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On the Effectiveness of Uniform Subsidies in Increasing Market Consumption

Retsef Levi
Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA 02142, retsef@mit.edu

Georgia Perakis
Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA 02142, georgiap@mit.edu

Gonzalo Romero
Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA 02142, gromeroy@mit.edu

We provide a new modeling framework to analyze a subsidy allocation problem with endogenous market response, under a budget constraint on the total amount of subsidies that the central planner can pay. The central planner's objective is to maximize the aggregated market consumption of a good, or equivalently, to maximize the consumer surplus. Using our framework, we identify sufficient conditions on the firms'marginal cost functions, such that *uniform subsidies are optimal*. That is, the simple policy that allocates the same subsidy to every firm is optimal, even if the firms are heterogeneous, and their efficiency levels are arbitrarily different. This is an important insight because uniform subsidies is a policy commonly used in practice, primarily because of its simplicity and perceived fairness. Moreover, we prove that, in many cases, uniform subsidies do not only obtain the optimal aggregated market consumption, but at the same time obtain the best social welfare solution. Furthermore, we show that the optimality of uniform subsidies is usually preserved, even if the central planner is uncertain about the specific market conditions. Finally, we present simulation results in relevant settings where uniform subsidies are not optimal. They suggest that the aggregated market consumption induced by uniform subsidies is relatively close to the one induced by optimal subsidies.

Key words: subsidies, budget constraint, cournot competition

1. Introduction

In this work, we study the important setting in which a central planer aims to impact a given market. Specifically, her goal is to increase the aggregated market consumption of a good, by providing co-payment subsidies that are paid, for each unit that is produced, to heterogeneous and selfish (profit maximizers) producers (firms), who compete in the market. The motivation to provide such subsidies stems from the positive societal externalities that can be obtained by increasing the aggregated market consumption, and from the fact that left alone the resulting market equilibrium induced by the selfish competing producers might not be socially optimal. A very current example, are the recent efforts around the production of infectious disease treatments to the developing world, such as malaria drugs (e.g., Arrow et al. (2004)), and vaccines (e.g., Brito et al. (1991)).

Furthermore, typically the central planer makes subsidy allocation in the presence of a budget constraint that is often determined prior to the actual subsidy allocation decision. For example, in some cases the central planner could be a foundation that raised a certain amount of money to address a related issue, and it is then facing the challenge of how to allocate the budget towards co-payment subsidies. Another challenge typically faced by the central planner, is that the intervention in the market through the allocation of subsidies will likely change the market equilibrium induced by the competing producers. Hence, to optimally allocate the subsidies, the central planer has to take into account these complex dynamics.

In this paper, we propose a novel modeling framework to study strategic and operational issues related to co-payment subsidy allocation. The models that we develop explicitly capture the setting of a central planner aiming to maximize the aggregated market consumption of a good, in the presence of a budget constraint and market competition between heterogeneous profit maximizing firms. The firms are heterogeneous in terms of their respective efficiency and cost structure. This is modeled through firm-specific marginal cost functions. The models that we develop fall into the class of *Mathematical Program with Equilibrium Constraints* (MPEC). They are relatively general and capture different cost structures, inverse demand functions, as well as a range of market dynamics of quantity competition that are typical to the settings being studied. For example, the models capture as special cases Cournot Competition with linear demand, as well as Cournot Competition under yield uncertainty with linear demand and linear marginal cost functions. MPEC models are typically computationally challenging, both to solve optimally and to analyze. However, by reformulating these problems, we are able to develop tractable mathematical programs that provide upper bounds on the optimal objective value, and allow the development of efficient algorithms. Even more importantly, they allow analyzing the effectiveness of practical policies. In particular, the paper focuses attention on the effectiveness of the commonly used *uniform co-payments*, in which the per-unit co-payment is the same for all competing firms in the market. The common use of uniform co-payments, in spite of the existence of heterogeneous firms, each with potentially different efficiency level, is primarily driven by the simplicity of implementation, as well as some notion of fairness. The paper addresses the important question of to what extent uniform co-payments are effective in increasing the aggregated market consumption, compared to potentially more sophisticated policies that could allow the co-payment to be firm-specific. Through the mathematical programming upper bound relaxation that we develop, the paper provides some surprising insights. First, we can show that for a large class of firm-specific cost structures, uniform co-payments are in fact *optimal*. That is, there is no loss of efficiency in using uniform co-payments in these settings. Second, this insight is maintained even if one considers the case in which there exists uncertainty about the future market state, and the central planner has to set up the subsidies prior to the realization of the market condition. Third, in many cases uniform subsidies do not only obtain the optimal (maximal) aggregated market consumption, but at the same time obtain the best social welfare solution. Finally, in other settings, where uniform subsidies are not optimal, extensive computational experiments suggest that they still perform, on average, very close to optimal.

To demonstrate the applicability of the model and the relevance of the issues studied in the paper, we next discuss in detail the case of malaria drugs.

Application: Global Subsidy for Malaria Drugs. A motivating example, where the setting modeled in this paper exists in practice, is the global fight against malaria. This has been a long standing challenge for the healthcare industry. About 300-500 million cases of malaria occur worldwide, and more than two million people die of malaria, per year. To make matters worse, recently chloroquine, the traditional drug for treating malaria, has become less effective due to growing resistance to this medication. Artemisinin combination therapies (ACT) have been the successor drugs to chloroquine in order to treat malaria; however, they are at least ten times more costly, see White (2008).

In 2004 the Institute of Medicine (IoM) reviewed the economics of antimalarial drugs. It identified that several manufacturers compete in an unregulated market, and concluded that the most effective way of ensuring access to ACTs for the greatest number of patients, would be to provide a centralized subsidy to the producers. The goal would be achieving high overall coverage of ACTs, therefore, the subsidized price to the end user should be at least as low as the chloroquine. Moreover, the IoM recognized that firms had not invested in producing ACTs on the scale needed to supply Africa, because there had been no assured market, therefore, the global capacity to produce ACTs was quite limited, see Arrow et al. (2004).

In this context, the Roll Back Malaria Finance and Resource Working Group, and the World Bank, developed in 2007 the Affordable Medicines Facility for malaria (AMFm), a concrete initiative to improve access to safe, effective, and affordable malaria medicines. The AMFm manages US $1.5 billion in funding, over five years, to pursue its main objective: increasing the consumption of ACTs, see their technical report online AMFm Task Force (2007).

As usually implemented in practice, the policy proposed by AMFm consisted of giving *a uniform co-payment*, see AMFm Task Force (2007). Namely, each firm receives the same co-payment for each unit sold. Giving the right incentives to the firms producing these drugs can increase access to them, hence, it has the potential to have a significant impact on this global problem, see Arrow et al. (2004).

Results and Contributions. The main contributions of this paper are the following:

New modeling framework for a subsidy allocation problem. We introduce a general optimization framework to analyze subsidy allocation problems with endogenous market response, under a budget constraint on the total amount of subsidies the central planner can pay. The central planner's objective is to maximize the aggregated market consumption of a good, or equivalently, to maximize the consumer surplus. Our models allow general inverse demand and marginal cost functions, assuming only that the inverse demand function is decreasing in the aggregated market consumption, and that the firms' marginal costs are increasing. These are standard assumptions in the literature. In fact, they are more general than assumptions usually considered.

Sufficient conditions for the optimality of uniform co-payments. We compare uniform co-payments to the optimal, and potentially differentiated, co-payment allocation, which provides more flexibility, but it is potentially significantly harder to implement. The main result in this paper shows that uniform co-payments are in fact *optimal* for a large family of marginal cost functions. This family of marginal cost functions includes homogeneous functions of the same degree as a special case. This result is surprising, considering that firms are heterogeneous, and particularly since the assumptions on the inverse demand function are very general (essentially only monotonicity and continuity). More importantly, it establishes sufficient conditions, such that the policy that is frequently being used in practice (see, for example, AMFm Task Force (2007)) is actually optimal. Additionally, we provide sufficient conditions for uniform co-payments to simultaneously maximize the social welfare. In particular, we show that homogeneous functions of the same degree satisfy these conditions as well.

Incorporate market state uncertainty. We extend the models by assuming that the central planner does not know the exact market state with certainty (i.e., the specific inverse demand function is uncertain), but she has a set of possible scenarios, and beliefs on the likelihood that each scenario will materialize. We model this setting as a stochastic MPEC, where the central planner decides her co-payment allocation policy with the objective of maximizing the expected aggregated market consumption. This model is considerably harder to analyze, see Patriksson and Wynter (1999). However, we show that uniform co-payments are still optimal in this setting, for a large family of firms' marginal cost functions. In particular, this family includes convex homogeneous functions of the same degree. Moreover, the analysis suggests that the central planner only has to consider the scenario with the highest aggregated market consumption at equilibrium, regardless of the exact distribution over the different market states.

Tractable upper bound problems. Based on an innovative mathematical programming reformulation of our model, we develop tractable upper bound problems. These are used extensively in the analysis mentioned above. In addition, we use them to conduct a numerical study of the performance of uniform co-payments in relevant settings where they are not optimal. We consider Cournot Competition with linear demand and constant marginal costs, and a more general setting with non-linear demand, and non-linear marginal cost. The results obtained on data generated at random suggest that the aggregated market consumption induced by uniform subsidies is on average 96% optimal. We believe that the innovative reformulation of the model, and the resulting upper bounds, would be useful to study additional interesting and important research questions.

The rest of the paper is structured as follows. Section 2 reviews related literature from operations management and economics. In Section 3 we present our model, the uniform co-payments allocation problem, and a relaxation of this problem. Section 4 presents the main result on sufficient conditions for the optimality of uniform co-payments in the deterministic model. In Section 5 we extend our model to consider the case when the central planner is uncertain about the market state. Section 6 considers the alternative objective of maximizing social welfare, and presents sufficient conditions for the optimality of uniform subsidies. Section 7 presents a numerical study of the relative performance of uniform subsidies in settings where they are not optimal. Finally, Section 8 provides concluding remarks.

2. Literature Review

The subject of taxes and subsidies allocation and incidence has a vast literature in the economics community. Fullerton and Metcalf (2002) present a thorough review of classical and recent result in this area. The main areas of research in this literature are imperfect competition, partial equilibrium models, and general equilibrium models. This paper is closely related to the study of subsidies in imperfect competition models. However, the traditional approach in this literature assumes homogeneous firms, and focuses on studying the impact of taxes, or subsidies, on the number of firms participating in the market in a symmetric equilibrium, see Fullerton and Metcalf (2002). Alternatively, models of differentiated products are considered, which give the firms some monopoly power, and the focus is again on the number of firms active in the market in equilibrium. The reason for this is that the number of competitors in the market is directly related to the ability to pass taxes forward to the consumer. More generally, when analyzing comparative static properties in oligopoly models, like the subsidy allocation in our case, it is fairly common to focus on symmetric equilibria with homogeneous firms in order to obtain more precise insights, see, for example, Vives (2001). In contrast, in our model we take an operational view: we assume heterogeneous firms that produce a commodity, and we focus on the specific subsidy allocation among them. Additionally, an important modeling characteristic we consider is the presence of a budget constraint, in terms of the total amount of funding that can be allocated to these subsidies. This feature allows us to investigate the interplay between the optimal subsidies structure, and the budget available.

Within economics, one particular area that studies a problem related to the one considered in this paper is the strategic trade policy literature, particularly the “third market model”, see Brander (1995). In this model, n home firms and n^* foreign firms export a commodity to a third market, where the market price is set through Cournot Competition, with constant marginal costs, among all the firms. The government can allocate subsidies to the home firms, increasing their profit at the expense of the foreign competitors. The government’s utility is equal to the profit earned by the home firms, minus the cost of the subsidy payments. Let us emphasize that the government does not face a budget constraint, and that the firms’ profit is equally weighted with the cost of the subsidy payments. An exception to the latter is found in Leahy and Montagna (2001), where the cost of the subsidy payments is weighted by a parameter δ, interpreted as the social cost of funds. An alternative interpretation of δ is to let it be the lagrange multiplier of a budget constraint for the government, relating it to our model. We focus here in the case with heterogeneous firms. In this setting, Collie (1993) and Long and Soubeyran (1997) *assume* a uniform subsidy and study its effect in the market shares of the firms. Later, Leahy and Montagna (2001) assume linear demand, and derive close form expressions for the optimal subsidies. They conclude that the optimal subsidy policy is generally not uniform; and if the social cost of funds is sufficiently low then the government should allocate higher export subsidies to more efficient firms. In contrast, in our model we assume more general increasing marginal cost functions, and find conditions under which uniform subsidies are optimal.

In the operations research and operations management communities, a growing literature has been devoted to analyzing oligopoly models with congestion, e.g., Acemoglu and Ozdaglar (2007), Johari et al. (2010). Recently, Correa et al. (2012) study markup equilibria, a particular case of supply function equilibria, with firms that have increasing marginal costs. In supply function equilibria firms are assumed to choose functions which map the quantity produced to prices, see Klemperer and Meyer (1989). In markup equilibria firms are restricted to choose a supply function of the form of a scalar times their marginal cost. Correa et al. (2012) find sufficient conditions for the existence of markup equilibria for marginal cost functions very similar to the ones were uniform co-payments are optimal in our model. The problem of controlling and reducing the contagion of infectious diseases has been studied in the operations management literature mainly focusing on the analysis of vaccine’s markets, particularly the influenza vaccine, its supply chain coordination -e.g. Chick et al. (2008) and Mamani et al. (2012)- and the market competition under yield uncertainty -e.g. Deo and Corbett (2009) and Arifoglu et al. (2012)- as opposed to our interest in subsidy allocation. In particular, we consider the case of allocating subsidies to Cournot competitors under yield uncertainty, and we show that if the demand and the marginal costs functions are linear, then uniform co-payments are optimal in this setting.

