The Volatility of International Capital Flows and Foreign Assets *

Winston Wei Dou and Adrien Verdelhan

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Abstract

This paper presents a two-good, two-country real model that replicates the basic stylized facts on equity excess returns and real interest rates. In the model, markets are incomplete. In each country, workers cannot participate in financial markets whereas investors trade domestic and foreign stocks, as well as an international bond. The investors' asset positions are subject to a borrowing constraint, along with a short-selling constraint on equity. Foreign and domestic agents differ in their elasticity of inter temporal substitution and in their risk-aversion. A time-varying probability of a global disaster implies time-varying risk premia in asset markets, and therefore large and time-varying expected valuation effects on international asset positions. The model highlights the role of market incompleteness and heterogeneity across countries in accounting for the volatility of equity and debt international capital flows.

Keywords: Capital Flows, Equity Premium, Exchange Rates, Home Equity Bias, Risk Sharing, Wealth-recursive Markov Equilibria, Incomplete Markets.

JEL: F31, G12, G15.

*Preliminary and Incomplete. Dou: MIT Sloan. Address: MIT Sloan School of Management, Department of Finance, E62-615, 100 Main Street, Cambridge, MA 02142; wdou@mit.edu. Verdelhan: MIT Sloan and NBER. Address: MIT Sloan School of Management, Department of Finance, E62-621, 100 Main Street, Cambridge, MA 02142; adrienv@mit.edu; Phone: (617) 253-5123; http://web.mit.edu/adrienv/www/. The authors thank Dave Backus, Hui Chen, Emmanuel Farhi, Leonid Kogan, Matteo Maggiori, Stan Zin, and Jiang Wang, as well as seminar participants at different institutions and conferences for helpful comments and suggestions.

1 Introduction

After decades of financial liberalization, foreign assets represent now a large fraction of aggregate wealth. For the U.S., the gross foreign equity and bond holdings amount to 83% of GDP in 2010 (Lane and Milesi-Ferretti, 2007, updated). Foreign holdings are volatile because their unit value changes, through valuation effects, and their quantities changes, through international capital flows. During the recent Great Recession for example, the value of the net U.S. foreign equity and bond holdings decreased by 51%, while at the same time, international capital flows dried up. From the perspective of the benchmark models in international economics, such large valuation changes and such volatile capital flows are puzzling. In this paper, we propose a two-good, two-country model that is consistent with the basic stylized facts in equity and interest rate markets. With a model consistent with asset prices in hand, we turn to the macroeconomic quantities: we use the model to assess the volatility of international capital flows and foreign assets.

Our model has four main characteristics: a rich endowment process, general recursive preferences with heterogenous agents, limited market participation, and short-selling and borrowing constraints.

The total endowment process has a global and a country-specific component. Both components are described by Markov processes. The growth rate of the global component is subject to disaster risk: with a small, time-varying probability, the world growth rate may fall. The country-specific endowment is persistent, but only subject to Gaussian risk. The total endowment is levered and divided into a labor income stream and a dividend stream. The leverage is also time-varying: as in the data, in bad times, leverage is large (Longstaff and Piazzesi, 2004). With these features and risk-averse agents, the model delivers large and time-varying risk premia in line with the empirical evidence on equity and bond markets.

The agents are characterized by Epstein and Zin (1989) preferences, which disentangle risk-aversion from the inter-temporal elasticity of substitution. The domestic (i.e. U.S.) agent is less risk-averse than her foreign (i.e. rest-of-the-world, denoted ROW) counterpart, but has a higher inter-temporal elasticity of substitution. The differences across agents lead to large gross foreign asset positions. As in the data, the U.S. tends to borrow from the ROW and invests in the foreign stock market, therefore providing insurance to the ROW. International trade is frictionless and each agent consumes both domestic and foreign goods.

In each country, some agents participate in international financial markets, while others do not. The workers, who do not participate, consume all of their labor income each period. The investors, who do participate, choose optimally the quantity of domestic and foreign stocks as well as their net borrowing or lending positions. Their investment decisions are subject to two constraints: they cannot short stocks and their borrowing is limited by the amount they can reimburse the next period in the worst state of the world. These constraints rule out defaults and ensure that the equilibrium solution of the model is stationary even if agents with Epstein and Zin (1989) preferences differ in their risk-aversion and inter-temporal elasticity of substitution. These constraints would not be necessary if agents would share the same preference parameters or if agents were characterized by constant relative risk-aversion preferences, but they are necessary in our model to obtain a stationary equilibrium.

In the model, markets are incomplete, even for the agents who participate in financial markets. There are five different endowment shocks (the global Gaussian growth rate, the global disaster state, the disaster probability shock, and two country-specific endowment shocks), but there are only three assets traded (two stocks and one bond). Moreover, borrowing and short-selling constraints sometimes, but not always, bind. Market incompleteness is a key feature of our model. While investors can choose optimally their portfolio positions to mitigate the impact of market incompleteness, workers can not work around their participation constraint.

Such a rich model has never been simulated before. Building on the results of Kubler and Schmedders (2003) and Duffie, Geanakoplos, Mas-Colell and McLennan (1994), we show that the model has a wealth-recursive Markov solution. The proof extends previous results on heterogenous agent models to the case of Epstein and Zin (1989) preferences and stochastic growth. Knowing that a wealth-recursive Markov solution exists, the model is simulated at the quarterly frequency. Our solution method relies on three ingredients: a time-shift, as proposed by Dumas and Lyasoff (2012), a wealth-recursive equilibrium, and a finite-period approximation of the infinite-horizon problem. The simulated moments are then compared to their empirical counterparts. The data sample focuses on the U.S. and an aggregate of the other G10 countries to build the ROW. The sample period is 1973.IV–2010.IV.

In the simulation, the model matches the characteristics of the U.S. and ROW GDP and aggregate consumption, as well the equity and risk-free bond returns. The endowment process matches the mean, standard deviation, and autocorrelation of the growth rates and H.P-filtered series of U.S. GDP, as well as its cross-country correlation with the ROW GDP. The model produces equity excess returns that are large and volatile in both countries. Equity excess returns are also predictable, using the price-dividend ratio and the wealth-consumption ratio, as in the data. The mean and the volatility of the risk-free rates are also in line with their empirical counterpart. The exchange rate is slightly less volatile than in the data, but the average return on the currency carry trade is in line with the data. The exchange rate change exhibits a low, negative correlation with relative consumption growth. The next exports, as a fraction of GDP, however, is less volatile in the model than in the data.

The model is used to assess the magnitude and volatility of international capital flows and foreign holdings. The model features not only unexpected valuation changes but also expected returns on foreign investments; the model can thus shed light on the current debate on the size of expected valuation effects and their importance in assessing the sustainability of the U.S. current account.

In the simulation, the U.S. invests in ROW equity and the ROW invests in U.S. equity. But the magnitudes of these gross positions differ: the U.S. holds more foreign equity assets than foreign equity liabilities. The reverse is true for bonds, and the U.S. in a net borrower. Overall the U.S. borrows from the ROW and invests in the ROW equity. The U.S. gross equity positions are even more volatile in the model than in the data, reflecting both the expected and unexpected valuation shocks. The bonds positions, to the contrary, are not volatile, in line with their empirical counterpart. The changes in expected excess returns lead to changes in optimal portfolio holdings and thus international capital flows. In our calibration, the gross equity flows are more volatile than in the data, although the volatility of the net equity positions is close to the one in the data. In comparison, the net debt flows are as smooth in the model as in the data.

The model thus highlights the key role of expected returns, i.e. expected valuation changes, in the volatility of international capital flows. The volatility of expected and unexpected equity returns seems to account, to a first order, for the volatility of international capital positions and flows in the data.

A study of the volatility of equity and bond assets and flows requires four features: (i) the markets must be incomplete such that equity and bond gross asset positions and flows can be defined separately in a meaningful way; (ii) portfolio holdings must be time-varying such that capital flows exist; (iii) expected returns must be large and time-varying for the model to be consistent with the prices of the underlying assets; and (iv) the model must be solved globally. A very large literature studies international holdings and capital flows, but few papers satisfy the four conditions above. Let us rapidly review the most relevant strands of the literature.

