an asymptotically vanishing probability but is relevant for the pricing of long-lived non-primitive state-contingent claims.

B.6 Proof of Propositions 5.6, 5.7, and 5.8

Since the utility function is of the DARA type, condition (8) implies that

$$|\ln(\lambda\xi_t)| \geq A(D_t) D_t |1-2w_t|. \tag{B.25}$$

Furthermore, using the fact that $xA(x)$ is an increasing function,

$$|\ln(\lambda\xi_t)| \geq A\left(D_t\right) D_t |1-2w_t| \geq A\left(\frac{D_t}{2}\right) D_t \left|\frac{1}{2} - w_t\right|, \tag{B.26}$$

which implies that

$$0 \leq \left(\frac{1}{2} - w_t\right)^2 A\left(\frac{D_t}{2}\right) D_t \leq \frac{\left(\ln\left(\lambda \xi_t\right)\right)^2}{A\left(\frac{D_t}{2}\right) D_t} \to 0, \ \mathbb{P} - \text{a.s.} \tag{B.27}$$

Because, by assumption, $\liminf_{t\to\infty} A(D_t/2)D_t > 0$, (B.27) implies $\lim_{t\to\infty} w_t = 1/2$, *a.s.* This proves Proposition 5.6.

Next, we prove Proposition 5.7. To show that there is price impact, we first verify that for a reference economy with $\lambda^\star = 1$, the difference

$$PI(t, s; 1) \equiv \int_{\frac{1}{2}D_{t+s}}^{D_{t+s}(1-w_{t+s})} A(x)\, dx - \int_{\frac{1}{2}D_t}^{D_t(1-w_t)} A(x)\, dx \tag{B.28}$$

does not converge to zero almost surely. The above expression corresponds to $\lambda^\star = 1$ in the definition of price impact.

We derive a set of bounds on $PI(t, s; 1)$. Consider an element of the probability space and the point in time such that $\lambda \xi_t < 1$. The DARA property of the utility function and condition (8) imply that

$$\int_{\frac{1}{2}D_t}^{D_t(1-w_t)} A(x)\, dx \leq \frac{1}{2} \int_{D_t w_t}^{D_t(1-w_t)} A(x)\, dx = -\frac{1}{2} \ln(\lambda \xi_t). \tag{B.29}$$

The same inequality holds if $\lambda \xi_t \geq 1$:

$$\int_{D_t(1-w_t)}^{\frac{1}{2}D_t} A(x)\, dx \geq \frac{1}{2} \int_{D_t w_t}^{D_t(1-w_t)} A(x)\, dx = \frac{1}{2} \ln(\lambda \xi_t). \tag{B.30}$$

Thus, for any value of $\lambda \xi_t$,

$$\int_{\frac{1}{2}D_t}^{D_t(1-w_t)} A(x)\, dx \leq -\frac{1}{2} \ln(\lambda \xi_t). \tag{B.31}$$

Next, we derive a lower bound on $\int_{\frac{1}{2}D_t}^{D_t(1-w_t)} A(x)\, dx$ assuming $\lambda \xi_t < 1$. Since

$A(x)x$ is increasing, a Taylor expansion of the log function shows that

$$\begin{aligned}
\int_{\frac{1}{2}D_t}^{D_t(1-w_t)} A(x)\,dx &\geq \int_{\frac{1}{2}D_t}^{D_t(1-w_t)} A\left(\frac{D_t}{2}\right)\frac{D_t}{2}\frac{1}{x}\,dx \\
&= A\left(\frac{D_t}{2}\right)\frac{D_t}{2}\ln\left(2-2w_t\right) \\
&= A\left(\frac{D_t}{2}\right)D_t\left[\left(\frac{1}{2}-w_t\right)-\frac{1}{\left(1-w_t^{\star}\right)^2}\left(1-2w_t\right)^2\right],
\end{aligned} \tag{B.32}$$

where $w_t^{\star} \in \left[w_t, \frac{1}{2}\right]$. Applying another Taylor expansion of the log function,

$$\begin{aligned}
-\ln\left(\lambda\xi_t\right) &= \int_{D_t w_t}^{D_t(1-w_t)} A(x)\,dx \\
&\leq A\left(\frac{D_t}{2}\right)D_t\left(\frac{1}{2}-w_t\right)+\int_{D_t w_t}^{\frac{D_t}{2}} A\left(\frac{D_t}{2}\right)\frac{D_t}{2}\frac{1}{x}\,dx \\
&= A\left(\frac{D_t}{2}\right)D_t\left(\frac{1}{2}-w_t\right)-A\left(\frac{D_t}{2}\right)\frac{D_t}{2}\ln\left(2w_t\right) \\
&= A\left(\frac{D_t}{2}\right)D_t\left(\frac{1}{2}-w_t\right)- \\
&\quad A\left(\frac{D_t}{2}\right)D_t\left[-\left(\frac{1}{2}-w_t\right)-\frac{1}{\left(w_t^{\star\star}\right)^2}\left(1-2w_t\right)^2\right],
\end{aligned} \tag{B.33}$$

where $w_t^{\star\star} \in \left[w_t, \frac{1}{2}\right]$. The inequality (B.27) implies that the last term in (B.33) approaches zero almost surely as t approaches ∞.