The motivation problem of allocating subsidies to increase the aggregated market consumption of new malaria drugs is also studied by Taylor and Xiao (2013), however they take a very different approach. Specifically, they consider the case of one firm and one retailer operating in a rural area with no market competition. Their analysis focuses on the placement of the subsidy in the supply chain, comparing the possibility of allocating co-payments to either the producers, or the retailers. On the other hand, there is a growing trend in the operations management literature that studies the problem of a central planner deciding rebates that are directed to the *consumers*, with the goal of incentivizing the adoption of some technology, such as green technology, see, for example, Lobel and Perakis (2012), Cohen et al. (2012), Chemama et al. (2013), Raz and Ovchinnikov (2013), and Cohen et al. (2013). In contrast, motivated by a different set of practical applications, we focus on co-payments that are allocated to the producers, for each unit sold in the market. More generally, our work is related to the operations management literature that analyzes the impact of contract design on the behavior of firms in a supply chain. A comprehensive overview of this literature is provided in Cachon (2003). However, the focus of this framework is set on firms designing contracts to maximize their profits, while we are interested in a central planner designing incentives to maximize the aggregated market consumption of a good.

3. Model

In this section, we introduce a mathematical programming formulation of the subsidies allocation problem. We then use this formulation to obtain a relaxation of the problem, which provides an upper bound on the largest aggregated market consumption that can be induced with the available budget.

We consider a market for a commodity composed by $n \geq 2$ heterogeneous competing firms. Each firm $i \in \{1, \ldots, n\}$ decides its output q_i independently, with the goal of maximizing its own profit. We assume that the introduction of subsidies in the market will induce an increase in the aggregated market consumption, and that the firms do not have the installed capacity to provide all of it. This implies that capacity is scarce in the market. We model this effect by assuming that the marginal cost of each firm is increasing. Specifically, *we assume that firms have a firm-specific non-negative, increasing and differentiable marginal cost function on its production quantity, denoted by $h_i(q_i)$.*

Consumers are described by an inverse demand function $P(Q)$, where $Q \equiv \sum_{i=1}^{n} q_i$ is the aggregated market consumption. *We assume that $P(Q)$ is non-negative, decreasing and differentiable in $[0, \bar{Q}]$, where $\bar{Q}$ is the smallest value such that $P(\bar{Q}) = 0$.* This is equivalent to assuming that the aggregated market demand for the good is bounded. This assumption captures the malaria drugs example, where even if the new malaria treatment was given away for free there would be a finite demand for it. Additionally, we assume that $P(0) = \bar{P} > 0$. This is equivalent to assuming that there exists a finite price such that the demand for the good becomes zero. This could be motivated by the consumers of the good switching to a substitute product, or simply not being able to afford it. In the malaria drugs example, there exist alternative treatments, which are less effective, that consumers may choose instead. Moreover, this is precisely the motivation for introducing a subsidy for the new malaria treatment in the first place.

The assumption on the market equilibrium dynamics is that each firm participating in the market equilibrium, produces up to the point where its marginal cost equals the market price; and firms that do not participate in the market equilibrium, must have a marginal cost of producing zero units, which is larger than the market price. This can be expressed in the following condition:

$$\text{For each } i,j, \text{ if } \ q_i > 0, \text{ then } \ h_i(q_i) = P(Q) \leq h_j(q_j). \tag{1}$$

At this level of generality, in both the firms' marginal costs and the inverse demand function, an interpretation for this equilibrium condition is that firms act as price takers and compete on quantity. However, we will show that for more specific families of marginal cost functions, or inverse demand functions, well known imperfect market competition models will be special cases of our model. These include Cournot Competition with linear demand, and Cournot Competition under yield uncertainty, with linear demand and linear marginal cost functions. Assuming a decreasing inverse demand function, and increasing firms' marginal cost functions, ensure that there exists a unique market equilibrium, see, for example, Marcotte and Patriksson (2007).

3.1. Co-payment Allocation Problem

We will refer to the problem faced by the central planner as the *co-payment allocation problem (CAP).* The co-payment allocation problem is a particular case of a Stackelberg game, or a bilevel optimization problem. In the first stage, the central planner allocates a given budget $B > 0$, in the form of co-payments $y_i \geq 0$, to each firm $i \in \{1,\dots,n\}$, per each unit provided in the market. Moreover, she anticipates that, in the second stage, the equilibrium output of each firm will satisfy a modified version of the equilibrium condition. The difference in the market equilibrium condition is given by the fact that, from firm i's perspective, the effective price, for each unit sold, is now $P(Q) + y_i$, or equivalently its marginal cost is reduced by y_i.

The central planner's objective is to maximize the equilibrium aggregated market consumption. This is equivalent to maximizing the consumer surplus. This is the appropriate objective in many applications, where the central planner is a supra-national authority, like the World Bank, whose main interest is effectively to maximize the aggregated market consumption, say of an infectious disease treatment, without taking into account the additional surplus obtained by local producers (see Arrow et al. (2004) for further discussion on this topic for the case of malaria drugs). Additionally, in Section 6, we will also analyze the case where the central planner's objective is to maximize the social welfare, including both the consumer and the producer surplus.

Finally, let us emphasize that the central planner can only allocate co-payments, and never charge a tax for the units produced in the market. In other words, the co-payments being allocated have to be non-negative. A formulation of the co-payment allocation problem is given in the following:

$$
\begin{aligned}
\max_{\mathbf{y},\mathbf{q},Q} \quad & Q \\
\text{s.}t. \quad & \sum_{i=1}^{n} q_i y_i \leq B && (2)\\
& y_i \geq 0, \text{ for each } i \in \{1,\dots,n\} && (3)\\
& \sum_{i=1}^{n} q_i = Q && (4)\\
& q_i \geq 0, \text{ for each } i \in \{1,\dots,n\} && (5)\\
& P(Q) + y_i = h_i(q_i), \text{ for each } i \in \{1,\dots,n\}. && (6)
\end{aligned}
$$

This is a valid formulation even if there are firms that have a positive marginal cost of producing zero units, which prevents them from participating in the market equilibrium. Namely, if for some firm i we have $h_i(0) \geq P(Q)$, then we can just set $q_i = 0$ and $y_i = h_i(0) - P(Q) \geq 0$. This is without loss of generality, because setting $q_i = 0$ ensures that firm i does not have any impact in the budget constraint (2), and the non-negativity constraint on the co-payment y_i ensures that the market equilibrium condition is satisfied. In other words, constraint (6) does *not* imply that every firm has to participate in the market equilibrium.

From the equilibrium condition given in constraint (6), it follows that we can replace all the co-payment variables y_i by $h_i(q_i) - P(Q)$. Namely, we can reformulate the co-payment allocation problem as if the central planner was deciding the output of each firm, as long as there exist feasible co-payments that can sustain the outputs chosen as the market equilibrium. The feasibility of the co-payments will be given by both the budget constraint (2), and the non-negativity of the co-payments (3). We summarize this observation in the following proposition.

PROPOSITION 1. *The co-payments allocation problem faced by the central planner can be formulated as follows:*

$$
\begin{aligned}
& \max_{q,Q} \quad Q \\
& \quad s.t. \quad \sum_{j=1}^{n} q_j h_j(q_j) - P(Q)Q \leq B && (7)\\
(CAP) \quad & \quad h_i(q_i) \geq P(Q), \ \textit{for each } i \in \{1,\dots,n\} && (8)
\end{aligned}
$$

$$\sum_{j=1}^{n} q_j = Q \tag{9}$$

$$q_i \geq 0,\ \textit{for each } i \in \{1,\ldots,n\}. \tag{10}$$

The co-payments that the central planner must allocate to induce outputs $\boldsymbol{q}$ *are,*

$$y_i(\boldsymbol{q}) = h_i(q_i) - P(Q),\ \textit{for each } i.$$

Constraint (7) is equivalent to the budget constraint (2). Note that it has a budget balance interpretation, namely, the total cost in the market, minus the total revenue in the market, has to be less or equal than the budget introduced by the central planner. Constraint (8) is equivalent to the non-negativity of the co-payments (3).

3.2. Special Cases

Our model is fairly general. In particular, in this section, we discuss some well known imperfect competition models that are captured as special cases.

Cournot Competition with Linear Demand. The classical oligopoly model proposed by Cournot is defined in a very similar setting. The only difference is that, given all the other firms production levels, each firm sets its output q_i at a level such that it maximizes their profit Π_i, where

$$\Pi_i = P(Q)q_i - \int_0^{q_i} h_i(x_i)dx_i.$$

If we assume $P(Q)$ is decreasing and $h_i(q_i)$ are increasing, for each i, as well as $P'(Q)+q_iP''(Q) \leq 0$, then there exists a unique market equilibrium defined by the solution to the first order conditions of the firms' profit maximization problem, see Vives (2001). Namely, at equilibrium, each firm sets its output at a level such that,

$$\text{For each } i, \text{ if } \ q_i > 0 \text{ then } \ \frac{\partial \Pi_i}{\partial q_i} = 0, \text{ or equivalently, } P(Q) = h_i(q_i) - P'(Q)q_i. \tag{11}$$

In the equilibrium condition (11), the marginal cost must be equal to the marginal revenue, while in the equilibrium condition (1), the marginal cost must be equal to the market price. Moreover, the term $P'(Q)q_i$ is not independent for each firm.

Now, for the commonly assumed special case where the inverse demand function is linear, namely $P(Q) = a - bQ$, it follows that $P'(Q) = -b$. Define $\tilde{h}_i(q_i) \equiv h_i(q_i) + bq_i$, for each i, then we can rewrite the equilibrium condition as follows:

$$\text{For each } i, \text{ if } \ q_i > 0 \text{ then } \ P(Q) = \tilde{h}_i(q_i).$$

This equilibrium condition is a special case of condition (1), but written for a modified cost function $\tilde{h}_i(q_i)$.

Cournot Competition under Yield Uncertainty with Linear Demand and Linear Marginal Costs. We consider the Cournot Competition under yield uncertainty model used in Deo and Corbett (2009). We assume that each firm $i \in \{1,\dots,n\}$ decides its production target $\bar{q}_i$, while the actual output is uncertain and given by $q_i = \alpha_i \bar{q}_i$, where α_i is a random variable reflecting the random yield for firm i. We assume that the random variables α_i are identically and independently distributed for all firms, with $\mathbb{E}[\alpha_i] = \mu$, and $\mathrm{Var}[\alpha_i] = \sigma^2$. Additionally, we assume a linear inverse demand function $P(Q) = a - bQ$, where $Q = \sum_{i=1}^n q_i$ is again the aggregated market consumption.

We consider two marginal costs: (i) $\bar{h}(\bar{q}_i)$ per unit of production target, and (ii) $h(q_i)$ per unit actually produced. The first cost is driven by the amount of raw materials needed for production, while the second cost corresponds to the cost of packaging the actual output. Finally, we assume Cournot Competition among the firms. Namely, given the production target of all the other firms, each firm sets its production target $\bar{q}_i$ to the level that maximizes its expected profit. We generalize the model used in Deo and Corbett (2009) in two ways. First, we consider heterogeneous firms while Deo and Corbett consider homogeneous firms. Second, Deo and Corbett assume a constant marginal cost function and a fixed cost to enter the market, while we assume more general marginal cost functions. Moreover, we extend the model to include a central planner allocating subsidies to the competing firms, anticipating the market reaction to the subsidy allocation, and facing a budget constraint. In order to do so, we assume that both marginal cost functions are linear. Namely, we assume that $\bar{h}(\bar{q}_i) = \bar{g}_i \bar{q}_i$, and $h(q_i) = g_i q_i$. Note that we consider heterogeneous firms, where some of them may be more efficient than the others, depending on the values of the firm specific parameters $\bar{g}_i$ and g_i.

Let us start by considering the second stage problem. Assume that the central planner allocates a co-payment $y_i \geq 0$ to each firm $i \in \{1,\dots,n\}$. Each firm sets it production target $\bar{q}_i$ to the level that maximizes its expected profit, given by

$$\mathbb{E}\left[P(Q)q_i + y_i q_i - \int_0^{\bar{q}_i} \bar{h}(x_i)dx_i - \int_0^{q_i} h(x_i)dx_i\right] = \mathbb{E}\left[\left(a - b\sum_{i=1}^n \alpha_i \bar{q}_i\right)\alpha_i \bar{q}_i + y_i \alpha_i \bar{q}_i - \bar{g}_i \frac{\bar{q}_i^2}{2} - g_i \frac{\alpha_i^2 \bar{q}_i^2}{2}\right].$$

The expectation is taken with respect to the random variables α_i. This is a concave maximization problem in $\bar{q}_i$, and therefore the first order condition is sufficient for optimality. In order to write the first order condition in a compact form, define

$$\tilde{g}_i \equiv \frac{\bar{g}_i}{\mu} + \frac{\sigma^2 + \mu^2}{\mu} g_i + b\mu + 2b\frac{\sigma^2}{\mu}, \quad \tilde{h}_i(\bar{q}_i) \equiv \tilde{g}_i \bar{q}_i, \quad \bar{Q} = \sum_{j=1}^n \bar{q}_j, \quad \bar{P}(\bar{Q}) = a - \mu b \bar{Q}.$$

Additionally, note that the expected market price has the following closed form expression,

$$\mathbb{E}\left[P(Q)\right] = a - \mu b \sum_{i=1}^n \bar{q}_i = a - \mu b \bar{Q} = \bar{P}(\bar{Q}).$$

Hence, we can write the first order condition of the firms' profit maximization problem as follows:

$$\bar{P}(\bar{Q}) = \tilde{h}_i(\bar{q}_i) - y_i. \tag{12}$$

In order to define the co-payment allocation problem in this setting, it remains to address how will the yield uncertainty be considered in the budget constraint. We consider two possible approaches that will lead to optimization problems with similar structure.