A large literature studies the equity home bias — a statement about the puzzlingly low amount of international diversification in the data compared to the one implied by standard neoclassical models. Important contributions include Baxter and Jermann (1997), Lewis (1999), Coeurdacier (2009), Nieuwerburgh and Veldkamp (2009), Coeurdacier and Gourinchas (2011) and Heathcote and Perri (2013). This literature is too large to be summarized here — the database Scopus returns more than 230 published articles over the last 25 years with the expressions "home bias" and "international" in the title or abstract; we refer the reader to the recent and excellent survey proposed by Coeurdacier and Rey (2013). Few papers in this literature feature large and time-varying risk premia: exceptions are Stathopoulos (2012), who considers habit preferences, and Benigno and Nisticò (2012), who introduce model uncertainty and long run consumption risk. Colacito, Croce, Ho and Howard (2014) study international capital flows in a production economy in the spirit of Backus, Kehoe and Kydland (1992).

Another large literature studies the sustainability of the current account imbalances and the size of potential valuation effects on foreign holdings. In a seminal paper, Gourinchas and Rey (2007) find a higher return on US external assets than on its external liabilities. Curcuru, Dvorak and Warnock (2010) offer alternative estimates. Ahmed, Curcuru, Warnock and Zlate (2015) describe the different components of international portfolio flows. Important contributions on the current account imbalances include Kraay and Ventura (2000), Ventura (2001), Caballero, Farhi and Gourinchas (2008), and Devereux and Sutherland (2010).

Finally, a recent literature studies the impact of market incompleteness on the capital flows and exchange rate puzzles, notably the Backus and Smith (1993) puzzle and the forward premium puzzle. The Backus and Smith (1993) puzzle refers to the perfect correlation between exchange rate changes and relative consumption in a complete market model with CRRA preferences. In the data, the correlation is small and negative. The forward premium puzzle refers to the deviations from the uncovered interest rate parity and the large currency carry trade excess returns (Tryon, 1979, Fama, 1984). Notable contributions in this literature include the work by Alvarez, Atkeson and Kehoe (2002), Chari, Kehoe and McGrattan (2002), Bacchetta and Wincoop (2006), Corsetti, Dedola and Leduc (2008), Alvarez, Atkeson and Kehoe (2009), Pavlova and Rigobon (2010), Pavlova and Rigobon (2012), Bruno and Shin (2014), Maggiori (2015), and Favilukis, Garlappi and Neamati (2015). Solving optimal portfolio problems in incomplete markets is challenging. Earlier solutions in the context of closed economies with specific preferences (e.g., log utility) or endowment processes include Dumas (1989), Wang (1996), Cochrane, Longstaff and Santa-Clara (2008), Longstaff and Wang (2012), and Martin (2013). Our model, existence theorem, and solution method can be used in the context of closed economies with heterogenous agents.

Recent attempts have been made to improve the solution method. Devereux and Sutherland (2011) and Tille and van Wincoop (2010) propose a second-order approximation method, subsequently used in several papers. In a key contribution, Rabitsch, Stepanchuk, and Tsyrennikov (2015), however, show that this solution method is inaccurate in the presence of heteroscedasticity and nonlinearities, which are key features of our model. Our solution method therefore is global and does not require any second-order approximation. Evans and Hnatkovska (2005) suggest a different approximation based on a constant wealth ratio, which is not applicable in our case.

The papers closer to ours are Gourinchas, Rey and Govillot (2010), Stepanchuk and Tsyrennikov (2015), Dumas, Lewis and Osambela (2014), Maggiori (2015), and Chien, Lustig and Naknoi (2015): the first two consider differences in risk-aversion across countries when markets are, respectively, complete or incomplete; the third one studies differences of opinion in complete markets; the last two papers feature incomplete markets to study respectively the impact of differences in financial development or the Backus and Smith puzzle (1993) puzzle. These authors only consider constant risk premia. Our work builds on these papers to deliver an incomplete market model with time-varying risk premia. The time-variation in expected return is key, as changes in expected returns translate into changes in optimal portfolio holdings and therefore capital flows.

The paper is organized as follows. Section 2 rapidly review the features of U.S. international capital flows and current account. Section 3 describes the model. Section 4 proves the existence of a wealth-recursive equilibrium. Section 5 presents the calibration of the model. Section 6 describes the simulation results of the benchmark calibration, with a particular focus on the comparison between international capital stocks and flows in the model and in the data. Section 7 concludes. A separate Appendix, available on our websites, details the proofs of our theoretical results, presents additional empirical results, describes the simulation method, and reports additional simulation results.[1]

2 Key Facts on U.S. International Capital Flows and Current Accounts

In this section, we review key facts on the U.S. current account and net foreign assets and then turn to the U.S. international capital flows.

[1]The separate Appendix is available at: http://web.mit.edu/adrienv/www/Research.html.

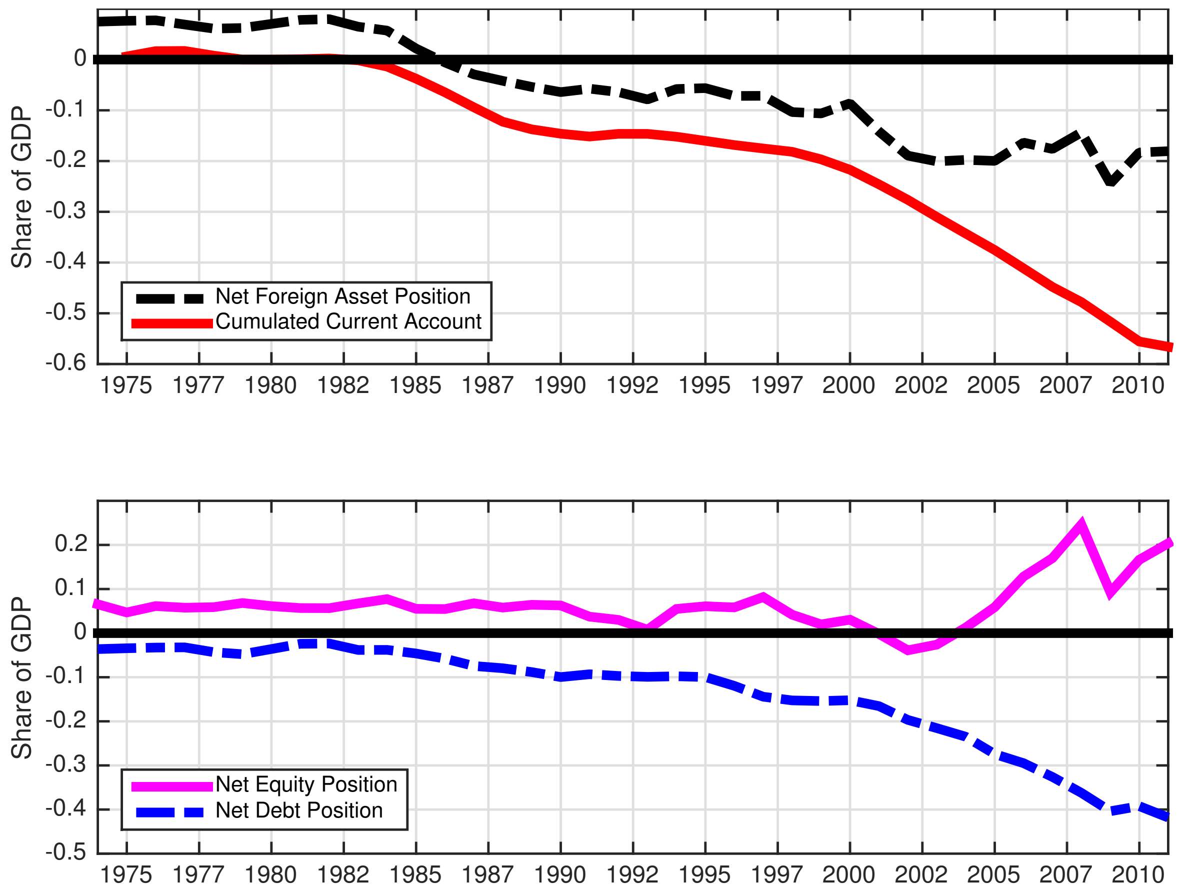

Figure 1: Cumulated Current Account, Net Foreign Assets, and Leverage

The upper panel of the figure presents the net foreign asset position and the sum of past current accounts (both scaled by U.S. GDP). The bottom panel presents the net equity and net debt U.S. positions (both scaled by U.S. GDP). All ”equity” stocks correspond to the sum of equity, foreign direct investment, and other investments. Net all ”equity” assets correspond to the difference between all “equity” assets and liabilities. Net debt assets correspond to the difference between debt portfolio assets and liabilities. Data are annual, from an updated and extended version of the Lane and Milesi-Ferretti (2007). The sample is 1973–2010.

