Internet Appendix on Decentralized Market Solution for

THE ECONOMIC AND POLICY CONSEQUENCES OF CATASTROPHES

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This Appendix derives the decentralized market equilibrium for our model, and shows that it is identical to the solution to the social planner's problem. First, we find the representative consumer's optimal consumption, portfolio choice and CIS demand. Second, we turn to firm value maximization taking prices as given. Finally, we conjecture and verify equilibrium prices and resource allocation.

Consumer Optimality. Let X denote the consumer's total marketable wealth and π the fraction allocated to the market portfolio. For catastrophe with recovery fraction in $(Z, Z+dZ)$, $\xi_t(Z)X_t dt$ gives the total demand for the CIS over time period $(t, t+dt)$. The total CIS premium payment in the time interval $(t, t+dt)$ is then $\left(\int_0^1 \xi_t(Z)p(Z)dZ\right) X_t dt$.

We conjecture that the cum-dividend return of the market portfolio is given by

$$\frac{dQ_t + D_t dt}{Q_{t-}} = \mu dt + \sigma dW_t - (1-Z)dJ_t\,, \tag{1}$$

where μ is the expected return on the market portfolio (including dividends) but without the effects of catastrophic risk (and will be determined in equilibrium). When a catastrophe occurs, the consumer's wealth changes from X_{t-} to X_t as follows:

$$X_t = X_{t-} - (1-Z)\pi_{t-}X_{t-} + \xi_{t-}(Z)X_{t-}\,. \tag{2}$$

The consumer's wealth accumulation is then given by

$$\begin{aligned} dX_t \;&=\; r\left(1-\pi_{t-}\right)X_{t-}dt + \mu\pi_{t-}X_{t-}dt + \sigma\pi_{t-}X_{t-}dW_t - C_{t-}dt \\ &\quad - \left(\int_0^1 \xi_{t-}(Z)p(Z)dZ\right)X_{t-}dt + \xi_{t-}(Z)X_{t-}dJ_t - (1-Z)\pi_{t-}X_{t-}dJ_t\,. \end{aligned} \tag{3}$$

The first four terms in (3) are standard in classic portfolio choice problems (with no insurance or catastrophes). The last three terms capture the effects of catastrophes on wealth accumulation. The fifth term is the total CIS premium paid before any catastrophe. The sixth term gives the CIS payments by the seller to the buyer when a catastrophe occurs. The last term is the loss of consumer wealth from exposure to the market portfolio.

The HJB equation for the consumer in the decentralized market setting is given by

$$\begin{aligned} 0 \;=\; \max_{C,\pi,\xi(\,\cdot\,)} &\left\{ f(C,J) + \left[rX(1-\pi) + \mu\pi X - \left(\int_0^1 \xi(Z)p(Z)dZ\right)X - C\right] J'(X) \right. \\ &\left. + \frac{1}{2}\sigma^2\pi^2X^2J''(X) + \lambda\mathcal{E}\left[J(X-(1-Z)\pi X + \xi(Z)X) - J(X)\right]\right\}\,. \end{aligned} \tag{4}$$

The FOCs for consumption C, market portfolio allocation as a fraction π of total wealth X, and the CIS demand $\xi(Z)$ for each Z are respectively:

$$
\begin{align}
f_C(C,J) &= J'(X) \tag{5}\\
(\mu - r)XJ'(X) &= -\sigma^2\pi X^2 J''(X) + \lambda\mathcal{E}\left[(1-Z)J'(X-(1-Z)\pi X + \xi(Z)X)\right] \tag{6}\\
0 &= -Xp(Z)J'(X) + \lambda X\left[J'(X-(1-Z)\pi X+\xi(Z)X)\right] f_Z(Z)\,. \tag{7}
\end{align}
$$