Assume now that $\lambda\xi_t \geq 1$. Proceeding as above,

$$\begin{aligned}
\int_{D_t(1-w_t)}^{\frac{1}{2}D_t} A(x)\,dx &\leq \int_{D_t(1-w_t)}^{\frac{1}{2}D_t} A\left(\frac{D_t}{2}\right)\frac{D_t}{2}\frac{1}{x}\,dx \\
&= -A\left(\frac{D_t}{2}\right)\frac{D_t}{2}\ln\left(2-2w_t\right) \\
&= -A\left(\frac{D_t}{2}\right)D_t\left[\left(\frac{1}{2}-w_t\right)-\frac{1}{\left(1-w_t^{*}\right)^2}\left(1-2w_t\right)^2\right],
\end{aligned} \tag{B.34}$$

where $w_t^* \in [1/2, w_t]$. Thus, we recover a similar inequality as for the case of $\lambda\xi_t < 1$:

$$\int_{\frac{1}{2}D_t}^{D_t(1-w_t)} A(x)\,dx \quad \geq \quad A\left(\frac{D_t}{2}\right) D_t \left[\left(\frac{1}{2} - w_t\right) - \frac{1}{(1-w_t^*)^2}(1-2w_t)^2\right]. \tag{B.35}$$

Furthermore,

$$\begin{aligned}
\ln(\lambda\xi_t) &= \int_{D_t(1-w_t)}^{D_t w_t} A(x)\,dx \geq A\left(\frac{D_t}{2}\right) D_t \left(w_t - \frac{1}{2}\right) + \int_{D_t(1-w_t)}^{D_t/2} A(x)\,dx, \\
\int_{D_t(1-w_t)}^{D_t/2} A(x)\,dx &= \int_{D_t(1-w_t)}^{D_t/2} (A(x)x)\; x^{-1}\,dx \\
&\geq -A(D_t(1-w_t))D_t(1-w_t)\ln(2(1-w_t)) \\
&= A(D_t(1-w_t))D_t(2-2w_t)\left[(w_t - 1/2) + \frac{1}{(1-w_t^{**})^2}(1-2w_t)^2\right] \\
&\geq A\left(\frac{D_t}{2}\right) D_t(2-2w_t)\left[(w_t - 1/2) + \frac{1}{(1-w_t^{**})^2}(1-2w_t)^2\right],
\end{aligned} \tag{B.36}$$

where $w_t^* \in [1/2, w_t]$. The first inequality in the above expression follows from $A(x)x$ being weakly increasing, and second from $A(x)$ being weakly decreasing. We now use the fact that $(w_t - 1/2)$ converges to zero almost surely.

We combine the inequalities in (B.32–B.36), and use the fact that $w_t - 1/2$ converges to zero almost surely, to conclude that

$$\lim_{t\to\infty} \int_{\frac{1}{2}D_t}^{(1-w_t)D_t} A(x) + \frac{1}{2}\ln(\lambda\xi_t) = 0, \text{ a.s.} \tag{B.37}$$

We have thus established asymptotic behavior of the price-impact measure $PI(t, s; 1)$, which corresponds to the reference economy with $\lambda^\star = 1$:

$$\lim_{t\to\infty} PI(t, s; 1) - \frac{1}{2}(\ln \xi_{t+s} - \ln \xi_t) = 0, \text{ a.s.} \tag{B.38}$$

We conclude that if the condition (39) fails to hold, then $PI(t, s; \lambda^\star = 1)$ vanishes asymptotically, and thus agent $\mathbb{B}$ does not exert long-run price impact.

This establishes that condition (39) is necessary for price impact. Moreover, (B.38) shows that the SDF does not depend on λ asymptotically, which means that the asymptotic behavior of the SDF in this economy is independent of the initial wealth distribution between agents $\mathbb{A}$ and $\mathbb{B}$.