2.1 Current Accounts and Net Foreign Assets

The current account is the sum of the trade balance (exports minus imports), the net dividend payments, and the net interest payments. In all but one of the last thirty years, the U.S. current account has been consistently negative, mostly because the U.S. imports more than it exports. As shown in the upper panel of Figure 1, the sum of the past cumulated current accounts is now close to 60% of GDP.

This alarming level contrasts with the net foreign asset position of the U.S. Consistent with a stream of negative current accounts, the net foreign asset position of the U.S. declined, reaching -20% of the U.S. GDP at the end of the sample. There is considerable uncertainty in the measure of the net foreign asset position. Yet, it appears much smaller than the cumulated past current accounts. As Gourinchas and Rey (2007, 2010, 2013) argue, this discrepancy suggests large valuation effects: the U.S. receives on average larger returns on their assets than they pay on their liabilities. While there is some uncertainty in the magnitude of the returns and their difference, it appears likely that the difference in returns at least partly compensates the deficit in the current account.

In this view, the sustainability of the current account relies on the ability of the U.S. to pocket large returns on its foreign investments. Such large returns in the past may have been unexpected and thus pure luck, or expected and thus reflecting differences in risk premia. As Gourinchas and Rey (2013) note and the bottom panel of Figure 1 illustrates, a difference in expected returns between U.S. assets and liabilities is consistent with the broad asset allocation of the country, since the U.S. is short domestic debt and long foreign equity. The U.S. may thus receive large expected returns on its levered equity investments, as a compensation for their risk, while paying low returns on its debt.

2.2 Equity and Bond Flows

The levered position of the U.S. economy has clear implications for the dynamics of its net foreign assets. In theory, the foreign asset positions can change either because their unit values change, a pure valuation effect, or because their quantities change, as a result of capital reallocation and thus international capital flows. In practice, a statistical gap exists between the changes in foreign assets on the one hand and the sum of the valuation effects and capital flows. Even after taking into account this statistical gap, a clear difference emerges between the dynamics of the U.S. foreign assets and liabilities.

Using the quarterly datasets of Bertaut and Tryon (2007) and Bertaut and Judson (2014), Figure 2 reports the changes in U.S. equity and bond assets and liabilities over the last twenty years. Three key results appear: (i) the volatility in foreign equity holdings is mostly due to valuation changes, not net capital flows; (ii) the volatility of foreign equity assets is much larger than the volatility of U.S equity liabilities; (iii) but the volatility of bond liabilities is mostly due to net capital flows, not valuation changes. The last recession illustrates these patterns vividly: the value of foreign equity held by U.S. investors plummeted, and so did the value of the foreign equity holdings in the U.S. But the magnitudes are different: in the worst quarter of the crisis, the foreign investors lost \$600 billions in U.S. equity wealth, while the U.S. investors lost \$1 trillion in foreign equity wealth, amounting to a wealth transfer of \$400 billions from the U.S. to the ROW in just one quarter. By comparison, bond values remain relatively stable. These patterns are intuitive — stocks tend to be more volatile than bonds — but they highlight the key difficulties in modeling international capital assets flows: the volatilities of holdings and flows are

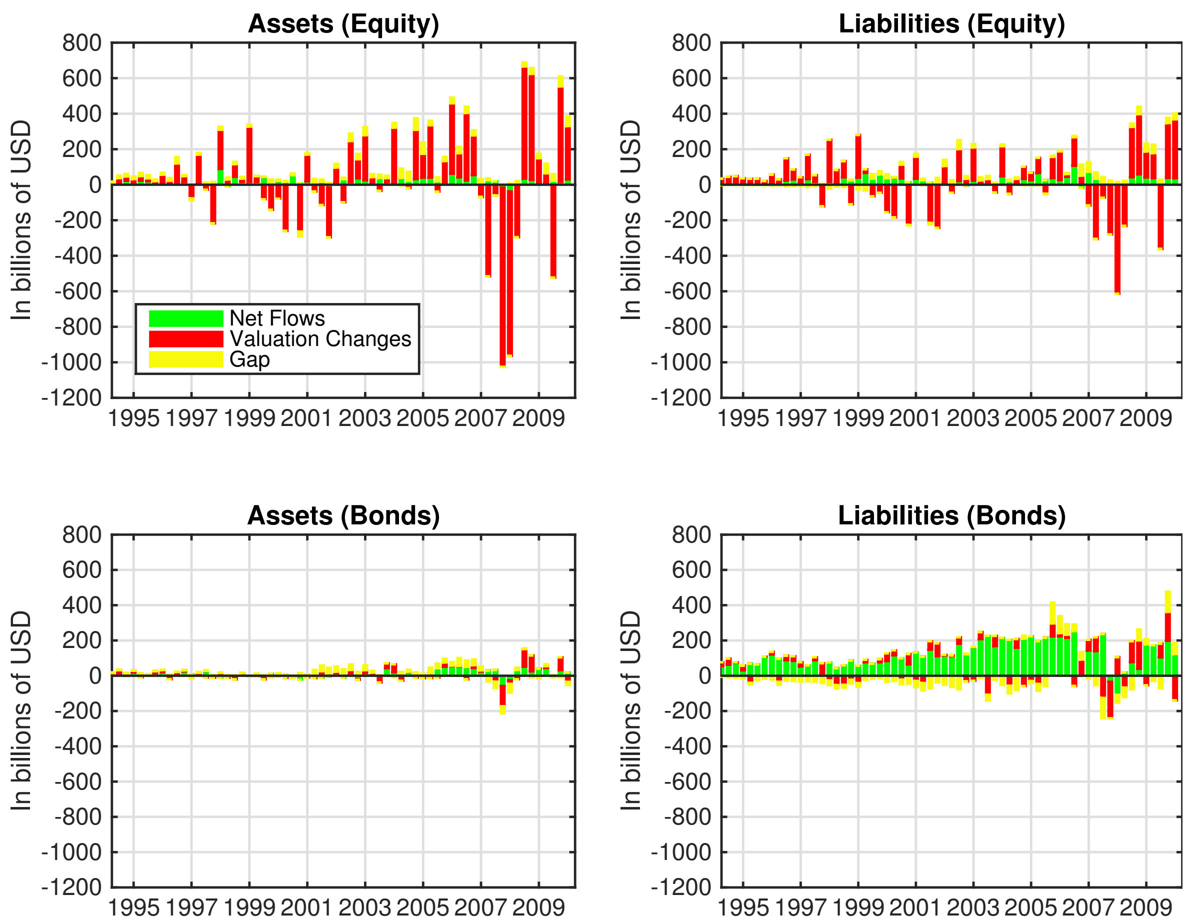

Figure 2: Changes in U.S. Foreign Assets and Liabilities: Capital Flows vs Valuation Effects

The figure presents the changes in U.S. equity and bond foreign assets and liabilities. The changes in holdings are decomposed into three components: net capital flows, valuation changes, and statistical gaps. Data are quarterly, from the Bertaut and Tryon (2007) and Bertaut and Judson (2014) datasets. The sample is 1995.I–2010.IV

country- and asset-specific.

We turn now to a model that can potentially assess the volatility of equity and bond holdings and flows. The model features both expected and unexpected valuation changes, as well as portfolio rebalancing.

3 Model

In this section, we describe the model, starting with the endowment processes and the preferences, before turning to the market frictions.

3.1 Endowments

The model features two endowment economies. In each country, the endowment has a world and a country-specific component.