The last FOC follows from the point-by-point optimization in (4) for the CIS demand and hence it holds for all levels of Z. Now conjecture that the consumer's value function is

$$
J(X) = \frac{1}{1-\gamma}(uX)^{1-\gamma}\,, \tag{8}
$$

where u is a constant to be determined. Using the consumption FOC (5) and the conjectured value function (8), we obtain the following linear consumption rule:

$$
C = \rho^{\psi} u^{1-\psi} X\,. \tag{9}
$$

Imposing the equilibrium outcome in which (1) $\pi = 1$; (2) $\xi(Z) = 0$ for all Z; and (3) the consumer's wealth equals the total value of the market portfolio, $X = Q$, we obtain:

$$
\begin{align}
0 &= (\mu - r)J'(Q) + \sigma^2 Q J''(Q) - \lambda\mathcal{E}\left[(1-Z)J'(ZQ)\right] \tag{10}\\
p(Z) &= \lambda\frac{J'(ZQ)}{J'(Q)} f_Z(Z) \tag{11}
\end{align}
$$

Using these equilibrium conditions, we can simplify the HJB equation as follows:

$$
\begin{align}
0 \;=\; & \frac{\rho}{1-\psi^{-1}}\left[\left(\frac{\rho}{u}\right)^{\psi-1} - 1\right] u^{1-\gamma}X^{1-\gamma} + \left(\mu - \rho^{\psi}u^{1-\psi}\right)(uX)^{1-\gamma} - \frac{\gamma}{2}\sigma^2 (uX)^{1-\gamma} \nonumber\\
& + \lambda\mathcal{E}\left[Z^{1-\gamma} - 1\right]\frac{1}{1-\gamma}(uX)^{1-\gamma} \tag{12}
\end{align}
$$

Eqn. (9) implies $c = \rho^{\psi}u^{1-\psi}q$ under the equilibrium condition $X = Q = qK$. Substituting $c = \rho^{\psi}u^{1-\psi}q$ into (12), we obtain

$$
0 = \frac{1}{1-\psi^{-1}}\left(\frac{c}{q} - \rho\right) + \left(\mu - \frac{c}{q}\right) - \frac{\gamma}{2}\sigma^2 + \lambda\mathcal{E}\left[Z^{1-\gamma} - 1\right]\frac{1}{1-\gamma} \tag{13}
$$

Firm Value Maximization. We assume financial markets are perfectly competitive and M-M holds. While the firm can hold financial positions (e.g., CIS contracts), equilibrium pricing implies that there is no value in doing so. We can thus ignore financial contracts and only focus on investment I when maximizing firm value, which is independent of financing. Taking the unique stochastic discount factor (SDF) implied by the equilibrium consumption process as given, the firm maximizes its value by choosing I to solve:

$$
\max_{I}\ \mathcal{E}\left[\int_0^{\infty} \frac{M_s}{M_0}\left(AK_s - I_s\right)ds\right], \tag{14}
$$

subject to capital accumulation, the production technology, and the transversality condition.

Using the homogeneity property of our model, we conjecture that the SDF is given by a geometric Brownian motion with constant drift, constant volatility and proportional jump for each possible recovery fraction Z, i.e.

$$dM_t = -rM_{t-}dt - \eta M_{t-}dW_t + M_{t-}\left[\left(Z^{-\gamma}-1\right)dJ_t - \lambda\mathcal{E}\left(Z^{-\gamma}-1\right)dt\right]. \tag{15}$$

The second and the third terms capture diffusion and catastrophic risk respectively. Both terms are martingales. Note that to make the catastrophe term a martingale, we must subtract the expected change of M due to all possible catastrophes. Finally, the first term gives the equilibrium drift of M, which must be $-rM_t$ from the no-arbitrage condition.