Next, we establish sufficiency of conditions (39) and (40) for price impact. According to (B.38), the SDFs in two economies with $\xi_t = 1$ and $\lambda > 0$ are the same asymptotically. Thus, the price impact measures $PI(t, s; \lambda^\star)$, $\lambda^\star > 0$, satisfy

$$\lim_{t\to\infty} PI(t, s; \lambda^\star) - \frac{1}{2}\left(\ln \xi_{t+s} - \ln \xi_t\right) = 0, \text{ a.s.} \tag{B.39}$$

Condition (39) then implies that $PI(t, s; \lambda)$ does not vanish asymptotically for any $\lambda^\star > 0$.

We now need to consider only the behavior of $PI(t, s; 0)$.

$$PI(t, s; 0) = \int_{(1-w_{t+s})D_{t+s}}^{D_{t+s}} A(x)\, dx - \int_{(1-w_t)D_t}^{D_t} A(x)\, dx. \tag{B.40}$$

Note that

$$\int_{(1-w_t)D_t}^{D_t} A(x)\, dx = \int_{D_t/2}^{D_t} A(x)\, dx - \int_{D_t/2}^{(1-w_t)D_t} A(x)\, dx. \tag{B.41}$$

Thus,

$$PI(t, s; 0) = -PI(t, s; 1) + \int_{D_{t+s}/2}^{D_{t+s}} A(x)\, dx - \int_{D_t/2}^{D_t} A(x)\, dx. \tag{B.42}$$

Using (B.38), together with (39) and (40), we conclude that $PI(t, s; 0)$ does not vanish asymptotically. This completes the proof of Proposition 5.7.

Proposition 5.8 follows as a special case from the proof of Proposition 5.7 above by setting $\xi_t = 1$.

C State-Dependent Preferences

In this section, we extend our results on survival and price impact to models with state-dependent preferences. Let the utility function take the form $u(C, H)$, where C is agent's consumption and H is the process for state variables affecting the agent's utility. We assume that H is an exogenous adapted process. This specification covers, as an important special case, models of external habit formation, or catching-up-with-the-Joneses preferences, as in Abel (1990) and Campbell and Cochrane (1999), in which case the process H is a function of lagged values of the aggregate endowment.

As we show below, the general lessons of the previous sections apply to state dependent preferences. However, the dependence of risk aversion variables other than consumption level affects the validity of MSH in economies with state-dependent preferences. The addition of state variables to the utility function means that the coefficient of risk aversion may be more variable, and, in particular, it may no longer be bounded even with CRRA-like preferences.

Theorem 4.1 extends to the case of state-dependent preferences. Let $A(C, H)$ and $\gamma(C, H)$ denote, respectively, the coefficients of absolute and relative risk aversion at consumption level C. Then we obtain an analog of Proposition 4.6:

Proposition C.1 *Assume that* $\lim_{t\to\infty} \xi_t = 0$*, a.s., and that the utility function* $u(C, H)$ *exhibits DARA: the coe cient of absolute risk aversion* $A(C, H)$ *is decreasing in* C*. Then, for agent* $\mathbb{B}$ *to go extinct it is su cient that there exists a sequence* $\epsilon_n \in (0, \frac{1}{2})$ *converging to zero such that for any* n

$$\lim_{t\to\infty} \frac{A(\epsilon_n D_t, H_t)D_t}{-\ln(\xi_t)} = 0, \quad \text{a.s.} \tag{C.43}$$

For agent $\mathbb{B}$ *to survive, it is su cient that for some* $\epsilon \in (0, \frac{1}{2})$

$$\text{Prob}\left[\limsup_{t\to\infty} \frac{A(\epsilon D_t, H_t)D_t}{-\ln(\xi_t)} = \infty\right] > 0. \tag{C.44}$$

If, in addition,

$$\lim_{t\to\infty} \frac{A(D_t, H_t)D_t}{-\ln(\xi_t)} = \infty, \quad \text{a.s.} \tag{C.45}$$

then $\lim_{t\to\infty} w_t = \frac{1}{2}$, a.s.

In a growing economy ($D_t \to \infty$, $a.s.$) with accumulating differences in beliefs and state-*independent* preferences, survival of agent $\mathbb{B}$ requires local utility curvature to decline slowly enough, so that $\limsup_{D\to\infty} A(D)D = \infty$, a.s. (see Proposition 4.6). This is not the case if preferences are state-dependent. In many common models with external habit formation, the process H_t is such that the process $A(D_t, H_t)D_t$ is stationary. In such cases, survival and price impact results are sensitive to the assumptions on beliefs and the aggregate endowment. As the following example illustrates, agent $\mathbb{B}$ may survive if the stationary distribution of risk aversion has a sufficiently heavy right tail, so that the condition (C.43) is violated.