World Endowment The world endowment, denoted e_t, is described by a Lucas tree whose stochastic growth follows a time-homogeneous Markov process. In the absence of disasters, the growth rate of the global component is g_t, which takes values in a discrete set $\mathcal{S}_g$ and is governed by a Markov transition matrix Π_g. But growth switches from "normal" times, denoted $\xi_t = 0$, to "disaster" times, denoted $\xi_t = -1$, with some probability p_t. The disaster probability p_t follows a homogeneous Markov process with values in $\mathcal{S}_p$ and transition matrix Π_p. Once the economy is in its disaster state, it remains there the next period with probability p_d. The global endowment growth is thus:

$$\log \frac{e_{t+1}}{e_t} = g_{t+1} + \varphi_d \xi_{t+1},$$

where φ_d denotes the size of the world disaster. Three state variables therefore describe the world endowment: the growth rate in normal times, g_t, the occurrence of a disaster, ξ_t, and the probability of a disaster, p_t.

Country-specific Endowments The country-specific endowments, $e_{i,t}$, follow independent time-homogeneous Markov processes, denoted $a_{1,t}$ and $a_{2,t}$. Both take values in the set $\mathcal{S}_a$ and share the same transition matrix Π_a. Thus, the exogenous state of the economy is summarized by $s_t = (a_{1,t}, a_{2,t}, p_t, g_t, \xi_t)$. The total endowment in each country is:

$$\log e_{i,t} = \log e_t + a_{i,t}, \quad \text{for } i = 1, 2,$$

and the log endowment growth of country i is equal to:

$$\log \frac{e_{i,t+1}}{e_{i,t}} = \underbrace{[g_{t+1} + \varphi_d \xi_{t+1}]}_{\text{Global Component}} + \underbrace{\Delta a_{i,t+1}.}_{\text{Country-specific Component}}$$

Note that the model features permanent shocks to the level of endowments.[2] This feature is key as Alvarez and Jermann (2005) and Hansen and Scheinkman (2014) show, in a preference-free setting, that permanent shocks account for most of the variance of the pricing kernel. Lustig, Stathopoulos and Verdelhan (2015), however, find that bond markets behave as if exchange rates are mostly driven by temporary components as if the permanent components were similar across countries. Our model features both a global permanent and two transitory components in the endowments. In other words, our economy is a Lucas-type economy with stochastic growth: the economy fluctuates around the stochastic trend governed by the world endowment e_t, whose sample path is driven by permanent shocks.

3.2 Preferences

In each country, there are two groups of agents: workers and investors. Both groups of agents in both countries maximize their utility over consumption. The utility function is recursive, following Kreps and Porteus (1978) and Epstein and Zin (1989). It is defined over a final consumption good that aggregates, with a constant elasticity of substitution (CES), the domestic and foreign goods. The value function of an agent in country i takes the following recursive form:

$$V_{i,t} = \left\{ C_{i,t}^{\frac{1-\gamma_i}{\theta_i}} + \beta \left[\mathbb{E}_t V_{i,t+1}^{1-\gamma_i} \right]^{\frac{1}{\theta_i}} \right\}^{\frac{\theta_i}{1-\gamma_i}}, \tag{1}$$

$$\text{where } C_{1,t} = \left[s \left(c_{1,t}^1 \right)^{\rho} + (1-s) \left(c_{1,t}^2 \right)^{\rho} \right]^{1/\rho} \text{ and } C_{2,t} = \left[(1-s) \left(c_{2,t}^1 \right)^{\rho} + s \left(c_{2,t}^2 \right)^{\rho} \right]^{1/\rho}. \tag{2}$$

The time discount factor is β, the risk aversion parameter is $\gamma_i \geq 0$, and the inter-temporal elasticity of substitution (EIS) is $\psi_i \geq 0$. The parameter θ_i is defined by $\theta_i \equiv (1-\gamma_i)/(1-\frac{1}{\psi_i})$. The consumption home bias parameter s is between 0.5 and 1, and the elasticity of substitution between the domestic and foreign goods is $\epsilon = 1/[1-\rho]$. The aggregate consumption of an agent in country 1 is denoted $C_{1,t}$: it includes the consumption of goods produced in country 1, denoted $c_{1,t}^1$, as well as the consumption of goods produced in country 2, denoted $c_{1,t}^2$. More generally, $c_{i,t}^j$ denotes the consumption of good j by agent i at time t.

[2] In many Markov economies used to study portfolio choices, such as Judd, Kubler and Schmedders (2003), Kubler and Schmedders (2003), and Stepanchuk and Tsyrennikov (2015), endowments, dividends and labor income depend on the current exogenous shock alone, i.e. $e^i : \mathcal{S} \to \mathbb{R}_{++}$ is a time-invariant function. In our model, because the shocks to the world component e_t are permanent, the endowments, dividends and labor income depend on both the current shock and the world component e_t. Heaton and Lucas (1996) and Brumm, Grill, Kubler and Schmedders (2013) also present models with stochastic growth and permanent shocks to study their asset pricing implications.

The CES consumption aggregators immediately imply the following price indices:

$$P_{1,t} = \left[s^{\epsilon} p_{1,t}^{1-\epsilon} + (1-s)^{\epsilon} p_{2,t}^{1-\epsilon}\right]^{1/(1-\epsilon)} \text{ and } P_{2,t} = \left[(1-s)^{\epsilon} p_{1,t}^{1-\epsilon} + s^{\epsilon} p_{2,t}^{1-\epsilon}\right]^{1/(1-\epsilon)}, \tag{3}$$

where $p_{1,t}$ and $p_{2,t}$ are the prices for goods produced by country 1 and country 2 respectively.[3] We normalize the price system assuming that:

$$p_{1,t} + p_{2,t} = 1.$$

Our calibration assumes a preference for an early resolution of uncertainty: for each agent $i \in \{1,2\}$, the EIS and risk-aversion parameters are above one ($\psi_i > 1$, $\gamma_i > 1$, and $\theta_i < 0$ for $i = 1,2$). After transformation, $U_i \equiv \frac{V_i^{1-\psi_i^{-1}}}{1-\psi_i^{-1}}$, the utility function can be re-written as:

$$U_{i,t} = \frac{C_{i,t}^{1-\psi_i^{-1}}}{1-\psi_i^{-1}} + \beta \mathbb{E}_t \left[U_{i,t+1}^{\theta_i}\right]^{1/\theta_i}.$$

As the notation above suggests, we assume that countries differ in their EIS and risk-aversion preference parameters: $\psi_1 > \psi_2$ and $\gamma_1 < \gamma_2$. Cross-country differences in risk-aversion are key in Gourinchas et al. (2010): in their model, the relatively less risk-averse U.S. agent insures the ROW agent by taking a levered position in ROW equity. The risky position of the U.S. accounts for the difference between the returns on its assets and liabilities. Differences in EIS have received some recent empirical support. Vissing-Jorgensen (2002) shows that the values of the EIS are larger for the U.S. households with larger financial positions; a similar reasoning at the aggregate level would suggest that the U.S. may have a higher EIS than the ROW. Likewise, Havranek, Horvath, Irsova and Rusnak (2013) find that households in richer countries and countries with higher asset market participation have higher values of EIS. Differences in preference parameters are also shortcuts for differences in financial sectors' sizes and skills as modeled in Mendoza, Quadrini and Rios-Rull (2009) and inMaggiori (2015).

[3] The terms of trade is $q \equiv p_2/p_1$, and hence the real exchange rate is:

$$Q \equiv \frac{P_2}{P_1} = \left[\frac{(1-s)^{\epsilon} q^{1-\epsilon} + s^{\epsilon}}{s^{\epsilon} q^{1-\epsilon} + (1-s)^{\epsilon}}\right]^{1/(1-\epsilon)}.$$

3.3 Limited Market Participation

Both workers and investors are characterized by the same preferences, but workers are hand-to-mouth, i.e. they do not have access to financial markets and consume their labor income every period, whereas investors participate in financial markets.

Financial Income Investors trade three assets: one stock in each country, as well as an international bond. The stocks are long-term assets, while the bond is one-period. The net supply of each stock is one, while the net supply of the bond is zero.