No arbitrage implies the drift of $M_t(AK_t - I_t)dt + d\left(M_tQ_t\right)$ is zero. From Ito's Lemma we have the following dynamics for $Q(K)$:

$$dQ(K) = \left(\Phi(I,K)Q_K + \frac{1}{2}Q_{KK}\sigma^2K^2\right)dt + \sigma KQ_KdW_t + \left(Q(ZK) - Q(K)\right)dJ_t\,. \tag{16}$$

Again using Ito's Lemma, we have

$$\begin{aligned} & M_{t-}(AK-I)dt + M_{t-}\left(Q_K\Phi(I,K)dt + \frac{1}{2}\sigma^2K^2Q_{KK}dt\right) + Q\left[-r - \lambda\mathcal{E}\left(Z^{-\gamma}-1\right)\right]M_{t-}dt \\ & -\eta M_{t-}\sigma KQ_Kdt + \lambda\mathcal{E}\left(Z^{-\gamma}Q(ZK) - Q(K)\right)M_{t-}dt = 0\,. \end{aligned} \tag{17}$$

Simplifying the above, we have

$$\begin{aligned} \left[r + \lambda\mathcal{E}\left(Z^{-\gamma}-1\right)\right]Q(K) &= (AK-I) + Q_K\left(\Phi(I,K) - \eta\sigma K\right) + \frac{1}{2}\sigma^2K^2Q_{KK} \\ & \quad + \lambda\mathcal{E}\left(Z^{-\gamma}Q(ZK) - Q(K)\right)\,. \end{aligned} \tag{18}$$

The FOC with respect to investment is therefore

$$1 = \Phi_I(I,K)Q_K\,. \tag{19}$$

Using the homogeneity assumption, we conjecture that firm value is $Q(K) = qK$, where Tobin's q is to be determined. We can thus simplify (18) as follows:

$$\left[r + \lambda\mathcal{E}\left(Z^{-\gamma}-1\right)\right]q = (A-i) + q\left(\phi(i) - \eta\sigma\right) + \lambda\mathcal{E}\left(Z^{1-\gamma}-1\right)q\,. \tag{20}$$

The equilibrium dynamic for firm value Q_t is then given by

$$dQ_t = gQ_{t-}dt + \sigma Q_{t-}dW_t - (1-Z)\,Q_{t-}dJ_t\,. \tag{21}$$

where $g = \phi(i)$ is the expected growth without the effects of catastrophes.

The FOC (19) can be simplified as follows:

$$q = \frac{1}{\phi'(i)}\,. \tag{22}$$

Market Equilibrium. We now verify that the conjectured prices and quantities are consistent with equilibrium market outcomes, and replicate equations (17)-(21) in the text. First, eqn. (17) in the text follows immediately from the goods market clearing condition, $Y = C + I$, and the homogeneity property. Second, eqn. (18) is the FOC for the producer under homogeneity. Third, we obtain eqn. (19) for consumption by comparing the dynamics for firm value on the consumer and firm sides, (1) and (21), to obtain the restriction:

$$\mu = \phi(i) + \frac{c}{q}\,. \tag{23}$$

The expected rate of return (without catastrophes) is $\phi(i)$ plus the dividend yield, which is also the consumption-wealth ratio. Substituting (23) into (13) gives eqn. (19) in the text.

Fourth, using the equilibrium consumption and evaluating the SDF via eqn. (A3) in Appendix A, we obtain the equilibrium interest rate r given by eqn. (20) in the text, and the equilibrium market price of diffusion risk $\eta = \gamma\sigma$.

Fifth, simplifying (10), we have the following result:

$$0 \;=\; (\mu - r) - \gamma\sigma^2 - \lambda\mathcal{E}\left[Z^{-\gamma}(1-Z)\right]. \tag{24}$$

Adding the expected loss due to the catastrophic risk, we obtain the following formula for the equity risk premium rp:

$$rp = \mu + \lambda\mathcal{E}(1-Z) - r = \gamma\sigma^2 + \lambda\mathcal{E}\left[(1-Z)\left(Z^{-\gamma}-1\right)\right], \tag{25}$$

which is eqn. (21) in the text. Finally, substituting (8) into (11) gives the CIS insurance premium $p(Z)$ of eqn. (32) in the text. We have verified that the conjectured equilibrium is indeed consistent with the social planner's solution.