First, assume that the central planner would like to find a co-payment allocation, such that it satisfies the budget constraint in expectation, then we can write the budget constraint as follows:

$$\mathbb{E}\left[\sum_{i=1}^{n} q_i y_i\right] = \mu \sum_{i=1}^{n} \bar{q}_i y_i \leq B.$$

Alternatively, assume that the central planner takes a robust approach. Namely, she would like to satisfy the budget constraint in each possible yield uncertainty realization. We will assume, for simplicity, that the i.i.d. random yields for each firm have a bounded support, that is $\alpha_i \in [\underline{\alpha}, \bar{\alpha}]$, for each i. Then, we can write the budget constraint as follows:

$$\bar{\alpha} \sum_{i=1}^{n} \bar{q}_i y_i \leq B.$$

Finally, assuming that the budget constraint must be satisfied in expectation (the robust approach is analogous), we can use Equation (12) to write the central planner's problem, like in Proposition 1, as follows:

$$\begin{aligned}
\max_{\mathbf{y},\bar{\mathbf{q}}} \quad & \mathbb{E}[Q] = \mu \sum_{i=1}^{n} \bar{q}_i = \mu \bar{Q} \\
\text{s.}t. \quad & \sum_{i=1}^{n} \bar{q}_i \tilde{h}_i(\bar{q}_i) - \bar{P}(\bar{Q})\bar{Q} \leq \frac{B}{\mu} && (13) \\
& \tilde{h}_i(\bar{q}_i) \geq \bar{P}(\bar{Q}), \text{ for each } i \in \{1,\ldots,n\} && (14) \\
& \sum_{j=1}^{n} \bar{q}_j = \bar{Q} && (15) \\
& \bar{q}_i \geq 0, \text{ for each } i \in \{1,\ldots,n\}. && (16)
\end{aligned}$$

The co-payments that the central planner must allocate to induce the production targets $\bar{\mathbf{q}}$, are $y_i = \tilde{h}_i(\bar{q}_i) - \bar{P}(\bar{Q})$, for each i. The resulting problem formulation is a special case of the co-payment allocation problem (CAP).

3.3. An Upper Bound Problem

Note that under our assumptions, the co-payment allocation problem (CAP) is not necessarily a convex optimization problem. In fact, we have only assumed that the marginal cost functions $h_i(q_i)$

are increasing, for each i, and that the inverse demand function $P(Q)$ is decreasing. In order to gain some insights into the structure of the optimal solution, we ignore the non-negativity of the co-payments and analyze the following relaxation, which provides an upper bound on the aggregated market consumption that can be induced with the available budget B.

$$
\begin{aligned}
& \max_{\mathbf{q},Q} \; Q \\
& \quad \text{s.}t. \quad \sum_{j=1}^{n} q_j h_j(q_j) - P(Q)Q \leq B && (17) \\
(UBP) \quad & \quad\quad\quad \sum_{j=1}^{n} q_j = Q && (18) \\
& \quad\quad\quad q_i \geq 0, \; \text{for each } i \in \{1,\dots,n\}. && (19)
\end{aligned}
$$

This upper bound problem may still be non-convex, because of the budget constraint (17). However, Lemma 1 below asserts that at optimality the budget constraint is tight (i.e. holds with equality), and each active firm i must have a value of $(h_i(q_i)q_i)'$ equal to each other, and no larger than any inactive firm. This property will be have a central role in proving the optimality of uniform subsidies.

LEMMA 1. *Assume that the marginal cost functions $h_i(q_i)$ are non-negative, increasing, and differentiable in $[0,\bar{Q})$; and that the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0,\bar{Q}]$. Then, any optimal solution to the upper bound problem (UBP) must satisfy the budget constraint (17) with equality, and also satisfy the following condition:*

$$\textit{If} \;\; q_i > 0, \;\; \textit{then} \;\; (h_i(q_i)q_i)' \leq (h_j(q_j)q_j)', \; \textit{for each } i,j \in \{1,\dots,n\}.$$

Proof. The feasible set of problem (UBP) is closed and bounded. It is bounded because $q_i \in [0,\check{q}_i]$, for each i, where $\check{q}_i$ is such that $h_i(\check{q}_i)\check{q}_i = B$. Similarly, $Q \in [0,\check{Q}]$, where $\check{Q} = \max_{i\in\{1,\dots,n\}}\{\check{q}_i\}$. On the other hand, it is closed because it is defined by inequalities on continuous functions. Additionally, the objective function of problem (UBP) is continuous. It follows that there exists an optimal solution to problem (UBP).

Let $(\mathbf{q}^*, Q^*)$ be an optimal solution to problem (UBP). Assume by contradiction that the budget constraint is not tight for $(\mathbf{q}^*, Q^*)$. Namely,

$$\sum_{j=1}^{n} q_j^* h_j(q_j^*) - P(Q^*)Q^* = \sum_{j=1}^{n} q_j^*(h_j(q_j^*) - P(Q^*)) < B.$$

Then, we can increase the value of q_i^*, for any index i, by $\epsilon > 0$ sufficiently small, maintain feasibility, and obtain a strictly larger objective value. This contradicts the optimality of $(\mathbf{q}^*, Q^*)$. Therefore, the budget constraint must be tight *for every optimal solution* to problem (UBP).

Assume by contradiction that there exist indexes i, j such that $q_i^* > 0$ and $(h_i(q_i^*)q_i^*)' > \left(h_j(q_j^*)q_j^*\right)'$. Then, we can decrease the value of q_i^*, and increase the value of q_j^*, both by the same $\epsilon > 0$ sufficiently small, and maintain feasibility. Specifically, the marginal change in the left hand side of the budget constraint (17) is $-(h_i(q_i^*)q_i^*)' + \left(h_j(q_j^*)q_j^*\right)' < 0$. Therefore, the budget constraint for this modified solution is satisfied, and not tight. However, this modified solution attains the same objective value Q^*, and it is therefore optimal. This is a contradiction to the fact that the budget constraint must be tight for every optimal solution to problem (UBP).

4. Optimality of Uniform Co-payments

The result obtained in this section asserts that uniform co-payments are optimal for the co-payment allocation problem (CAP), for a large class of marginal costs functions $h_i(q_i)$. Specifically, we show that if the marginal cost functions satisfy Property 1 below, then uniform subsidies are optimal.

PROPERTY 1. For each i, j and each $q_i, q_j \geq 0$, if $h_i(q_i) > h_j(q_j)$ then $(h_i(q_i)q_i)' \neq (h_j(q_j)q_j)'$.

Next, we show that there exists a large class of marginal cost functions that satisfy Property 1 above. Consider the case, in which $h_i(q_i) = h(g_i q_i)$, where $h(x)$ is non-negative, increasing, and differentiable over $x \geq 0$, and $g_i > 0$ is a firm specific parameter. This captures the setting, where all firms use a similar technology, but can differ in their efficiency. Specifically, $h(x)$ models the industry specific marginal cost function, while $g_i > 0$ models the efficiency of firm i.

In this setting there is no loss of generality in assuming $h(0) = 0$. Specifically, any positive value for $h(0)$ will affect each firm in the same way, and therefore will only shift the market price by a constant that can be re-scaled to zero. This assumption implies that all firms have a positive output in the market equilibrium, for any positive market price. Therefore, the underlying assumption is that all firms have already entered the market *before* the subsidy is decided, and there is no subsequent entry or exit of firms into the market. This assumption is reasonable in our setting, where the subsidy is not permanent (it only applies until the budget is exhausted), and it is paid ex-post to the firms, for each unit already sold.

In this setting, any function $h(x)$, such that $h(x) + h'(x)x$ is monotone will satisfy Property 1. Specifically, for each such function we would have that $h_i(q_i) > h_j(q_j)$ is equivalent, by definition, to $h(g_i q_i) > h(g_i q_j)$. However, $h(x)$ increasing implies $g_i q_i > g_i q_i$. Moreover, $h(x) + h'(x)x$ monotone implies $h(g_i q_i) + h'(g_i q_i) g_i q_i \neq h(g_i q_i) + h'(g_i q_i) g_i q_i$. Which is, again by definition, equivalent to $(h_i(q_i)q_i)' \neq (h_j(q_j)q_j)'$. Some functions that satisfy this condition, and the respective marginal cost functions associated to them, are:

- $h(x) = e^x - 1$, $h_i(q_i) = e^{g_i q_i} - 1$.
- $h(x) = x^u$, for $u > 0$, $h_i(q_i) = g_i q_i^u$.
- $h(x) = \ln(x + 1)$, $h_i(q_i) = \ln(g_i q_i + 1)$.

- Any polynomial with positive coefficients.

Specifically, all these functions have the property that $h(x)x$ is convex over $x \geq 0$, therefore, $h(x) + h'(x)x$ is increasing. Note that the marginal cost functions $h(x)$ are allowed to be concave, e.g., $h(x) = x^u$ for $0 < u < 1$, and $h(x) = \ln(x+1)$. Moreover, note that $h_i(q_i) = g_i q_i^u$ corresponds exactly to the only homogeneous function of degree $u > 0$ in one variable.

4.1. Sufficient Optimality Condition

The next one is the main result in this section.

THEOREM 1. *Assume that the marginal cost functions $h_i(q_i)$ are non-negative, increasing, and differentiable in $[0, \bar{Q})$; the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0, \bar{Q}]$. If the marginal cost functions satisfy Property 1, then uniform co-payments are optimal for the co-payment allocation problem (CAP).*

Proof. The existence of an optimal solution to problem (UBP) was shown in Lemma 1. Let $(\mathbf{q}, Q)$ be an optimal solution to problem (UBP). We will show that if the marginal cost functions satisfy Property 1, then $(\mathbf{q}, Q)$ induces uniform co-payments for every firm with a positive output $q_i > 0$. Moreover, $(\mathbf{q}, Q)$ is feasible for the co-payment allocation problem (CAP), and therefore optimal.

From Lemma 1 it follows that $(\mathbf{q}, Q)$ is such that the budget constraint is binding, and for each i, j with $q_i > 0$ and $q_j > 0$, we must have $(h_i(q_i)q_i)' = (h_j(q_j)q_j)'$. The assumption that the marginal cost functions satisfy Property 1 implies $h_i(q_i) = h_j(q_j)$, which implies that uniform subsidies are optimal. Specifically, because the budget constraint is tight, it follows that $h_i(q_i) - P(Q) = \dfrac{B}{Q} > 0$ for every i such that $q_i > 0$.

In order to show that the solution $(\mathbf{q}, Q)$ is feasible for the co-payment allocation problem (CAP), it remains to show that the firms that do not participate in the market equilibrium effectively have a marginal cost of producing zero units, which is larger than the induced market price. Specifically, $(\mathbf{q}, Q)$ is such that, for each i, j with $q_i > 0$ and $q_j = 0$, we have,

$$h_j(0) - P(Q) \geq h_i(q_i) + h_i'(q_i)q_i - P(Q) \geq h_i(q_i) - P(Q) = \frac{B}{Q} > 0.$$

The first inequality follows from Lemma 1, and the second inequality follows from $h_i(q_i)$ increasing. The equality follows from the fact that the budget constraint is tight, and $q_i > 0$.

Hence, $(\mathbf{q}, Q)$ is also feasible for the co-payment allocation problem (CAP), and therefore optimal. Moreover, $(\mathbf{q}, Q)$ induces uniform co-payments. Therefore, uniform co-payments are optimal for the co-payment allocation problem (CAP).

This result is surprising, considering that the assumptions on the inverse demand function are very general, and particularly since firms can be heterogeneous and the central planner has the freedom to allocate differentiated co-payments to each firm.

The intuition behind this result comes from the market equilibrium condition and the budget constraint. Essentially, if the central planner allocates a larger co-payment to a firm, then its resulting market share will increase, which is exactly the rate at which it will consume budget. This will in turn make less budget available to the rest of the firms, and therefore, their co-payments would have to decrease. Theorem 1 shows that if the marginal cost functions of the firms satisfy Property 1, then the net effect of this change will never be positive.

In particular, Theorem 1 applies for the special cases we considered in Section 3.2. For Cournot Competition with linear demand, uniform co-payments are optimal for any marginal cost functions $h_i(q_i)$, such that the functions $\tilde{h}_i(q_i) = h_i(q_i) + bq_i$ satisfy Property 1. Specifically, if the marginal cost functions are linear, that is $h_i(q_i) = g_i q_i$, for each i, then uniform co-payments are optimal. Similarly, for Cournot Competition under yield uncertainty with linear demand, if both marginal costs are linear, then uniform subsidies are optimal. Note that in both cases we allow for heterogeneous firms, where some of them can be significantly more efficient than others.

5. Incorporating Market State Uncertainty

A natural extension of the model discussed in Section 3, is to consider the setting where the central planner does not know the market state (defined by the inverse demand function) with certainty, but generally she will have a set of possible market state scenarios, and beliefs on the likelihood that each scenario will materialize.

In more details, we assume that she has a discrete description of the market state uncertainty, where each scenario $s \in \{1, \ldots, m\}$ is realized with probability p_s. Each scenario s is characterized by a scenario dependent inverse demand function $P^s(Q^s)$. For each scenario $s \in \{1, \ldots, m\}$, we make assumptions like in Section 3. Namely, we assume that, each inverse demand function $P^s(Q)$ is non-negative, decreasing and differentiable in $[0, \bar{Q}^s]$, where $\bar{Q}^s$ is the smallest value such that $P^s(\bar{Q}^s) = 0$. Similarly, for the market equilibrium condition we assume that if scenario s realizes, then firms set their output q_i^s at a level such that, for each i, j, if $q_i^s > 0$, then $h_i(q_i^s) = P^s(Q^s) \leq h_j(q_j^s)$.