The international bond, bought at price q_t^b at date t, is a claim on e_{t+1} units of a composite good, which is a bundle of α goods from country 1 and $1-\alpha$ goods from country 2, with $\alpha = 1/2$. The price of the composite good at date $t+1$ is equal to: $p_{\alpha,t+1} = \alpha p_{1,t+1} + (1-\alpha) p_{2,t+1}$. We model only one instead of two bonds for computational reasons: an equilibrium with two bonds is more difficult to determine. Note that adding a second bond would not be enough for the markets to be complete, and our simplification thus appears innocuous.

In each country, a stock is a claim to a stream of dividends $d_{i,t} > 0$ measured in units of good i. Stocks are traded at the ex-dividend prices, denoted $q_{1,t}$ and $q_{2,t}$. The dividends are leveraged payoffs of endowments:

$$d_{i,t} = e_t \left[\bar{d} + s_\xi \left(\exp(\varphi_d \xi_t) - 1 \right) + s_g \left(\exp(g_t) - 1 \right) + s_a \left(\exp(a_{i,t}) - 1 \right) \right].$$

The leverage is time-varying, as in Longstaff and Piazzesi (2004). As a result, the dividend growth rate is not perfectly correlated to the endowment growth rate.

Labor Income Labor income in country i, denoted $\omega_{i,t}$, is the fraction of the total endowment not distributed as dividends:

$$\omega_{i,t} = e_t \left[1 - \bar{d} - s_\xi \left(\exp(\varphi_d \xi_t) - 1 \right) - s_g \left(\exp(g_t) - 1 \right) - s_a \left(\exp(a_{i,t}) - 1 \right) \right].$$

In the model, since leverage is time-varying, the income share is also time-varying. Unlike the dividend cash flow that can be traded by buying and selling long-lived equities, the future labor income cash flow cannot be traded: potential reasons include financial frictions, capital income taxation, or poor enforcement of property rights. Workers thus face a hard constraint: they cannot participate in financial markets and cannot work around this constraint.

Since workers are hand-to-mouth, their consumption can be easily obtained. Let I_I denote the share of labor income received by investors in each country. The workers in country i receive a total income of $(1-I_I)\omega_{i,t}$ in terms of their domestic goods. Their budget constraint implies that $(1-I_I)\omega_{i,t}p_{i,t} = P_{i,t}C_{w,i,t}$, and their consumption levels are:

$$
\begin{aligned}
c^1_{w,1,t} &= s^{\epsilon}\left[\frac{p_{1,t}}{P_{1,t}}\right]^{-\epsilon}\frac{(1-I_I)\omega_1 p_{1,t}}{P_{1,t}}, \quad \text{and} \quad c^2_{w,1,t} = (1-s)^{\epsilon}\left[\frac{p_2}{P_{1,t}}\right]^{-\epsilon}\frac{(1-I_I)\omega_1 p_{1,t}}{P_{1,t}}, \\
c^1_{w,2,t} &= (1-s)^{\epsilon}\left[\frac{p_{1,t}}{P_{2,t}}\right]^{-\epsilon}\frac{(1-I_I)\omega_{2,t} p_{2,t}}{P_{2,t}}, \quad \text{and} \quad c^2_{w,2,t} = s^{\epsilon}\left[\frac{p_{2,t}}{P_{2,t}}\right]^{-\epsilon}\frac{(1-I_I)\omega_{2,t} p_{2,t}}{P_{2,t}},
\end{aligned}
$$

where, again, $c^j_{i,t}$ denotes the consumption of good j by agent i at time t. The investors' optimal consumption solves a more complicated optimal portfolio problem.

3.4 Borrowing and Short-Selling Constraints

In the model, investors face two specific constraints: (i) they cannot short equity and (ii) their borrowing ability is limited.

The short-selling constraint on equity positions and the presence of labor income together imply that some risk cannot be hedged. This plays a crucial role in determining the portfolio position of the agents since the perfect conditional correlation between non-tradable income and dividends gives investors an incentive to short their own equity. Let $\vartheta^j_{i,t}$ denote the holding of stock j by agent i at date t: the subscript characterizes the country holder and the superscript characterizes the goods in which the asset is denominated. Formally, the short-selling constraint is:

$$
\vartheta^j_{i,t} \geq 0, \quad \text{for } i,j = 1,2. \tag{4}
$$

The borrowing constraint is such that debt can always be repaid since the amount due is always above or equal to the financial wealth of the borrower in the worst state of the world next period:

$$
\begin{aligned}
b_{i,t} \quad &\geq \quad -B_{i,t}, \text{ for } i,j=1,2, &(5)\\
\text{where } B_{1,t} \quad &\equiv \quad \min_{s^{t+1}\succcurlyeq s^t}\left\{w_{1,t+1}\frac{p_{1,t+1}}{p_{\alpha,t+1}}+\sum_{j=1}^{2}\vartheta^j_{1,t}\frac{q_{j,+1}+p_{j,t+1}d_{j,t+1}}{p_{\alpha,t+1}}\right\}, &(6)\\
B_{2,t} \quad &\equiv \quad \min_{s^{t+1}\succcurlyeq s^t}\left\{w_{2,t+1}\frac{p_{2,t+1}}{p_{\alpha,t+1}}+\sum_{j=1}^{2}\vartheta^j_{2,t}\frac{q_{j,t+1}+p_{j,t+1}d_{j,t+1}}{p_{\alpha,t+1}}\right\}, &(7)
\end{aligned}
$$

where the minimum is taken on all possible states the next period: the symbol $\succcurlyeq$ denotes the partial order on the tree S such that node $s^{t_1} \succcurlyeq s^{t_2}$ if s^{t_1} is a descendant of s^{t_2}. The right hand side of Equations (6) and (7) describe the lowest possible sum of labor income and equity wealth for investors in countries 1 and 2 respectively next period. Labor income and equity wealth are thus collateral, securing international debt. Bonds cannot be used as collateral as there is a unique bond in the model: if one country lends, the other must borrow. As a result, the country that borrows has no bond to post as collateral. The borrowing constraint remains potentially binding even in the long run because investors cannot become rich enough to forget it: the non-participation of workers to financial markets prevents investors from lending money to workers, accumulating wealth up to the point when the borrowing constraints are no longer relevant.

The short-selling and borrowing constraints are key: they rule out defaults and address the survivorship or degenerated stationary distribution issue highlighted in Lucas and Stokey (1984) and Anderson (2005). In our model, despite the heterogeneity in agents' preferences, both agents survive in the long run because the collateral and short-sale constraints prohibit them from assuming more and more debt over time. The consumption of investors satisfy the following budget constraint:

$$
\begin{aligned}
\sum_{j=1}^{2}p_{j,t}c^j_{i,t} \quad &+ \quad \sum_{j=1}^{2}q_{j,t}\vartheta^j_{i,t}+q^b_t b_{i,t}\\
&= \quad p_{i,t}\omega_{i,t}+\sum_{j=1}^{2}\left[q_{j,t}+p_{j,t}d_{j,t}\right]\vartheta^j_{i,t-1}+p_{\alpha,t}b_{i,t-1}. &(8)
\end{aligned}
$$

In the next section, we define the competitive equilibrium in the model and prove that a wealth-recursive equilibrium exists. This proof is not purely formal: as pointed out by Kubler and Polemarchakis (2004), the approximate equilibria obtained by numerical methods may exist even when no exact equilibrium exists. The following section guarantees that the wealth-recursive Markov equilibrium exists. The reader mostly interested by the simulation results can skip this section.

4 Equilibrium

Before characterizing the equilibrium, we formulate the country's optimization Bellman equation into a compact and manageable form.