Example C.2 *Consider two discrete-time economies with external habit formation. Let the utility curvature, $A(x, H)$, of the two agents be*

$$\mathbb{A}(x, H) = \frac{1}{x} + \frac{H}{x^2}, \tag{C.46}$$

where $H_t = D_{t-1}$. Assume that the disagreement process follows

$$\ln(\xi_t) = -\frac{1}{2}t + \sum_{n=1}^{t} Z_n, \tag{C.47}$$

where Z_n are distributed according to a standard normal distribution and are independent of the endowment process. Assume that $\lim_{t\to\infty} \xi_t = \infty$, a.s.

Endowment growth is independently and identically distributed over time in both economies. Assume that the endowment process D_t is independent of the disagreement process ξ_t, which means that agents $\mathbb{A}$ and $\mathbb{B}$ disagree on probabilities of payo -irrelevant states.[1,2] *In the first economy, endowment growth has bounded support, $0 < \underline{g} \leq \frac{D_t}{D_{t-1}} \leq \overline{g} < \infty$. In the second economy, $\underline{g} = 0$. Moreover, in the second economy, the distribution of endowment growth is such that*

$$\text{Prob}\left[\frac{D_t}{D_{t-1}} < x\right] > x^{1/3} \tag{C.48}$$

for su ciently small x. Then, agent $\mathbb{B}$ becomes extinct in the first economy, and survives in the second economy.

In the first economy, it is clear that condition (C.43) is satisfied for any positive ϵ_n, and thus agent $\mathbb{B}$ does not survive. In the second economy, the distribution of endowment growth is such that relative risk aversion exhibits frequent large spikes,

[1]Another example of an economy in which belief differences are independent of the aggregate endowment is a multi-sector economy in which agents agree on the distribution of the aggregate endowment, but disagree about the distribution of sectors' shares in the aggregate endowment.

[2]Survival results in this example do not depend on the joint distribution of the endowment process and the disagreement process. Thus, one may assume that the two agents disagree about the probabilities of payoff-relevant states by specifying $\frac{D_t}{D_{t-1}}$ to be a nonlinear function of Z_t. The assumption of independence of endowment and beliefs makes it easy to establish price impact results below.

namely

$$\begin{aligned}\text{Prob}\left(A(\epsilon D_t, H_t)\epsilon D_t > t^3\right) &= \text{Prob}\left(1 + \frac{D_{t-1}}{\epsilon D_t} > t^3\right)\\ &\geq \text{Prob}\left(\epsilon \frac{D_t}{D_{t-1}} < t^{-3}\right)\\ &> \epsilon^{-1/3} t^{-1}.\end{aligned}$$

Such spikes in risk aversion occur frequently enough that the condition (C.44) holds. Specifically, since $\sum_{t=1}^{\infty} t^{-1} = \infty$, the Borel-Cantelli lemma implies that

$$\limsup_{t\to\infty} A(\epsilon D_t, H_t)\epsilon D_t t^{-3} \geq 1 \ a.s. \tag{C.49}$$

Since $\lim_{t\to\infty}(-\ln(\xi_t))t^{-3} = 0$ a.s., (C.44) follows.

The following propositions extend some of our results on price impact to economies with state-dependent preferences. Their proofs follow closely the results of Sections 4 and 5.

Proposition C.3 *There is no price impact or survival in models with* $\lim_{t\to\infty} \xi_t = 0$ *a.s., and utility curvature such that* $A(x, H) \leq \overline{C} x^{-1}$.

In the model with state-independent preferences, bounding the endowment process implies bounding risk aversion. This, in turn, implies a lack of price impact. With state-dependent preferences, a bounded endowment no longer implies that risk aversion is bounded, and therefore there is no analog to the Corollary 4.3 and Proposition 5.2 for the economies with state-dependent preferences.

Returning to Example C.2, note that agent $\mathbb{B}$ has no price impact in the first economy but exerts price impact in the second economy. The first result follows the same argument as the proof of Proposition C.3, since bounded dividend growth implies that $A(x, H) \leq \overline{C} x^{-1}$ over the interval $(D_t(1 - w_t), D_t)$. The second result can be established using a slight modification of the proof of Proposition 5.7.[3]

[3] As we show above, $\limsup_{t\to\infty} \frac{D_t}{D_{t-1}} t^{-3} \geq 1$, a.s., while $\lim_{t\to\infty}(\ln(\xi_t))^2 t^{-3} = 0$, a.s., implying that $\limsup_{t\to\infty} A(D_t, H_t)D_t/|\ln(\xi_t)|^2 = \infty$, a.s. The price impact result then follows from independence of D_t and ξ_t and the assumption that increments $\ln(\xi_t) - \ln(\xi_{t-1})$ are independent across time.