Similar to Section 3, a formulation of the co-payments allocation problem under market state uncertainty can be written as follows:

$$\max_{(\mathbf{q}^s, Q^s)_{s \in \{1, \ldots, m\}}, \mathbf{y}} \quad \sum_{s=1}^{m} Q^s p_s$$

$$\text{s.}t. \quad \sum_{j=1}^{n} q_j^s y_j \leq B, \text{ for each } s \in \{1,\dots,m\} \tag{20}$$

$$y_i \geq 0, \text{ for each } i \in \{1,\dots,n\} \tag{21}$$

$$\sum_{j=1}^{n} q_j^s = Q^s, \text{ for each } s \in \{1,\dots,m\} \tag{22}$$

$$q_i^s \geq 0, \text{ for each } i \in \{1,\dots,n\}, \ s \in \{1,\dots,m\} \tag{23}$$

$$h_i(q_i^s) - P^s(Q^s) - y_i \geq 0, \text{ for each } i \in \{1,\dots,n\}, \ s \in \{1,\dots,m\} \tag{24}$$

$$q_i^s\left(h_i(q_i^s) - P^s(Q^s) - y_i\right) = 0, \text{ for each } i \in \{1,\dots,n\}, \ s \in \{1,\dots,m\}. \tag{25}$$

The objective is to maximize the expected aggregated market consumption. Constraint (20) is the budget constraint, for each market state scenario. Constraint (21) corresponds to the non-negativity of the co-payments. Constraint (22) defines the aggregated market consumption for each scenario. Finally, constraints (23)-(25) are the complementarity constraints, which tie together the different scenarios. They state that, in each scenario, each firm either participates in the market equilibrium, in which case it produces the quantity that equates its marginal cost with the market price plus the co-payment; or its marginal cost of producing zero units is strictly larger than the market equilibrium price plus the co-payment, in which case the firm is inactive. Naturally, each firm must get the same co-payment in each possible scenario.

In other words, constraints (23)-(25) correspond to the non-anticipativity constraints, and they state that co-payments are a first stage decision made by the central planner *before* the uncertainty is realized. This is precisely what prevents us from using the co-payments to eliminate the complementarity constraints from the model formulation, similarly to Proposition 1. This makes the problem significantly harder to analyze. In order to somewhat simplify this formulation, we make the additional assumption that producing zero units has a marginal cost of zero, as stated in the following Proposition.

Proposition 2. *If we additionally assume*

- $h_i(0) = 0$ *for each* $i \in \{1,\dots,n\}$

Then, the co-payments allocation problem under market state uncertainty faced by the central planner can be re-written as follows:

$$\max_{(\boldsymbol{q}^s, Q^s)_{s=1,\dots,m}} \sum_{s=1}^{m} Q^s p_s$$

$$s.t. \quad \sum_{j=1}^{n} q_j^s h_j(q_j^s) - P^s(Q^s)Q^s \leq B, \textit{ for each } s \in \{1,\dots,m\} \tag{26}$$

$$h_i(q_i^s) \geq P^s(Q^s), \textit{ for each } i \in \{1,\dots,n\}, \ s \in \{1,\dots,m\} \tag{27}$$

$$(SCAP) \qquad \sum_{j=1}^{n} q_j^s = Q^s, \textit{ for each } s \in \{1,\dots,m\} \tag{28}$$

$$q_i^s \geq 0, \text{ for each } i \in \{1,\ldots,n\}, \ s \in \{1,\ldots,m\} \tag{29}$$

$$h_i(q_i^s) - P^s(Q^s) = h_i(q_i^{s'}) - P^{s'}(Q^{s'}), \text{ for each } {}_{s,s' \in \{1,\ldots,m\}.}^{i \in \{1,\ldots,n\},} \tag{30}$$

The co-payments that the central planner must allocate to induce outputs $\{\boldsymbol{q}^s\}_{s\in\{1,\ldots,m\}}$ *are,*

$$y_i = h_i(q_i^s) - P^s(Q^s), \text{ for each } i \in \{1,\ldots,n\}, \ s \in \{1,\ldots,m\}. \tag{31}$$

Proposition 2 states that, if the marginal cost of producing zero units is zero, then every firm will participate in the market equilibrium, for any non-negative market price. Therefore, Equation (31) holds, and we can eliminate the variables y_i from the problem formulation.

Like in Proposition 1, constraint (26) corresponds to the budget constraint. Namely, for each scenario s, the total cost minus the total revenue in the market has to be less or equal than the budget introduced by the central planner. Constraint (27) is the non-negativity of the co-payments, it ensures that the solution proposed by the central planner can be sustained as a market equilibrium by allocating only subsidies, and not taxes. Like before, the only constraint that ties all the scenarios together is the non-anticipativity constraint (30), which states that each firm must get the same co-payment in each possible scenario.

This problem is still hard to analyze directly, which motivates us to develop a relaxation that provides an upper bound on the expected aggregated market consumption that can be induced with the available budget, as shown below. All the proofs in this section are presented in Appendix A.

5.1. An Upper Bound Problem

We start with a simple observation that is derived from the structure of problem (SCAP).

Lemma 2. *For any feasible solution to the co-payments allocation problem under market state uncertainty (SCAP), without loss of generality, the scenarios can be renumbered, such that the following inequalities are true:*

$$P^1(Q^1) \geq P^2(Q^2) \geq \ldots \geq P^m(Q^m), \tag{32}$$

$$h_i(q_i^1) \geq h_i(q_i^2) \geq \ldots \geq h_i(q_i^m), \text{ for each } i, \tag{33}$$

$$q_i^1 \geq q_i^2 \geq \ldots \geq q_i^m, \text{ for each } i, \tag{34}$$

$$Q^1 \geq Q^2 \geq \ldots \geq Q^m, \tag{35}$$

$$\sum_{j=1}^n q_j^1 y_j \geq \sum_{j=1}^n q_j^2 y_j \geq \ldots \geq \sum_{j=1}^n q_j^m y_j, \tag{36}$$

where $\sum_{j=1}^n q_j^s y_j$ *is the total amount spent in co-payments in scenario* s.

Let $(\mathbf{q}^s, Q^s)^*_{s=1,\dots,m}$ be an optimal solution to the co-payment allocation problem under scenario uncertainty (SCAP), and assume that the scenarios are numbered such that Equations (32)-(36) above hold. Then, we claim that the solution to problem (SUBP) below provides an upper bound on the expected aggregated market consumption that can be induced with the available budget. Specifically, problem (SUBP) is derived from problem (SCAP) by adding constraints (41) and (42) below, and replacing the non-anticipativity constraint (30), with the relaxed version (43).

$$
(SUBP) \quad
\begin{aligned}
\max_{\mathbf{q}^s, Q^s} \quad & \sum_{s=1}^{m} Q^s p_s \\
\text{s.}t. \quad & \sum_{j=1}^{n} q_j^s h_j(q_j^s) - P^s(Q^s)Q^s \le B, \text{ for each } s \in \{1,\dots,m\} && (37)\\
& h_i(q_i^s) \ge P^s(Q^s), \text{ for each } i \in \{1,\dots,n\}, \ s \in \{1,\dots,m\} && (38)\\
& \sum_{j=1}^{n} q_j^s = Q^s, \text{ for each } s \in \{1,\dots,m\} && (39)\\
& q_i^s \ge 0, \text{ for each } i \in \{1,\dots,n\}, \ s \in \{1,\dots,m\} && (40)\\
& P^1(Q^1) \ge P^s(Q^s), \text{ for each } s \in \{1,\dots,m\} && (41)\\
& Q^1 \ge Q^s, \text{ for each } s \in \{1,\dots,m\} && (42)\\
& h_i(q_i^s) - P^s(Q^s) \le h_i(q_i^1) - P^1(Q^1), \text{ for each } {}^{i \in \{1,\dots,n\},}_{s \in \{1,\dots,m\}}. && (43)
\end{aligned}
$$

Problem (SUBP) is a valid relaxation of problem (SCAP). Specifically, the optimal solution of problem (SCAP), $(\mathbf{q}^s, Q^s)^*_{s=1,\dots,m}$, is feasible for problem (SUBP), and attains the same objective value. To argue the feasibility of solution $(\mathbf{q}^s, Q^s)^*_{s=1,\dots,m}$ for problem (SUBP), recall from Lemma 2 that $s = 1$ is the scenario that attains the largest value for both the induced market price (see (32)), and the induced aggregated market consumption (see (35)), in solution $(\mathbf{q}^s, Q^s)^*_{s=1,\dots,m}$. It follows that adding constraints (41) and (42), does not cut-off solution $(\mathbf{q}^s, Q^s)^*_{s=1,\dots,m}$. Finally, solution $(\mathbf{q}^s, Q^s)^*_{s=1,\dots,m}$ satisfies constraint (43) with equality.

5.2. Optimality of Uniform Co-payments

In this section, we consider again the setting where all firms use a similar technology, but they can differ in their respective efficiency, similar to the assumptions in Section 4. Specifically, we consider the case, in which $h_i(q_i) = h(g_i q_i)$, where h(x) is non-negative, increasing, and differentiable over $x > 0$, and $g_i > 0$ is a firm specific parameter. The function $h(x)$ models the industry specific marginal cost function, while g_i models the efficiency of firm i. Recall from Section 4 that we can assume, without loss of generality, that $h(0) = 0$, therefore, we will refer to the co-payment allocation problem under market state uncertainty (SCAP), and its upper bound problem (SUBP).

We now present sufficient conditions, which ensure that uniform subsidies maximize the expected aggregated market consumption in this setting. Specifically, we show that if the firms' marginal cost functions satisfy Property 2 below, then uniform subsidies are optimal for the co-payments allocation problem under market state uncertainty (SCAP).

PROPERTY 2. The function $h(x)$ is *convex*, and such that for any $x_1 > x_2 \geq 0$, and $x_1 > x_3 > x_4 \geq 0$, if $\dfrac{h(x_2)}{h(x_1)} > \dfrac{h(x_4)}{h(x_3)}$ then $\dfrac{h'(x_2)}{h'(x_1)} > \dfrac{h'(x_4)}{h'(x_3)}$.

Note that Property 2 implies Property 1, discussed in Section 4. Specifically, $h(x)$ increasing and convex implies that $h(x)+h'(x)x$ is increasing. This is a sufficient condition for Property 1 to hold.

REMARK 1. The functions $h_i(q_i) = g_i q_i^m$, for $m \geq 1$, and $h_i(q_i) = e^{g_i q_i} - 1$, satisfy Property 2.

Note, from Remark 1, that from the examples of marginal cost functions that satisfy Property 1 given in Section 4, all the ones that are also *convex* satisfy Property 2 as well. In this sense, the extra requirements in Property 2, with respect to Property 1, are mainly driven by the convexity assumption. Finally, note that functions $h_i(q_i) = g_i q_i^m$, for $m \geq 1$, are the unique convex homogeneous functions in one variable.

Theorem 2 below shows that there exists an optimal solution to the upper bound problem (SUBP), such that the co-payments induced in scenario $s = 1$, the one that attains the largest aggregated market consumption (see (35)), and the largest amount spent in co-payments (see (36)), are uniform. This result will play a central role in proving the main result in this section.

THEOREM 2. *Assume that the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0, \bar{Q}]$. Assume that the marginal costs functions are given by $h_i(q_i) = h(g_i q_i)$ for each i, for any increasing and continuously differentiable function $h(x)$, such that $h(0) = 0$. If $h(x)$ satisfies Property 2, then there exists an optimal solution to the upper bound problem (SUBP), $(\hat{\boldsymbol{q}}^s, \hat{Q}^s)_{s=1,\ldots,m}$, such that, $h_i(\hat{q}_i^1) - P^1(\hat{Q}^1) = y^1$ for each $i \in \{1,\ldots,n\}$, for some value $y^1 > 0$.*

To prove Theorem 2, we show the following lemmas that will be useful in the analysis. Specifically, we will consider the optimal solution to problem (SUBP) with the smallest difference between $(\max_{i\in\{1,\ldots,n\}} \{h_i(q_i^1)\})$ and $(\min_{i\in\{1,\ldots,n\}} \{h_i(q_i^1)\})$ (from Lemma 3 below we know that such a solution does exist). Note that proving Theorem 2, is equivalent to showing that this difference is zero. We will assume by contradiction that this difference is strictly positive, and show that then we can construct another optimal solution with an even smaller difference, a contradiction.

When constructing the modified optimal solution, Lemma 4 allows us to focus only on constraint (43). On the other hand, using the convexity assumption on $h(x)$, Lemma 5 provides bounds on the impact that the modifications to the optimal solution have on constraint (43). These bounds will allow us to complete the proof by arguing that the modified solution is feasible and optimal, while attaining a smaller difference between the maximum marginal cost in scenario $s = 1$, and the minimum marginal cost in scenario $s = 1$.

LEMMA 3. *Under the assumptions of Theorem 2, there exists an optimal solution to problem (SUBP) that attains the minimum of the gaps between the maximum marginal cost in scenario* $s = 1$*, and the minimum marginal cost in scenario* $s = 1$*, induced by any optimal solution.*

LEMMA 4. *Under the assumptions of Theorem 2, for any feasible solution to problem (SUBP),* $(\boldsymbol{q}^s, Q^s)_{s=1,\dots,m}$*, if* $h_i(q_i^s) > h_j(q_j^s)$ *for some* i*,* j*,* s*, then we can transfer a sufficiently small* $\epsilon^s > 0$*, from* q_i^s *to* q_j^s*, without violating any of the constraints (37)-(42) related to scenario* s*.*

LEMMA 5. *Under the assumptions of Theorem 2, for any feasible solution to problem (SUBP),* $(\boldsymbol{q}^s, Q^s)_{s=1,\dots,m}$*, for any* $\epsilon^1 > 0$*, and for any scenario* $s \neq 1$*. The following conditions must hold:*

$$\text{If } \epsilon^s \geq 0 \text{ satisfies } \epsilon^1 \leq \frac{h'(g_i(q_i^s - \epsilon^s))}{h'(g_i q_i^1)} \epsilon^s, \text{ then } h_i(q_i^s - \epsilon^s) - P^s(Q^s) \leq h_i(q_i^1 - \epsilon^1) - P^1(Q^1).$$

$$\text{If } \epsilon^s \geq 0 \text{ satisfies } \frac{h'(g_i(q_i^s + \epsilon^s))}{h'(g_i q_i^1)} \epsilon^s = \epsilon^1, \text{ then } h_i(q_i^s + \epsilon^s) - P^s(Q^s) \leq h_i(q_i^1 + \epsilon^1) - P^1(Q^1).$$

Theorem 3 below concludes this section characterizing a family of firms' marginal cost functions such that uniform co-payments are optimal, even if the central planner is uncertain about the market state. This family includes convex homogeneous functions of the same degree.