4.1 Time-Shift

We appeal to the "time shift" proposed by Dumas and Lyasoff (2012). We translate the combined borrowing constraints in Equations (5), (6), and (7) into a group of separate constraints as follows, for each date t and $t+1$:

$$\mathbb{C}_1(t,t+1) \equiv p_{1,t+1}\omega_{2,t+1} + \sum_{j=1}^{2} \vartheta^j_{1,t}\left[q_{j,t+1} + p_{j,t+1}d_{j,t+1}\right] + b_{1,t}p_{\alpha,1,t+1} \geq 0,$$

$$\mathbb{C}_2(t,t+1) \equiv p_{2,t+1}\omega_{2,t+1} + \sum_{j=1}^{2} \vartheta^j_{2,t}\left[q_{j,t+1} + p_{j,t+1}d_{j,t+1}\right] + b_{2,2}p_{\alpha,2,t+1} \geq 0.$$

The Lagrangian multiplier for each of the $|\mathcal{S}|$ borrowing constraints is $\mu^b_{i,t,s_{t+1}}$. The $|\mathcal{S}|$ Lagrangian multipliers are endogenous variables in period t. Likewise, each short-selling constraint is associated with a multiplier $\mu^j_{i,t}$. The recursive form of the value function leads to the following Bellman equation with Lagrangian multipliers, for every $t \geq 0$:

$$\begin{aligned} U_i(W_{i,t}; s^t) &= \\ \min_{\mu^j_{i,t}\geq 0,\ \mu^b_{i,t,s_{t+1}}\geq 0} \ \max_{c^j_{i,t},\vartheta^j_{i,t},b_{i,t}} \ \frac{C_{i,t}^{1-\psi_i^{-1}}}{1-\psi_i^{-1}} &+ \beta\mathbb{E}_t\left[U_i(W_{i,t+1}; s^{t+1})^{\theta_i}\right]^{1/\theta_i} + \sum_{j=1}^{2}\mu^j_{i,t}\vartheta^j_{i,t} + \sum_{s_{t+1}\in\mathcal{S}}\mu^b_{i,t,s_{t+1}}\mathbb{C}_i(t,t+1), \end{aligned}$$

subject to the inter-temporal budget constraints:

$$\begin{aligned} W_{i,t} &= p_{1,t}c^1_{i,t} + p_{2,t}c^2_{i,t} + \vartheta^1_{i,t}q_{1,t} + \vartheta^2_{i,t}q_{2,t} + b_{i,t}q_{\alpha,t}, \\ \text{and } W_{i,t+1} &= p_{i,t+1}\omega_{i,t+1}I_I + \sum_{j=1}^{2}\vartheta^j_{i,t}\left(q_{j,t+1} + p_{j,t+1}d_{j,t+1}\right) + b_{i,t}p_{\alpha,t+1}. \end{aligned}$$

4.2 Definitions

Let us now define formally the competitive equilibrium.

Definition 1. *A competitive equilibrium with initial asset holdings $\{\vartheta_i(s^{-1}), b_i(s^{-1})\}_{i=1,2}$ and initial shock s_0 is a* *collection of prices* $\mathcal{P}^{\mathbb{S}} = \left\{ \left(p_i(s^t), q_i(s^t), q^b(s^t) \right)_{i=1,2} \right\}_{s^t \in \mathbb{S}}$, *consumption allocations* $\mathcal{C}^{\mathbb{S}} = \left\{ \left(c_i^1(s^t), c_i^2(s^t) \right)_{i=1,2} \right\}_{s^t \in \mathbb{S}}$, *and asset holdings* $\mathcal{A}^{\mathbb{S}} = \left\{ \left(\vartheta_i^1(s^t), \vartheta_i^2(s^t), b_i(s^t) \right)_{i=1,2} \right\}_{s^t \in \mathbb{S}}$ *such that*

(i) given the price system $\mathcal{P}^{\mathbb{S}}$*, each investor in country* $i \in \{1,2\}$ *solves the optimization problem* $U_i(\mathcal{C}_i^{\mathbb{S}})$ *with the consumption plan* $\mathcal{C}_i^{\mathbb{S}}$ *and the asset holdings* $\mathcal{A}_i^{\mathbb{S}}$ *lying in the sequential budget set* $\mathbb{B}_{\mathbb{S}}\left(\mathcal{P}^{\mathbb{S}}\right)$ *described in Equation (8) under the short-selling constraint described in Equation (4) and the borrowing constraints described in Equations (5), (6), and (7);*

(ii) given the same price system $\mathcal{P}^{\mathbb{S}}$*, each worker in country* $i \in \{1,2\}$ *maximizes her utility under her budget constraint;*

(iii) equity markets and bond markets clear, i.e. for $j = 1,\ 2$ *and for all dates* t*:*

$$\begin{aligned} \vartheta_{1,t}^j + \vartheta_{2,t}^j &= 1, \\ b_{1,t} + b_{2,t} &= 0. \end{aligned}$$

(iv) goods markets clear, i.e. for $j = 1,\ 2$ *and for all dates* t

$$c_{1,t}^j + c_{2,t}^j = e_{j,t}.$$

The borrowing constraints in the agents' optimization problem not only constitute a market imperfection but also ensure the existence of a solution to the agents' optimization problem (see e.g., Levine and Zame, 1996; Magill and Quinzii, 1996; and Hernandez and Santos, 1996). Although the proof of the existence of a competitive equilibrium in Lucas-type infinite-horizon exchange economies with heterogeneous agents and incomplete markets exists, it is impossible to compute the equilibrium in general because it is not unique and the equilibria are mathematically equivalent to an infinite number of equilibrium prices – a infinite dimensional problem. Duffie et al. (1994) show that if the exogenous shocks' dynamics can be characterized by a finite-valued time-homogeneous Markov process, then there exists a competitive equilibrium in which the endogenous variables can be summarized by a finite number of endogenous state variables as well as the exogenous state variables. The endogenous state variables follow a time-homogeneous Markov process having a time invariant transition with an ergodic measure. This type of equilibrium is called recursive Markov equilibria. A recursive Markov equilibrium in which the wealth distribution summarizes all the endogenous state variables is called a wealth-recursive Markov equilibrium. Duffie et al. (1994) show that a recursive Markov equilibrium is a competitive equilibrium under general regularity conditions. Under mild regularity conditions, Kubler and Schmedders (2003) in their Lemma 2 show that a wealth-recursive Markov equilibrium is a competitive equilibrium. Their proof does not apply to our model, but we show how to extend their result. In order to do so, let us first rigorously define the wealth-recursive Markov equilibrium.

Because we have two heterogeneous representative investors in the economy, the wealth portion of the agent 1 fully characterizes the wealth distribution. The wealth share of country 1 is denoted w:

$$w_t \equiv \frac{W_{1,t}}{W_{1,t} + W_{2,t}},$$

where the total wealth in the economy is $W_{1,t} + W_{2,t} = \sum_{j=1}^{2} \left[p_{j,t} e_{j,t} + q_{j,t} \right]$. Let $\mathcal{Y}$ denote the space of all possible endogenous variables that occur in the economy at some node s^t. That is, $\mathcal{Y}$ consists of all vectors:

$$\left\{ \left(c_i^1, c_i^2 \right)_{i=1,2}, \left(\vartheta_i^1, \vartheta_i^2, b_i \right)_{i=1,2}, \left(p_i, q_i, q_i^b \right)_{i=1,2}, \left(\mu_i^1, \mu_i^2, \mu_{i,\tilde{s}}^b \right)_{i=1,2; \tilde{s} \in \mathcal{S}} \right\} \tag{9}$$

such that, for $i, j \in \{1, 2\}$:

$$c_i^j, p_j, q_j, q^b, \mu_i^j, \mu_{i,\tilde{s}}^b \in \overline{\mathbb{R}}_+, \quad \text{and} \quad \vartheta_i^j, b_i^j \in \mathbb{R}_+,$$
$$p_1 + p_2 = 1, \quad \text{and} \quad \vartheta_i^j \mu_i^j = 0, \quad \text{and} \quad \vartheta_1^j + \vartheta_2^j = 1, \quad \text{and} \quad b_1 + b_2 = 0.$$

The Lagrangian multiplier μ_i^j corresponds to the short-selling constraint of the agent in the country i on the stock j, for $i, j \in \{1, 2\}$, while the Lagrangian multiplier $\mu_{i,\tilde{s}}^b$ corresponds to agent i's borrowing constraint. The space of endogenous variables $\mathcal{Z}$ is a closed subset of $\overline{\mathbb{R}}^{2 \times (11 + |\mathcal{S}|)}$. The space of both exogenous and endogenous variables is $\mathcal{Z} \equiv \mathcal{Y} \times \mathcal{S}$. Let $\widehat{\mathcal{Z}} \equiv [0, 1] \times \mathcal{Y} \times \mathcal{S} \times \overline{\mathbb{R}}_+$.

The expectation correspondence maps the variables $\widehat{z} \in \widehat{\mathcal{Z}}$ in the current period to a subset of the space of endogenous variables in next period $([0, 1] \times \mathcal{Y})^{|\mathcal{S}|}$, where $([0, 1] \times \mathcal{Y})^{|\mathcal{S}|}$ is the Cartesian product of $|\mathcal{S}|$ copies of $[0, 1] \times \mathcal{Y}$. More precisely, the expectation correspondence is denoted by

$$\Phi \; : \; \widehat{\mathcal{Z}} \rightrightarrows ([0, 1] \times \mathcal{Y})^{|\mathcal{S}|},$$

such that for a given state in current period $\widehat{z} \equiv (w, y, s, e) \in \widehat{\mathcal{Z}}$, the country 1's wealth share $\{ w(\tilde{s}) \; : \; \tilde{s} \in \mathcal{S} \}$

of next period and the vector of endogenous variables $\{\tilde{y}(\tilde{s}) \, : \, \tilde{s} \in \mathcal{S}\}$ in the next period lies in the set $\Phi(\widehat{z})$ if and only if they are consistent with the inter-temporal budget constraints, the first-order conditions and market clearing conditions.

Definition 2. *A wealth-recursive Markov equilibrium consists of a (nonempty valued) "policy correspondence" $\Pi : [0,1] \times \mathcal{S} \times \overline{\mathbb{R}}_+ \rightrightarrows \mathcal{Y}$, where $\mathcal{Y}$ is the space of endogenous policy variables defined in (9) - (10) and a "transition map" $\Omega : [0,1] \times \mathcal{S} \to [0,1]^{|\mathcal{S}|}$ such that for any given $(w,s,e) \in [0,1] \times \mathcal{S} \times \overline{\mathbb{R}}_+$ with $(\tilde{w}(\tilde{s}))_{\tilde{s}\in\mathcal{S}} = \Omega(w,s)$, it holds that $\forall\, y \in \Pi(w,s,e)$ and $\forall\, \tilde{y}(\tilde{s}) \in \Pi(\tilde{w}(\tilde{s}),\tilde{s},\tilde{e}))$ with $\tilde{e} \equiv e \times \zeta(\tilde{s})$ and $\tilde{s} \in \mathcal{S}$,*

$$(\tilde{w}(\tilde{s}), \tilde{y}(\tilde{s}))_{\tilde{s}\in\mathcal{S}} \in \Phi(w,y,s,e).$$

For notational simplicity, we denote $\tilde{w}(\tilde{s}) = \Omega(w,s;\tilde{s})$.

We now turn to our main theorem.

4.3 Existence of a Wealth-Recursive Markov Equilibrium

Theorem 1. *Assuming that there exists $d_m > 0$ and $\omega_m > 0$ such that $d_i(s_t)/e(s_t) > d_m$ and $\omega_i(s_t)/e(s_t) > \omega_m$ for all $i = 1,2$ and $s_t \in \mathcal{S}$, there exists a wealth-recursive Markov equilibrium in the economy with heterogenous agents with recursive utility described in Section 3.*

Proof. The assumption guarantees that the dividend and wage incomes, as percentages of world GDP, are bounded from below. The proof of the theorem is reported in Appendix C. It consists of three main steps. First, we show that for any T-truncated economy, the competitive equilibrium's policy functions are uniformly bounded if a competitive equilibrium exists.[4] In this step, we generalize the results of Kubler and Schmedders (2003) and Duffie et al. (1994) to allow stochastic growth in the economy, lower-bounded utility functions and Epstein-Zin-Weil preferences. Second, we show the existence of competitive equilibrium for each T-truncated economy. Third, we show the existence of wealth-recursive Markov equilibrium exists for the infinite-horizon economy by backward induction. $\square$

Theorem 1 extends the results of Kubler and Schmedders (2003) to a large class of preferences and to stochastic growth. Duffie et al. (1994) and Kubler and Schmedders (2003) crucially assume that the utility

[4] The T-truncated economy is defined to be a finite-horizon economy built on an event tree, denoted by $\mathbb{S}^T$, which consists of all the nodes and edges along the path $s^T = (s_0, s_1, \cdots, s_T)$ in the original event tree $\mathbb{S}$. The endowments and asset payoffs at the nodes of the truncated tree, as well as agents' preferences and portfolio constraints at these nodes, are the identical to the original infinite-horizon economy.

is not bounded from below, which guarantees that the equilibrium variables are all uniformly bounded. They focus on the time-separable CRRA utility function whose coefficient of relative risk aversion is not smaller than one. However, for the Epstein-Zin-Weil preferences with an EIS parameter bigger than one, the utility function is not bounded from below, and thus their arguments do not go through. We use the results inGeanakoplos and Zame (2013), who show the existence of a competitive equilibrium for a two-period incomplete-market model, and combine them with the proofs in Kubler and Schmedders (2003) in order to extend their results.

The wealth recursive formulation of the agent's optimization problem makes it natural to consider wealth-recursive Markov equilibrium of the economy. The intuition is that the wealth distribution among agents at the beginning of each period presumably influences prices and allocations in that period. Intuitively, one would expect that the wealth distribution constitutes a sufficient endogenous state space. The argument would be that the initial distribution of wealth is the only endogenous variable that influences the equilibrium behavior of the economy. However, as pointed by Kubler and Schmedders (2002), the wealth distribution alone does not always constitute a sufficient endogenous state space, mainly because the equilibrium decisions at time t also must be consistent with expectations at time $t-1$ and that these expectations at time $t-1$ cannot always be summarized in the wealth distribution alone. Our existence result allow us to proceed further in simulating the model. Theorem 1, however, does not guarantee the uniqueness of the equilibrium or the existence of non-degenerate ergodic measure. But Theorem 1 offers a key characteristic of the solution method.

Corrolary 1. *Under the same assumptions as in Theorem 1, the policy correspondence Π and value functions U_i in a wealth-recursive Markov equilibrium have the following forms, for $i, j \in \{1, 2\}$,*

$$c_i^j(w,s,e) \equiv c_i^j(w,s)e, \quad \vartheta_i^j(w,s,e) \equiv \vartheta_i^j(w,s), \quad b_i^j(w,s,e) \equiv b_i^j(w,s), \tag{10}$$

$$p_i(w,s,e) \equiv p_i(w,s), \quad q_i(w,s,e) \equiv q_i(w,s)e, \quad q_i^b(w,s,e) \equiv q_i^b(w,s)e, \tag{11}$$

$$\mu_i^j(w,s,e) \equiv \mu_i^j(w,s)e^{1-\psi_i^{-1}}, \quad \mu_{i,\tilde{s}}^b(w,s,e) \equiv \mu_{i,\tilde{s}}^b(w,s)e^{-\psi_i^{-1}}, \quad U_i(w,s,e) = U_i(w,s)e^{1-\psi_i^{-1}}. \tag{12}$$

Proof. The proof is in Appendix B. □

Corollary 1 suggests that the components of the policy correspondence in equilibrium are homogeneous in terms of the size of the global economy e to different degrees, because the level of the global tree e controls the scale of the economy and shocks on the size of global tree are permanent shocks. For example, the consumption, the bond holdings and the equity prices are degree-one homogeneous in the size of the economy, which is intuitive because only the consumption shares between agents and the debt ratios of each agent matter for the economy and the size of the economy is proportional to the amount of commodities attached to equity. Furthermore, the equity shares and the bond prices are invariant to the scale of the economy, because the total amount of the equity is normalized to be one and by definition the claim of a unit of bond is always assumed to be one unit of commodity. As a standard property, the Epstein-Zin-Weil preference U_i is homogeneous in $1-\psi_i^{-1}$ degrees in term of wealth that is proportional to the size of the economy. The shadow values are also homogeneous in term of economy scale according to the value functions. Thus, without loss of generality, we can assume that the endowment level of the global tree in current period is one, i.e. $e=1$. Therefore, when solving for the equilibrium, we only need to focus on the wealth share w and the exogenous shock s .

5 Calibration

This section describes our data set and the key statistics on GDP, consumption, international trade, and asset prices that define our calibration.