THEOREM 3. *Assume that the inverse demand function* $P(Q)$ *is non-negative, decreasing, and differentiable in* $[0, \bar{Q}]$*. Assume that the marginal costs functions are given by* $h_i(q_i) = h(g_i q_i)$ *for each* i*, for any increasing and continuously differentiable function* $h(x)$*, such that* $h(0) = 0$*. If* $h(x)$ *satisfies Property 2, then allocating the largest feasible uniform co-payment is an optimal solution for the co-payment allocation problem under market state uncertainty (SCAP).*

This result is surprising, as it shows that, with some additional conditions, the optimality of uniform subsidies is preserved, even if the central planner is uncertain about the market state. Different market states induce different inverse demand functions, which can be arbitrarily different. Moreover, the assumption on the inverse demand functions of each scenario are very mild. Specifically, we only assume that they are decreasing. This is a very relevant setup, as it corresponds to a more realistic representation of the problem faced in practice, where there are large uncertainties about different characteristics of the market state, which ultimately define the effective response of the demand side to different market prices.

Moreover, the analysis suggests that the central planner only needs to consider the scenario with the highest aggregated market consumption at equilibrium (see (35)), i.e., scenario $s = 1$, regardless of the exact distribution over the different market states. Specifically, Theorem 2 shows that uniform subsidies are optimal for scenario $s = 1$ in the relaxed upper bound problem (SUBP), while Theorem 3 shows that the uniform subsidies induced by scenario $s = 1$, are in fact optimal for the co-payment allocation problem under market state uncertainty (SCAP). This insight suggests that the central planner only needs to identify the scenario with the highest aggregated market consumption at equilibrium, and implement the uniform subsidies induced by it, as opposed to taking into consideration her beliefs on the likelihood that each market state will be realized, and the effect that the subsidy allocation will have on each possible market state scenario.

6. Maximizing Social Welfare

In this section, we assume that the central planner's objective is in fact to maximize social welfare. Given some $\delta \in (0,1]$, which represents the social cost of funds, the central planner problem of allocating subsidies to maximize social welfare can be written as follows:

$$\begin{aligned} \max_{\mathbf{y},\mathbf{q},Q} \quad & \int_0^Q P(x)dx - \sum_{i=1}^n \int_0^{q_i} (h(x_i) - y_i)dx_i - \delta\left(\sum_{i=1}^n q_i y_i\right) \\ \text{s.}t. \quad & y_i \geq 0, \text{ for each } i \in \{1,\dots,n\} && (44) \\ & \sum_{i=1}^n q_i = Q && (45) \\ & q_i \geq 0, \text{ for each } i \in \{1,\dots,n\} && (46) \\ & P(Q) + y_i = h_i(q_i), \text{ for each } i \in \{1,\dots,n\}. && (47) \end{aligned}$$

The first two terms in the objective function correspond to the sum of the consumer and producer surplus, including the co-payments y_i. The third term in the objective function corresponds to the social cost of financing the subsidies. Note that in this problem there is no budget constraint. Specifically, the social cost of funds $\delta \in (0,1]$ will induce a total amount invested in subsidies at optimality, which can be interpreted as the implicit budget available. Constraint (44) states that the central planner is only allowed to allocate subsidies, and not taxes, to the firms. Like in Section 3, constraint (47) does *not* imply that every firm has to participate in the market equilibrium.

From the equilibrium condition given in constraint (47), it follows again that we can replace all the co-payment variables y_i by $h_i(q_i) - P(Q)$, as stated in the proposition below.

Proposition 3. *The social welfare maximization problem can be written as follows:*

$$\begin{aligned} & \max_{\boldsymbol{q},Q} \ \mathrm{SW}(\boldsymbol{q},\mathrm{Q}) \equiv \int_0^Q P(x)dx - \sum_{i=1}^n \int_0^{q_i} h(x_i)dx_i + (1-\delta)\left(\sum_{i=1}^n h(q_i)q_i - P(Q)Q\right) \\ & s.t. \quad h(q_i) \geq P(Q), \ \textit{for each } i \in \{1,\dots,n\} && (48) \\ (CAP - SW) \quad & \sum_{i=1}^n q_i = Q && (49) \\ & q_i \geq 0, \ \textit{for each } i \in \{1,\dots,n\}. && (50) \end{aligned}$$

The co-payments that the central planner must allocate to induce outputs $\boldsymbol{q}$ *are,*

$$y_i(\boldsymbol{q}) = h_i(q_i) - P(Q), \ \textit{for each } i.$$

The first two terms in the objective function correspond to the sum of the consumer and producer surplus, with no subsidies. The third term corresponds to the increase in social welfare induced by subsidies, minus the social cost of financing them. Constraint (48) states that the central planner is only allowed to allocate subsidies, and not taxes, to the firms.

We will make the natural assumption that the social cost of funds, $\delta \in (0,1]$, is such that objective function of problem (CAP-SW), $SW(\mathbf{q},Q)$, is coercive[1], and therefore there exists an optimal solution, see, for example, Bertsekas (1999). Then, the budget B that the central planner spends in subsidies, in order to maximize social welfare, can be written as follows. Let $(\mathbf{q}^*,Q^*)$ be an optimal solution of problem (CAP-SW), then

$$B \equiv \sum_{i=1}^{n} h(q_i^*)q_i^* - P(Q^*)Q^*.$$

6.1. Optimality of Uniform Co-payments

We conclude this section by characterizing settings where, in addition to maximizing the aggregated market consumption, uniform co-payments also maximize social welfare. Specifically, we show that if the marginal cost functions satisfy Property 3 below, then uniform subsidies are optimal for the problem (CAP-SW).

Property 3. For each i,j and each $q_i,q_j \geq 0$, if $h_i(q_i) > h_j(q_j)$ then $\dfrac{h_i(q_i)}{h_i'(q_i)q_i} \geq \dfrac{h_j(q_j)}{h_j'(q_j)q_j}$.
Two examples of marginal cost functions that satisfy Property 3, are

- $h_i(q_i) = g_i q_i^u$ for $u > 0$.
- $h_i(q_i) = \ln(g_i q_i + 1)$.

Note that these marginal cost functions also satisfy Property 1. Therefore, for these two marginal cost functions uniform subsidies maximize both the consumer surplus and social welfare.

This leads to the main result of this section.

Theorem 4. *Assume that the marginal cost functions $h_i(q_i)$ are non-negative, increasing, and differentiable in $[0,\bar{Q})$; the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0,\bar{Q}]$; and the social cost of funds $\delta \in (0,1]$ is such that it induces a finite central planner's budget B. If the marginal cost functions satisfy Property 3, then uniform co-payments are optimal for the social welfare maximization problem (CAP-SW).*

Proof. Let $(\mathbf{q}^*,Q^*)$ be the optimal solution of problem (CAP-SW). First, we show that $(\mathbf{q}^*,Q^*)$ must satisfy,

$$(1-\delta) < \frac{h_i(q_i^*)}{h_i(q_i^*) + h_i'(q_i^*)q_i^*}, \text{ for each } i. \tag{51}$$

[1] $SW(\mathbf{q},Q)$ is coercive if $SW(\mathbf{q}^k,Q^k) \longrightarrow -\infty$ for any feasible sequence such that $||(\mathbf{q}^k,Q^k)|| \longrightarrow \infty$.

Specifically, the expression in the objective function of problem (CAP-SW) related to the aggregated market consumption Q, is strictly increasing in Q. Namely,

$$\frac{\partial}{\partial q_i}\left(\int_0^Q P(x)dx - (1-\delta)P(Q)Q\right) = \delta P(Q) - (1-\delta)P'(Q)Q > 0.$$

Where the inequality follows from the inverse demand function $P(Q)$ being non-negative and decreasing. On the other hand, the remaining expression in the objective function of problem (CAP-SW), related to firm i's output q_i, is such that,

$$\frac{\partial}{\partial q_i}\left((1-\delta)\sum_{i=1}^n h(q_i)q_i - \sum_{i=1}^n \int_0^{q_i} h(x_i)dx_i\right) = (1-\delta)(h_i(q_i) + h_i'(q_i)q_i) - h_i(q_i).$$

Therefore, if Equation (51) does not hold. Namely, if there exists an index i such that, $(1-\delta)(h_i(q_i^*) + h_i'(q_i^*)q_i^*) - h_i(q_i^*) \geq 0$, then we can increase q_i^* by $\epsilon > 0$ sufficiently small, and obtain a feasible solution that attains a strictly larger objective value. This is a contradiction to the optimality of $(\mathbf{q}^*, Q^*)$.

Second, assume by contradiction that there exist indexes i, j, with $q_i^* > 0$ and $q_j^* > 0$, such that $h_i(q_i^*) > h_j(q_j^*)$. The fact that the marginal cost functions satisfy Property 3 implies that $\frac{h_i(q_i^*)}{h_i'(q_i^*)q_i^*} \geq \frac{h_j(q_j^*)}{h_j'(q_j^*)q_j^*}$. From direct algebraic manipulations, it follows that,

$$\frac{h_i(q_i^*)}{h_i(q_i^*) + h_i'(q_i^*)q_i^*} \leq \frac{h_i(q_i^*) - h_j(q_j^*)}{h_i(q_i^*) + h_i'(q_i^*)q_i^* - h_j(q_j^*) - h_j'(q_j^*)q_j^*}. \tag{52}$$

Now, Equations (51) and (52) imply that

$$(1-\delta) < \frac{h_i(q_i^*) - h_j(q_j^*)}{h_i(q_i^*) + h_i'(q_i^*)q_i^* - h_j(q_j^*) - h_j'(q_j^*)q_j^*}.$$

Therefore, we can transfer $\epsilon > 0$ sufficiently small from q_i^* to q_j^*, and obtain the following positive marginal change in the objective function, $h_i(q_i^*) - h_j(q_j^*) - (1-\delta)(h_i(q_i) + h_i'(q_i)q_i - h_j(q_j^*) - h_j'(q_j^*)q_j^*) > 0$. Namely, there exists a feasible solution with a strictly larger objective value. This contradicts the optimality of $(\mathbf{q}^*, Q^*)$.

Hence, we conclude that for each i, j, with $q_i^* > 0$ and $q_j^* > 0$, it must be the case that $h_i(q_i^*) = h_j(q_j^*)$. Therefore, uniform subsidies maximize social welfare.

7. Empirical Performance

In Section 4, we have identified conditions on the firms' marginal cost functions that guarantee the optimality of uniform co-payments to maximize the aggregated market consumption of a good. In this section we study the performance of uniform co-payments, in relevant setting where they are sub-optimal. More precisely, we consider two experimental settings. On the one hand, we consider Cournot Competition with linear demand and constant marginal cost. On the other hand, we study

$Q^{\mathrm{U}}/Q^{\mathrm{OPT}}$	n=2	n=3	n=10	n=20
Min.	0.9360	0.9175	0.9182	0.9442
1st Qu.	0.9776	0.9734	0.9785	0.9828
Median	0.9919	0.9845	0.9849	0.9876
Mean	0.9860	0.9806	0.9834	0.9866
3rd Qu.	0.9983	0.9930	0.9902	0.9912
Max.	1.0000	1.0000	1.0000	0.9991

Table 1 Relative Performance of Uniform Co-payments - Cournot Constant MC

a more general setting with non-linear demand, and non-linear marginal costs. Our goal here is to study empirically the performance of uniform co-payments on problems with data generated at random.

Note that in order to evaluate the relative performance of uniform subsidies, we need to be able to compute the aggregated market consumption induced by them. Proposition 4 below addresses this issue.

Proposition 4. *Assume that the marginal cost functions $h_i(q_i)$ are non-negative, increasing, and differentiable in $[0,\bar{Q})$; and the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0,\bar{Q}]$. Then, the market equilibrium induced by the largest feasible uniform co-payment can be computed as the solution to the following convex optimization problem,*

$$\begin{aligned} &\min_q \sum_{j=1}^{n} \int_0^{q_j} h_j(x_j)dx_j - \int_0^{q_{n+1}} P(\bar{Q}-x_{n+1})dx_{n+1} - B\ln(\bar{Q}-q_{n+1}) \\ &s.t. \quad \sum_{j=1}^{n} q_j + q_{n+1} = \bar{Q} \\ (UCAP) \qquad &q_i \geq 0, \ \textit{for each } i. \end{aligned}$$

Assuming that the inverse demand function $P(Q)$ is decreasing, and that the firms'marginal cost functions $h_i(q_i)$ are increasing, implies that problem (UCAP) is a convex optimization problem. On the other hand, in the experimental settings we consider, it will always be the case that at least the upper bound problem (UBP) is a convex optimization problem. To solve these problems we used CVX, a package for specifying and solving convex programs, see Grant and Boyd (2012). We will denote by Q^{U}, Q^{OPT} and Q^{UB} the aggregated market consumption component of the optimal solutions to problems (UCAP), (CAP) and (UBP), respectively.

Cournot Competition with Linear Demand and Constant Marginal Costs The model presented in Section 3 captures Cournot Competition with linear demand and non-decreasing marginal cost functions $h_i(q_i)$. Specifically, this implies that the modified marginal cost function defined in Section 3.2, $\tilde{h}_i(q_i) \equiv h_i(q_i) + bq_i$, is increasing. In particular, in this section we consider *constant* marginal costs. Although the constant marginal costs case moves away from the our scarce installed capacity

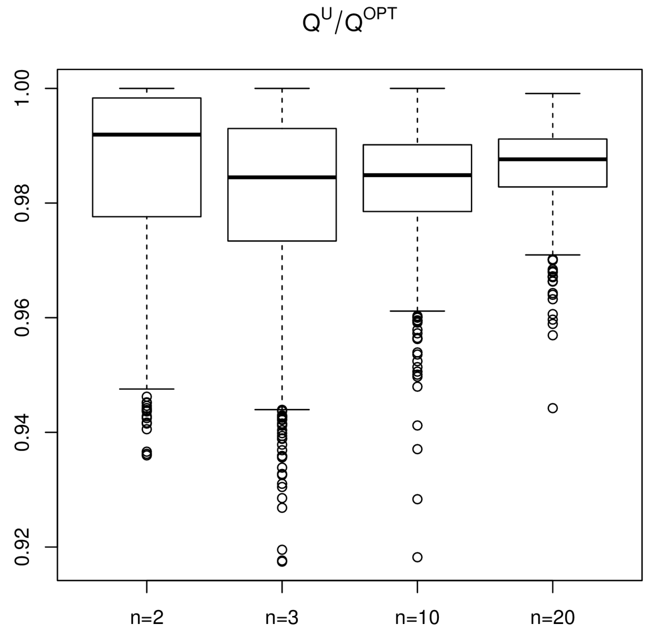

Figure 1 Relative Performance of Uniform Co-payments - Cournot Constant MC

assumption, it is a well understood model where uniform co-payments are not optimal. Therefore, it is interesting to study the performance of uniform co-payments in this setting.

Specifically, in this section we assume $P(Q) = a - bQ$, and $h_i(q_i) = c_i$, for each i. Therefore, the modified marginal cost is $\tilde{h}_i(q_i) = c_i + bq_i$, for each i. Under these assumptions, the co-payment allocation problem (CAP) is a convex optimization problem. Therefore, we solve both the uniform co-payments allocation problem (UCAP) and the co-payment allocation problem (CAP), and we compare their objective functions directly. We consider four cases in the number of firms participating in the market, $n \in \{2, 3, 10, 20\}$. For each one of this four cases, we solve 1,000 instances of the problem. These instances are randomly generated, with parameters sampled from the following distributions: a, b are uniformly distributed in $[0, 50]$, c_i are independent and uniformly distributed in $[0, a]$, for each i.

Figure 1 presents a boxplot of the results for the ratio $Q^{\mathrm{U}}/Q^{\mathrm{OPT}}$, while Table 1 presents some summary statistics. It is interesting that the minimum value of the ratio $Q^{\mathrm{U}}/Q^{\mathrm{OPT}}$ never went below 91% in the simulation results. Moreover, the mean and median values are above 98%, for each value of the number of firms participating in the market n. This suggests that, in most cases, the aggregated market consumption induced by uniform co-payments is fairly close to the aggregated market consumption induced by the optimal co-payment allocation.

Price Taking Firms with Non-linear Demand and Marginal Costs Now we consider a more general experimental setup, with non-linear demand, and non-linear marginal costs, where the firms act as price takers. In this setting we assume $P(Q) = a - bQ^{m_0}$, and $h_i(q_i) = c_i + g_i q_i^{m_i}$ for each i.

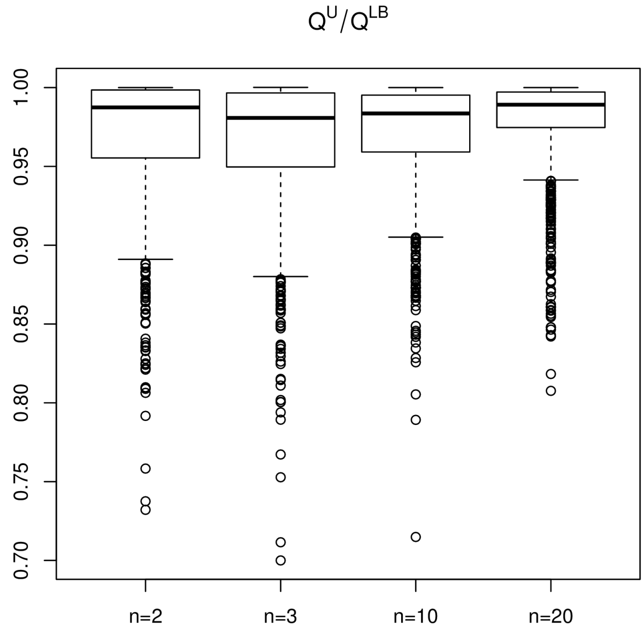

Figure 2 **Relative Performance of Uniform Co-payments - Nonlinear Instance**

$Q^{\mathrm{U}}/Q^{\mathrm{LB}}$	n=2	n=3	n=10	n=20
Min.	0.7321	0.7000	0.7149	0.8076
1st Qu.	0.9554	0.9497	0.9592	0.9747
Median	0.9874	0.9808	0.9836	0.9892
Mean	0.9698	0.9661	0.9710	0.9784
3rd Qu.	0.9985	0.9966	0.9952	0.9972
Max.	1.0000	1.0000	1.0000	1.0000

Table 2 **Relative Performance of Uniform Co-payments - Nonlinear Instance**

Under these assumptions, the co-payment allocation problem (CAP) is a non-convex optimization problem. However, the upper bound problem (UBP) is a convex optimization problem. Therefore, we solve both the uniform co-payments allocation problem (UCAP) and the upper bound problem (UBP), and we compare their objective functions.

We consider again four cases in the number of firms participating in the market, $n \in \{2, 3, 10, 20\}$. For each one of this four cases, we solve 1,000 instances of the problem. These instances are randomly generated, with parameters sampled from the following distributions: a, b are uniformly distributed in $[0, 50]$. For each i, c_i are independent and uniformly distributed in $[0, a]$, g_i are independent and uniformly distributed in $[0, 50]$, and m_i are independent and uniformly distributed in $(0, 20]$. Finally, m_0 is uniformly distributed in $(0, 3]$.

Note that $P(Q) = a - bQ^{m_0}$, $m_0 \in (0, 3]$, captures both convex and concave decreasing inverse demand functions. Similarly, $h_i(q_i) = c_i + g_i q_i^{m_i}$, $m_i \in (0, 20]$ for each i captures both convex and concave marginal cost firms. The results for the ratio $Q^{\mathrm{U}}/Q^{\mathrm{LB}}$ are displayed in Figure 2 and in

Table 2. The minimum value of the ratio $Q^{\mathrm{U}}/Q^{\mathrm{LB}}$ never went below 70% in the simulation results. Moreover, the mean and median values are above 96%, for each value of the number of firms participating in the market n, where in this case we are not comparing directly to the optimal solution, but to an upper bound. This suggests that again, in most cases, the aggregated market consumption induced by uniform co-payments is fairly close to the aggregated market consumption induced by the optimal co-payment allocation.

8. Conclusions

We provide a new modeling framework to analyze the problem of a central planner injecting a budget of subsidies into a competitive market, with the objective of maximizing the aggregated market consumption of a good. This is equivalent to maximizing the consumer surplus. The co-payment allocation policy that is usually implemented in practice is uniform, in the sense that every firm gets the same co-payment. A central question in this paper is how efficient uniform co-payments are compared to the optimal subsidy allocation, assuming that some firms could be significantly more efficient than others.

Using our framework, we show that uniform co-payments are in fact *optimal* for a large a family of marginal cost functions. Moreover, we show that the optimality of uniform co-payments is preserved, under less general conditions, in the case where the central planner is uncertain about the market state. Furthermore, we show that uniform co-payments also maximize the social welfare for a large a family of marginal cost functions. Finally, we study the performance of uniform co-payments in relevant settings where they are not optimal. Our simulation results suggest that the aggregated market consumption induced by uniform co-payments is relatively close to the aggregated market consumption induced by the optimal co-payment allocation. It is an interesting research question to explore whether there exist theoretical bounds on the effectiveness of uniform subsidies in these settings.

In summary, we present interesting evidence that gives theoretical support to the use of uniform co-payments in practice. Therefore, decision makers facing the problem of allocating subsidies to increase the aggregated market consumption of a good, should not spend time and resources developing sophisticated allocation policies, as it is very likely that the very simple uniform subsidy policy will attain most of the potential benefits. Future research on this topic should study whether these insights are preserved in dynamic models, where the subsidy allocation may change over time, or under different market equilibrium conditions, such as supply function equilibria.

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References

Acemoglu, D., A. Ozdaglar. 2007. Competition and efficiency in congested markets. *Mathematics of Operations Research* **32**(1) 1 – 31.

AMFm Task Force, Roll Back Malaria Partnership. 2007. *Affordable Medicines Facility for malaria (AMFm): Technical Design*. World Health Organization, Geneva, Switzerland. URL `http://rbm.who.int`.

Arifoglu, K., S. Deo, S. Iravani. 2012. Consumption externality and yield uncertainty in the influenza vaccine supply chain: Interventions in demand and supply sides. *Management Science* **58**(6) 1072 – 1091.

Arrow, K. J., C. B. Panosian, H. Gelband. 2004. *Saving Lives, Buying Time: Economics of Malaria Drugs in an Age of Resistance*. National Academies Press, Washington, DC.

Bertsekas, D. P. 1999. *Nonlinear programming*. Athena Scientific, Belmont, MA.

Brander, J. A. 1995. Chapter 27 strategic trade policy. Gene M. Grossman, Kenneth Rogoff, eds., *Handbook of International Economics*, vol. 3. Elsevier, 1395 – 1455. doi:10.1016/S1573-4404(05)80007-3. URL `http://www.sciencedirect.com/science/article/pii/S1573440405800073`.

Brito, D. L., E. Sheshinski, M. D. Intriligator. 1991. Externalities and compulsary vaccinations. *Journal of Public Economics* **45**(1) 69 – 90.

Cachon, G. P. 2003. *Handbooks in Operations Research and Management Science: Supply Chain Management*, chap. Supply chain coordination with contracts. North - Holland, Amsterdam, 229 – 339.

Chemama, J., M. Cohen, R. Lobel, G. Perakis. 2013. Designing subsidies with industry response dynamics: Commitment vs. flexibility. *Working Paper* .

Chick, S.E., H. Mamani, D. Simchi Levi. 2008. Supply chain coordination and influenza vaccination. *Operations Research* **56**(6) 1493 – 1506.

Cohen, M., R. Lobel, G. Perakis. 2012. Consumer subsidies with industry response for green technology adoption. *Under review* .

Cohen, M., G. Perakis, C Thraves. 2013. The effects of competition in green technology adoption. *Working Paper* .

Collie, D. 1993. Strategic trade policy under asymmetric oligopoly. *European Journal of Political Economy* **9** 275 – 280.

Correa, J. R., N. Figueroa, R. Lederman, N. E. Stier Moses. 2012. Pricing with markups in industries with increasing marginal costs. *Under Review* .

Deo, S., C. J. Corbett. 2009. Cournot competition under yield uncertainty: The case of the u.s. influenza vaccine market. *Manufacturing and Service Operations Management* **11**(4) 563 – 576.

Fullerton, D., G. E. Metcalf. 2002. Chapter 26 tax incidence. Alan J. Auerbach, Martin Feldstein, eds., *Handbook of Public Economics*, vol. 4. Elsevier, 1787 – 1872. doi:10.1016/S1573-4420(02)80005-2. URL `http://www.sciencedirect.com/science/article/pii/S1573442002800052`.

Grant, M., S. Boyd. 2012. CVX: Matlab software for disciplined convex programming, version 2.0 beta. `http://cvxr.com/cvx`.

Johari, R., G. Y. Weintraub, B. Van Roy. 2010. Investment and market structure in industries with congestion. *Operations Research* **58**(5) 1303 – 1317.

Klemperer, P. D., M. Meyer. 1989. Supply function equilibria in oligopoly under uncertainty. *Econometrica* **57**(6) 1243 – 1277.

Leahy, D., C. Montagna. 2001. Strategic trade policy with heterogeneous costs. *Bulletin of Economic Research* **53**(3) 177 – 182.

Lobel, R., G. Perakis. 2012. Consumer choice model for forecasting demand and designing incentives for solar technology. *Under review* .

Long, N. Van, A. Soubeyran. 1997. Cost heterogeneity, industry concentration and strategic trade policy. *Journal of International Economics* **43** 207 – 220.

Mamani, H., E. Adida, D. Dey. 2012. Vaccine market coordination using subsidy. *IIE Transactions on Healthcare Systems Engineering* **2**(1) 78 – 96.

Marcotte, P., M. Patriksson. 2007. *Handbooks in Operations Research and Management Science, Volume 14, Transportation*, chap. Traffic Equilibrium. Elsevier, Amsterdam, North - Holland, 623 – 713.

Patriksson, M., L. Wynter. 1999. Stochastic mathematical programs with equilibrium constraints. *Operations Research Letters* **25** 159 – 167.

Raz, G., A. Ovchinnikov. 2013. Coordinating pricing and supply of public interest goods using rebates and subsidies. *Under review* .

Taylor, T. A., W. Xiao. 2013. Subsidizing the distribution channel: Donor funding to improve the availability of malaria drugs. *Working Paper* .

Vives, X. 2001. *Oligopoly Pricing: Old Ideas and New Tools*. The MIT press, Cambridge, MA.

White, N. J. 2008. Qinghaosu (artemisinin): the price of success. *Science* **320** 330 – 334.

Appendix. Online Appendix

A. Proofs of Section 5

LEMMA 2. *For any feasible solution to the co-payments allocation problem under market state uncertainty (SCAP), without loss of generality, the scenarios can be renumbered, such that the following inequalities are true:*

$$P^1(Q^1) \geq P^2(Q^2) \geq \ldots \geq P^m(Q^m), \tag{53}$$

$$h_i(q_i^1) \geq h_i(q_i^2) \geq \ldots \geq h_i(q_i^m), \text{ for each } i, \tag{54}$$

$$q_i^1 \geq q_i^2 \geq \ldots \geq q_i^m, \text{ for each } i, \tag{55}$$

$$Q^1 \geq Q^2 \geq \ldots \geq Q^m, \tag{56}$$

$$\sum_{j=1}^n q_j^1 y_j \geq \sum_{j=1}^n q_j^2 y_j \geq \ldots \geq \sum_{j=1}^n q_j^m y_j, \tag{57}$$

where $\sum_{j=1}^n q_j^s y_j$ *is the total amount spent in co-payments in scenario* s.

Proof. Assume, without loss of generality, the first chain of inequalities (53). Using Equation (31), and given that the co-payments y_i are the same for each scenario, we conclude the second set of inequalities (54). From here, $h_i(q_i)$ increasing implies the third set of inequalities (55). Summing over all firms gives us the fourth set of inequalities (56). Finally, given that the co-payments y_i are the same for each scenario, from the third set of inequalities we get,

$$q_i^1 y_i \geq q_i^2 y_i \geq \ldots \geq q_i^M y_i, \text{ for each } i,$$

and summing over all firms gives us the fifth set of inequalities (57).

LEMMA 3. *Under the assumptions of Theorem 2, there exists an optimal solution to problem (SUBP) that attains the minimum of the gaps between the maximum marginal cost in scenario* $s = 1$*, and the minimum marginal cost in scenario* $s = 1$*, induced by any optimal solution.*

Proof. The feasible set of problem (SUBP) is closed and bounded. It is bounded because $q_i^s \in [0, \check{q}_i]$, for each i, s, where $\check{q}_i$ is such that $h_i(\check{q}_i)\check{q}_i = B$. Similarly, $Q^s \in [0, \check{Q}]$, for each s, where $\check{Q} = \max_{i\in\{1,\ldots,n\}}\{\check{q}_i\}$. On the other hand, it is closed because it is defined by inequalities on continuous functions. Additionally, the objective function of problem (SUBP) is continuous. It follows that there exists an optimal solution.

Define the set Γ, as the set of all the optimal solutions to problem (SUBP). The set Γ is closed and bounded. It is bounded because it is a subset of the feasible set, which is bounded. On the other hand, denote by z^* the optimal value of the objective function in problem (SUBP). Then, the set Γ is closed because it is the intersection of the feasible set, which is closed, and the set $\{(\mathbf{q}^s, Q^s)_{s=1,\ldots,m} \,:\, \sum_{s=1}^m Q^s p_s \geq z^*\}$, which is closed because the functions Q^s are continuous.

Define the set $X(\Gamma)$, as the set of all the gaps between the maximum marginal cost in scenario $s = 1$, and the minimum marginal cost in scenario $s = 1$, induced by any optimal solution. Namely,

$$X(\Gamma) \equiv \left\{ x \;\middle/\; \text{There exists } (\mathbf{q}^s, Q^s)_{s=1,\ldots,m} \in \Gamma \quad \text{s.t.} \quad x = \max_{i\in\{1,\ldots,n\}} \{h_i(q_i^1)\} - \min_{i\in\{1,\ldots,n\}} \{h_i(q_i^1)\} \right\}.$$

The set $X(\Gamma)$ is also closed and bounded. Specifically, the maximum and the minimum of continuous functions are continuous, therefore $X(\Gamma)$ is the image of a compact set under a continuous mapping. Hence, $\hat{x} \equiv \min_{x\in X(\Gamma)} x$ is well defined. Namely, the minimum of the gaps between the maximum marginal cost in scenario $s=1$, and the minimum marginal cost in scenario $s=1$, induced by any optimal solution, is attained.

LEMMA 4. *Under the assumptions of Theorem 2, for any feasible solution to problem (SUBP), $(\boldsymbol{q}^s, Q^s)_{s=1,\dots,m}$, if $h_i(q_i^s) > h_j(q_j^s)$ for some i, j, s, then we can transfer a sufficiently small $\epsilon^s > 0$, from q_i^s to q_j^s, without violating any of the constraints (37)-(42) related to scenario s.*

Proof. The modified solution generates the same aggregated market consumption Q^s. Therefore, we only need to check that the budget constraint (37) for scenario s, and the non-negativity of the co-payments (38) for scenario s, are still satisfied.

Specifically, from $h_i(q_i) = h(g_i q_i)$ with $h(x)$ increasing it follows that $h_i(q_i^s) > h_j(q_j^s)$ implies $g_i q_i^s > g_j q_j^s$. Together with $h(x)$ convex, they imply $(h_i(q_i^s)q_i^s)' = h(g_i q_i^s) + g_i q_i^s h'(g_i q_i^s) > h(g_j q_j^s) + g_j q_j^s h'(g_j q_j^s) = \left(h_j(q_j^s)q_j^s\right)'$. It follows that the modified solution has a smaller total cost, $\sum_{j=1}^n h_j(q_j^s)q_j^s$, while generating the same aggregated market consumption Q^s. Hence, it satisfies the scenario s budget constraint (37).

Additionally, $(\mathbf{q}^s, Q^s)_{s=1,\dots,m}$ feasible for problem (SUBP), and constraint (38), imply $h_i(q_i^s) > h_j(q_j^s) \geq P^s(q^s)$. Therefore, $h_i(q_i^s - \epsilon^s) \geq h_j(q_j^s + \epsilon^s) \geq P^s(q^s)$ holds for $\epsilon^s > 0$ sufficiently small. Namely, the modified solution also satisfies the non-negativity of the co-payments (38) related to scenario s.

LEMMA 5. *Under the assumptions of Theorem 2, for any feasible solution to problem (SUBP), $(\boldsymbol{q}^s, Q^s)_{s=1,\dots,m}$, for any $\epsilon^1 > 0$, and for any scenario $s \neq 1$. The following conditions must hold:*

$$\text{If } \epsilon^s \geq 0 \text{ satisfies } \ \epsilon^1 \leq \frac{h'(g_i(q_i^s - \epsilon^s))}{h'(g_i q_i^1)}\epsilon^s, \ \text{ then } \ h_i(q_i^s - \epsilon^s) - P^s(Q^s) \leq h_i(q_i^1 - \epsilon^1) - P^1(Q^1). \tag{58}$$

$$\text{If } \epsilon^s \geq 0 \text{ satisfies } \ \frac{h'(g_i(q_i^s + \epsilon^s))}{h'(g_i q_i^1)}\epsilon^s = \epsilon^1, \ \text{ then } \ h_i(q_i^s + \epsilon^s) - P^s(Q^s) \leq h_i(q_i^1 + \epsilon^1) - P^1(Q^1). \tag{59}$$

Proof. First, from $h_i(q_i) = h(g_i q_i)$, it follows that the left hand side of Equation (58) is equivalent to $h_i'(q_i^1)\epsilon^1 \leq h_i'(q_i^s - \epsilon^s)\epsilon^s$. Moreover, from this inequality and $h(x)$ convex, it follows that $h_i(q_i^1) - h_i(q_i^1 - \epsilon^1) \leq h_i'(q_i^1)\epsilon^1 \leq h_i'(q_i^s - \epsilon^s)\epsilon^s \leq h_i(q_i^s) - h_i(q_i^s - \epsilon^s)$.

Therefore, on the one hand we have $h_i(q_i^1) - h_i(q_i^1 - \epsilon^1) \leq h_i(q_i^s) - h_i(q_i^s - \epsilon^s)$. On the other hand, from constraint (43) it follows that $h_i(q_i^s) - P^s(Q^s) \leq h_i(q_i^1) - P^1(Q^1)$. By adding up these two inequalities we conclude,

$$h_i(q_i^s - \epsilon^s) - P^s(Q^s) \leq h_i(q_i^1 - \epsilon^1) - P^1(Q^1).$$

Second, from $h_i(q_i) = h(g_i q_i)$, it follows that the left hand side of Equation (59) is equivalent to $h_i'(q_i^s + \epsilon^s)\epsilon^s = h_i'(q_i^1)\epsilon^1$. Moreover, from this inequality and $h(x)$ convex, it follows that $h_i(q_i^s + \epsilon^s) - h_i(q_i^s) \leq h_i'(q_i^s + \epsilon^s)\epsilon^s = h_i'(q_i^1)\epsilon^1 \leq h_i(q_i^1 + \epsilon^1) - h_i(q_i^1)$.

Therefore, on the one hand we have $h_i(q_i^s + \epsilon^s) - h_i(q_i^s) \leq h_i(q_i^1 + \epsilon^1) - h_i(q_i^1)$. On the other hand, from constraint (43) it follows that $h_i(q_i^s) - P^s(Q^s) \leq h_i(q_i^1) - P^1(Q^1)$. By adding up these two inequalities we conclude,

$$h_i(q_i^s + \epsilon^s) - P^s(Q^s) \leq h_i(q_i^1 + \epsilon^1) - P^1(Q^1).$$

THEOREM 2. *Assume that the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0, \bar{Q}]$. Assume that the marginal costs functions are given by $h_i(q_i) = h(g_i q_i)$ for each i, for any increasing and continuously differentiable function $h(x)$, such that $h(0) = 0$. If $h(x)$ satisfies Property 2, then there exists an optimal solution to the upper bound problem (SUBP), $(\hat{\mathbf{q}}^s, \hat{Q}^s)_{s=1,\ldots,m}$, such that, $h_i(\hat{q}_i^1) - P^1(\hat{Q}^1) = y^1$ for each $i \in \{1, \ldots, n\}$, for some value $y^1 > 0$.*

Proof. Let $\hat{x}$ be as defined in the proof of Lemma 3. Namely, let $\hat{x}$ be the minimum of the gaps between the maximum marginal cost in scenario $s = 1$, and the minimum marginal cost in scenario $s = 1$, induced by any optimal solution. The statement in the Theorem is equivalent to showing $\hat{x} = 0$.

Assume by contradiction that $\hat{x} > 0$. Moreover, denote the optimal solution that induces $\hat{x}$ by $(\hat{\mathbf{q}}^s, \hat{Q}^s)_{s=1,\ldots,m}$. Let the indexes min and max be such that, $h_{\min}(\hat{q}_{\min}^1) \le h_i(\hat{q}_i^1)$ for each i, and $h_{\max}(\hat{q}_{\max}^1) \ge h_i(\hat{q}_i^1)$ for each i. The assumption $\hat{x} > 0$ is equivalent to $h_{\max}(\hat{q}_{\max}^1) > h_{\min}(\hat{q}_{\min}^1)$. We will show that we can construct an optimal solution $(\tilde{\mathbf{q}}^s, \tilde{Q}^s)_{s=1,\ldots,m}$, such that it induces a strictly smaller gap $\tilde{x} = \max_{i\in\{1,\ldots,n\}} \{h_i(\tilde{q}_i^1)\} - \min_{i\in\{1,\ldots,n\}} \{h_i(\tilde{q}_i^1)\} < \hat{x}$, contradicting the definition of $\hat{x}$.

Specifically, from Lemma 4, it follows that if we transfer an arbitrarily small $\epsilon^1 > 0$, from $\hat{q}_{\max}^1$ to $\hat{q}_{\min}^1$, then all the constraints (37)-(42) related to scenario $s = 1$ are still satisfied. Therefore, this modified solution could only become infeasible due to violating the relaxed non-anticipativity constraints (43). We can avoid this infeasibility as follows. We will show that for an arbitrarily small $\epsilon^1 > 0$, and for each scenario $s \ne 1$, there exists $\epsilon^s \ge 0$ such that,

$$h_{\max}(\hat{q}_{\max}^s - \epsilon^s) - P^s(\hat{Q}^s) \le h_{\max}(\hat{q}_{\max}^1 - \epsilon^1) - P^1(\hat{Q}^1), \tag{60}$$

and

$$h_{\min}(\hat{q}_{\min}^s + \epsilon^s) - P^s(\hat{Q}^s) \le h_{\min}(\hat{q}_{\min}^1 + \epsilon^1) - P^1(\hat{Q}^1). \tag{61}$$

Namely, we will show that we can transfer some $\epsilon^s \ge 0$ from $\hat{q}_{\max}^s$ to $\hat{q}_{\min}^s$, for each scenario $s \ne 1$, such that the modified solution satisfies constraint (43). Additionally, we will show that the modified solution also satisfies constraints (37)-(42), for each scenario $s \ne 1$. Hence, the modified solution is feasible for problem (SUBP). Moreover, it is an optimal solution, and it attains a smaller gap than $\hat{x}$.

From Lemma 5 it follows that, for an arbitrarily small $\epsilon^1 > 0$, and for each scenario $s \ne 1$, it is enough to show that there exists an $\epsilon^s \ge 0$ such that it satisfies the following stronger condition,

$$\frac{h'(g_{\min}(\hat{q}_{\min}^s + \epsilon^s))}{h'(g_{\min}\hat{q}_{\min}^1)}\epsilon^s = \epsilon^1 \le \frac{h'(g_{\max}(\hat{q}_{\max}^s - \epsilon^s))}{h'(g_{\max}\hat{q}_{\max}^1)}\epsilon^s. \tag{62}$$

Specifically, from Equation (58) it follows that the inequality in (62) implies condition (60). Additionally, from Equation (59) it follows that the equality in (62) implies condition (61).

Now we show that for an arbitrarily small $\epsilon^1 > 0$, and for each scenario $s \ne 1$, there exists an $\epsilon^s \ge 0$ such that conditions (60) and (61) are satisfied, and constraints (37)-(42) for scenario s are also satisfied. We do so by considering all possible cases. Specifically, if scenario s is such that $h_{\max}(\hat{q}_{\max}^s) - P^s(\hat{Q}^s) < h_{\max}(\hat{q}_{\max}^1) - P^1(\hat{Q}^1)$, then, for an arbitrarily small $\epsilon^1 > 0$, taking $\epsilon^s = 0$ satisfies conditions (60) and (61), and constraints (37)-(42) for scenario s, and we are done with this case.

It follows that, without lost of generality, we can focus on an scenario s such that $h_{\max}(\hat{q}^s_{\max}) - P^s(Q^s) = h_{\max}(\hat{q}^1_{\max}) - P^1(\hat{Q}^1)$. From constraint (41) it follows that $P^1(Q^1) \geq P^s(Q^s)$. Note that if $P^1(\hat{Q}^1) = P^s(\hat{Q}^s)$, then $h_{\max}(\hat{q}^s_{\max}) = h_{\max}(\hat{q}^1_{\max}) > h_{\min}(\hat{q}^1_{\min}) \geq h_{\min}(\hat{q}^s_{\min})$, where the last inequality follows from Lemma 2. Therefore, the convexity of $h(x)$ implies that taking an arbitrarily small $\epsilon^s = \epsilon^1 > 0$ satisfies conditions (60) and (61). Moreover, Lemma 4 ensures that constraints (37)-(42) for scenario s are also satisfied, and we are done with this case.

Therefore, without lost of generality, assume $P^s(\hat{Q}^s) < P^1(\hat{Q}^1)$. This implies,

$$\frac{h_{\max}(\hat{q}^s_{\max})}{h_{\min}(\hat{q}^s_{\min})} \geq \frac{h_{\max}(\hat{q}^1_{\max}) - \overbrace{\left(P^1(\hat{Q}^1) - P^s(\hat{Q}^s)\right)}^{>0}}{h_{\min}(\hat{q}^1_{\min}) - \left(P^1(\hat{Q}^1) - P^s(\hat{Q}^s)\right)} > \frac{h_{\max}(\hat{q}^1_{\max})}{h_{\min}(\hat{q}^1_{\min})}.$$

The first inequality follows from $h_{\max}(\hat{q}^s_{\max}) - P^s(Q^s) = h_{\max}(\hat{q}^1_{\max}) - P^1(\hat{Q}^1)$ and constraint (43). The second inequality follows from $h_{\max}(\hat{q}^1_{\max}) > h_{\min}(\hat{q}^1_{\min})$. Hence, from $h_i(q_i) = h(g_i q_i)$, the fact that $h(x)$ satisfies Property 2, and the strict inequality above, we conclude that scenario s satisfies,

$$\frac{h'(g_{\max}\hat{q}^s_{\max})}{h'(g_{\max}\hat{q}^1_{\max})} > \frac{h'(g_{\min}\hat{q}^s_{\min})}{h'(g_{\min}\hat{q}^1_{\min})}. \tag{63}$$

From Equation (63) it follows that the stronger condition (62) is satisfied for $\epsilon^s > 0$ sufficiently small. Therefore, conditions (60) and (61) hold, and we are done with this case. This completes the analysis of all possible cases.

To summarize, we have shown that there exist $\epsilon^1 > 0$, and $\epsilon^s \geq 0$ for each scenario $s \neq 1$, such that the modified solution $(\tilde{\mathbf{q}}^s, \tilde{Q}^s)_{s=1,\dots,m}$, defined by,

$$\tilde{q}^s_{\min} = \hat{q}^s_{\min} + \epsilon^s, \text{ for each } s \in \{1,\dots,m\},$$

$$\tilde{q}^s_{\max} = \hat{q}^s_{\max} - \epsilon^s, \text{ for each } s \in \{1,\dots,m\},$$

$$\tilde{q}^s_i = \hat{q}^s_i, \text{ for each } i \notin \{\min,\max\}, s \in \{1,\dots,m\}.$$

is an optimal solution to the upper bound problem (SUBP). Specifically, it is feasible and it attains the same objective value as the optimal solution $(\hat{\mathbf{q}}^s, \hat{Q}^s)_{s=1,\dots,m}$. Moreover, by potentially repeating this argument for the finite number of pair of indexes $i,j \in \{1,\dots,n\}$, we conclude that its gap $\tilde{x} = \max_{i\in\{1,\dots,n\}} \{h_i(\tilde{q}^1_i)\} - \min_{i\in\{1,\dots,n\}} \{h_i(\tilde{q}^1_i)\}$ is strictly smaller than $\hat{x}$. This contradicts the definition of $\hat{x}$.

Hence, we conclude that $\hat{x} = 0$. Therefore, $h_i(\hat{q}^1_i) = h_j(\hat{q}^1_j)$, for each i,j. Or equivalently, $h_i(\hat{q}^1_i) - P^1(\hat{Q}^1) = y^1$ for each $i \in \{1,\dots,n\}$.

Theorem 3. *Assume that the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0,\bar{Q}]$. Assume that the marginal costs functions are given by $h_i(q_i) = h(g_i q_i)$ for each i, for any increasing and continuously differentiable function $h(x)$, such that $h(0) = 0$. If $h(x)$ satisfies Property 2, then allocating the largest feasible uniform co-payment is an optimal solution for the co-payment allocation problem under market state uncertainty (SCAP).*

Proof. We will show that there exists an optimal solution to the upper bound problem (SUBP) that induces uniform co-payments. Moreover, this solution is feasible for the co-payment allocation problem under market state uncertainty (SCAP). Therefore, uniform co-payments are optimal for problem (SCAP).

From Theorem 2 it follows that there exists an optimal solution to the upper bound problem (SUBP) $(\hat{\mathbf{q}}^s, \hat{Q}^s)_{s=1,\ldots,m}$ such that $h_i(\hat{q}_i^1) - P^1(\hat{Q}^1) = y^1$ for each i. We will show first that there exists an optimal solution for problem (SUBP), $(\tilde{\mathbf{q}}^s, \hat{Q}^s)_{s=1,\ldots,m}$, such that $h_i(\tilde{q}_i^s) - P^s(\hat{Q}^s) = y^s$ for each i, for each scenario $s \neq 1$, for some value $y^s > 0$. Then, we will conclude by showing that we must have $y^s = y^1$ for each s.

Plugging in y^1 in the budget constraint for scenario $s = 1$ we obtain $y^1 \leq \dfrac{B}{\hat{Q}^1}$. Moreover, for this solution we can decompose the upper bound problem (SUBP) for each scenario $s \neq 1$, and obtain the following independent problem,

$$
(SLBP - s) \qquad
\begin{aligned}
\min_{\mathbf{q},Q} \quad & Qp_s \\
\text{s.}t. \quad & \sum_{j=1}^{n} q_j h_j(q_j) - P^s(Q)Q \leq B && (64)\\
& h_i(q_i) \geq P^s(Q), \text{ for each } i \in \{1,\ldots,n\} && (65)\\
& \sum_{j=1}^{n} q_j = Q && (66)\\
& q_i \geq 0, \text{ for each } i \in \{1,\ldots,n\} && (67)\\
& P^1(\hat{Q}^1) \geq P^s(Q) && (68)\\
& \hat{Q}^1 \geq Q && (69)\\
& h_i(q_i) - P^s(Q) \leq y^1, \text{ for each } i \in \{1,\ldots,n\}. && (70)
\end{aligned}
$$

It follows that the components of the optimal solution to the upper bound problem (SUBP) corresponding to scenario s, $(\hat{\mathbf{q}}^s, \hat{Q}^s)$, must be an optimal solution for problem (SLBP-s) as well. Note that the budget constraint (64) is redundant for this problem. Specifically, we have,

$$\sum_{i=1}^{n} q_i h_i(q_i) - P^s(Q)Q \leq Qy^1 \leq Q\frac{B}{\hat{Q}^1} \leq B.$$

The first inequality follows from constraint (70), the second inequality follows from $y^1 \leq \frac{B}{\hat{Q}^1}$, and the third inequality follows from constraint (69). Therefore, without loss of generality, we can drop the budget constraint in scenario $s \neq 1$ (64).

Exactly as in Lemma 3, the feasible set of problem (SLBP-s) is closed and bounded, and its objective function is continuous. It follows that there exists an optimal solution. Now we show that there exists an optimal solution for problem (SLBP-s), $(\hat{\mathbf{q}}^s, \hat{Q}^s)$, such that $h_i(\hat{q}_i^s) - P^s(\hat{Q}^s) = y^s$, for each i, for some value $y^s > 0$. Specifically, assume by contradiction that this is not the case. It follows that there must exist indexes min and max such that $h_{\min}(\hat{q}^s_{\min}) \leq h_i(\hat{q}_i^s)$ for each i, $h_{\max}(\hat{q}^s_{\max}) \geq h_i(\hat{q}_i^s)$ for each i, and $h_{\min}(\hat{q}^s_{\min}) < h_{\max}(\hat{q}^s_{\max})$. On the other hand, let $\tilde{\mathbf{q}}^s$ be the optimal solution to the following optimization problem.

$$
\begin{aligned}
\min_{\mathbf{q}} \quad & \sum_{j=1}^{n} q_j h_j(q_j) \\
\text{s.}t. \quad & \sum_{j=1}^{n} q_j = \hat{Q}^s \\
& q_i \geq 0, \text{ for each } i \in \{1,\ldots,n\}
\end{aligned}
$$

We show that $(\tilde{\mathbf{q}}^s, \hat{Q}^s)$ is feasible for problem (SLBP-s). Because budget constraint (64) is redundant, and the aggregated market consumption $\hat{Q}^s$ is fixed, it follows that we only need to check that constraints (65) and (70) are satisfied. From $h_i(q_i) = h(g_i q_i)$, and $h(x)$ convex and increasing, it follows that the objective function of this problem is convex. The first order conditions are $(h_i(\tilde{q}_i^s)\tilde{q}_i^s)' = (h_j(\tilde{q}_j^s)\tilde{q}_j^s)'$ for each i, j. Moreover, because $h(x)$ satisfy Property 2, we conclude $h_i(\tilde{q}_i^s) = h_j(\tilde{q}_j^s)$ for each i, j.

Additionally, we claim that $h_{\min}(\hat{q}^s_{\min}) < h_i(\tilde{q}_i^s) < h_{\max}(\hat{q}^s_{\max})$ for each i. In fact, if $h_{\max}(\hat{q}^s_{\max}) > h_{\min}(\hat{q}^s_{\min}) \geq h_i(\tilde{q}_i^s)$, for each i, then we must have, $\sum_{j=1}^n \tilde{q}_j^s < \sum_{j=1}^n \hat{q}_j^s = \hat{Q}^s$. This is a contradiction to the feasibility of solution $(\tilde{\mathbf{q}}^s, \hat{Q}^s)$. Similarly, if $h_i(\tilde{q}_i^s) \geq h_{\max}(\hat{q}^s_{\max}) > h_{\min}(\hat{q}^s_{\min})$, for each i, then we must have, $\sum_{j=1}^n \tilde{q}_j^s > \sum_{j=1}^n \hat{q}_j^s = \hat{Q}^s$. This is a contradiction to the feasibility of solution $(\tilde{\mathbf{q}}^s, \hat{Q}^s)$. This implies, together with the feasibility of $(\hat{\mathbf{q}}^s, \hat{Q}^s)$ for problem (SLBP-s), that,

$$h_i(\tilde{q}_i^s) > h_{\min}(\hat{q}^s_{\min}) \geq P^s(\hat{Q}^s), \text{ for each } i,$$

and,

$$h_i(\tilde{q}_i^s) - P^s(\hat{Q}^s) < h_{\max}(\hat{q}^s_{\max}) - P^s(\hat{Q}^s) \leq y^1, \text{ for each } i.$$

Namely, constraints (65) and (70) are satisfied. Therefore, $(\tilde{\mathbf{q}}^s, \hat{Q}^s)$ is feasible for problem (SLBP-s). Moreover, it attains the same objective value than $(\hat{\mathbf{q}}^s, \hat{Q}^s)$, therefore it is also optimal. Finally, from $h_i(\tilde{q}_i^s) = h_j(\tilde{q}_j^s)$ for each i, j, it follows that $h_i(\tilde{q}_i^s) - P^s(\hat{Q}^s) = y^s$ for each $i \in \{1, \ldots, n\}$ for some value $y^s > 0$.

Finally, we show that we must have $y^s = y^1$ for each scenario s. From $h_i(q_i) = h(g_i q_i)$, it follows that, for any given value of $y^s \geq 0$, $\hat{Q}^s$ is uniquely determined by the solution of the equation,

$$\hat{Q}^s(y^s) = \sum_{i=1}^n \frac{h^{-1}\left(P^s(\hat{Q}^s(y^s)) + y^s\right)}{g_i}.$$

It follows that, $\hat{Q}^s(y^s)$ is increasing in y^s. Assume by contradiction that $y^s < y^1$, then we can increase y^s by $\epsilon > 0$ sufficiently small, and obtain a strictly better objective value while keeping feasibility. In fact, the only constraint that might prevent this increase is the budget constraint (64), which is not tight. This contradicts the optimality of $(\tilde{\mathbf{q}}^s, \hat{Q}^s)$.

We have shown that there exists an optimal solution to the upper bound problem (SUBP) $(\tilde{\mathbf{q}}^s, \hat{Q}^s)_{s=1,\ldots,m}$ such that, $h_i(\hat{q}_i^s) - P^s(\hat{Q}^s) = y^1$ for each $i \in \{1, \ldots, n\}$, and for each $s \in \{1, \ldots, m\}$, for some value $y^1 \geq 0$. That is, it satisfies the relaxed non-anticipativity constraints with equality. Therefore, it is feasible in the co-payment allocation problem under market state uncertainty (SCAP). Hence, uniform co-payments are optimal for problem (SCAP).