5.1 Data

Our data come from different sources. At the quarterly frequency, GDP, consumption and international trade series are from the OECD, while international capital stocks and flows are from the International Monetary Fund (IMF). International capital flows come from Bluedorn, Duttagupta, Guajardo and Topalova (2013); the balance of payments of each country is the primary source of the data. Foreign equity return indices are built by Datastream; for the U.S., the equity return series come from CRSP. Interest rates correspond to Treasury Bills or money market rates from the IMF. At the annual frequency, long time-series of capital stocks come from Lane and Milesi-Ferretti (2007).

This dataset is used to characterize two countries, the U.S. and the ROW. The ROW is defined as the aggregate of the G10 countries, excluding the U.S. (i.e., Belgium, Canada, Japan, France, Germany, Italy, Netherlands, Sweden, Switzerland, and U.K.). Each period, the ROW GDP and consumption growth rates are obtained by weighting each country-specific real growth rates by the share of its real GDP (measured at purchasing power parity) in total GDP. Indices are built from the growth rates and HP-filtered with a smoothing coefficient of 1600, as it is usual for quarterly series (Hodrick and Prescott, 1997). The sample period is 1973.1–2010.4.

5.2 Macroeconomic and Financial Variables

Let us now rapidly review the properties of macroeconomic and financial variables in the U.S. and ROW.

Production, Consumption, and International Trade Table 1 reports the mean, standard deviation, and autocorrelation of U.S. GDP and consumption growth rates, as well as their rest-of-the-world (ROW) counterparts. The table also reports similar summary statistics on the U.S. net exports and trade openness. Net exports are obtained as the difference between exports and imports, both scaled by GDP. Trade openness corresponds to the average of imports and exports, also scaled by GDP.

The macroeconomic data exhibit classic features of real business cycles. In both the US and the ROW, consumption appears less volatile than GDP, a common finding among developed countries. GDP and consumption are less volatile in the ROW than in the US as some of the foreign shocks average out across foreign countries. GDP growth rates are more correlated across countries than consumption growth rates. These characteristics appear on growth rates as well as on HP-filtered series. Trade openness is around 10%, while net exports are on average -2%; both measures are very persistent.

Interest Rates, Equity, and Currency Returns Panel A of Table 2 reports the mean, standard deviation, and autocorrelation of U.S. and rest-of-the-world (ROW) real interest rates, dividend yields, real equity returns and excess returns, as well as their cross-country correlation coefficients. Over the last forty years, the average real equity returns in the U.S. and ROW are respectively equal to 8.4% and 4.7% per year, leading to average equity excess returns respectively equal to 6.4% and 2.7%.[5] The dividend yields are 3.1% and 2.8% in the U.S. and ROW, implying price dividend ratios of 32 and 37. The price-dividend ratios are volatile, and thus either future dividend growth or future equity excess returns must be predictable (Campbell and Shiller, 1988). Equity returns are volatile both in the U.S. and in the ROW aggregate (18% on an annual basis) but appear largely correlated (0.8) among the most developed countries. In the model, the wealth consumption ratio is large and volatile, as it is in the data (Lustig, et al., 2013).

[5]The Datastream series understate the aggregate equity return: for the U.S., the difference between the CRSP and Datastream estimates is equal to 2.7% on average over our sample period. The discrepancy is certainly related to the Datastream focus on only a subset of large firms. The equity premium for the ROW is thus likely much higher than reported here.

Table 1: GDP, Consumption, and International Trade

	Data				Model			
	Mean	Std	AC(1)	Corr(ROW,US)	Mean	Std	AC(1)	Corr(ROW,US)
	Panel A: Raw Series (Growth Rates and Ratios)							
US GDP	0.68	0.83	0.39		0.68	0.87	0.44	
	(0.09)	(0.07)	(0.08)					
US Consumption	0.74	0.68	0.34		0.68	0.87	0.46	
	(0.07)	(0.06)	(0.08)					
ROW GDP	0.53	0.62	0.48	0.45	0.68	0.87	0.44	0.47
	(0.07)	(0.09)	(0.08)	(0.10)				
ROW Consumption	0.54	0.51	0.04	0.34	0.68	0.85	0.44	0.50
	(0.04)	(0.05)	(0.11)	(0.09)				
US Net Exports/GDP	-2.13	1.71	0.98		-1.74	0.41	0.96	
	(0.24)	(0.13)	(0.05)					
US Trade Openness	10.44	1.92	0.98		8.28	0.23	0.96	
	(0.27)	(0.16)	(0.05)					
	Panel B: HP-Filtered Series							
US GDP		1.53	0.87			1.08	0.82	
		(0.15)	(0.06)					
US Consumption		1.21	0.88			1.07	0.82	
		(0.10)	(0.06)					
ROW GDP		1.13	0.88	0.65		1.08	0.82	0.43
		(0.13)	(0.05)	(0.06)				
ROW Consumption		0.72	0.80	0.47		1.05	0.82	0.47
		(0.07)	(0.07)	(0.09)				
US Net Exports/GDP		0.46	0.77			0.14	0.69	
		(0.04)	(0.06)					
US Trade Openness		0.53	0.81			0.08	0.69	
		(0.07)	(0.06)					

Notes: Panel A reports the mean, standard deviation, and autocorrelation of U.S. rest-of-the-world (ROW) GDP and consumption growth rates, as well as their cross-country correlation coefficients. It also reports the mean, standard deviation, and autocorrelation of U.S. net exports and trade openness. Net exports are obtained as the difference between exports and imports, both scaled by GDP. Trade openness corresponds to the average of imports and exports, also scaled by GDP. Panel B reports the same test statistics (except for the mean) for HP-filtered series in levels. Standard errors are reported in parentheses; they are obtained by block-boostrapping. Data are quarterly, from the OECD database. All variables are reported in percentage points, except for the autocorrelation and cross-country correlation coefficients. The sample period is 1973.1–2010.4. The simulated moments correspond to samples without disasters.

Table 2: Dividend Yields, Equity Returns, and Interest Rates

	Data				Model			
	Panel A: Moments							
	Mean	Std	AC(1)	Corr(ROW,US)	Mean	Std	AC(1)	Corr(ROW,US)
US Dividend Yield	4.36	1.35	0.94		5.37	0.95	0.90	
	(0.20)	(0.11)	(0.05)					
ROW Dividend Yield	2.76	0.88	0.95	0.72	3.12	0.43	0.86	0.96
	(0.12)	(0.07)	(0.05)	(0.07)				
US Real Equity Returns	8.37	17.03	0.12		9.36	16.55	-0.05	
	(2.84)	(1.55)	(0.10)					
ROW Real Equity Returns	4.73	17.76	0.13	0.08	6.62	12.78	-0.04	0.95
	(3.12)	(1.34)	(0.08)	(0.09)				
US Real Money Market	1.87	2.61	0.82		0.86	3.38	0.88	
	(0.34)	(0.20)	(0.06)					
ROW Real Money Market	2.07	2.39	0.96	0.63	-1.46	5.76	0.88	1.00
	(0.32)	(0.22)	(0.05)	(0.07)				
US Equity Excess Returns	6.39	16.99	0.12		8.50	17.72	-0.01	
	(2.83)	(1.60)	(0.10)					
ROW Equity Excess Returns	2.69	17.34	0.12	0.08	8.09	15.19	0.09	0.95
	(2.88)	(1.48)	(0.07)	(0.09)				
	Panel B: Predictability Tests							
	β_{pd}	R^2	β_{cay}	R^2	β_{pd}	R^2	β_{cay}	R^2
US Pred.	0.37	0.09	0.54	0.23	1.44	0.44	0.52	0.36
	(0.18)	(0.03)	(0.14)	(0.03)				
ROW Pred.	1.24	0.31			2.84	0.39		
	(0.34)	(0.03)						
	Panel C: Expected Equity Excess Returns							
	Mean	Std	AC(1)		Mean	Std	AC(1)	
US Exp. ER (D/P)	4.28	1.28	0.98		8.50	7.14	0.90	
	(2.68)	(0.82)	(0.05)					
US Exp. ER (cay)	4.28	2.66	0.93		8.50	3.21	0.94	
	(2.68)	(1.20)	(0.05)					

Notes: Panel A of the table reports the mean, standard deviation, and autocorrelation of U.S. and rest-of-the-world (ROW) real interest rates, dividend yields, real equity returns and excess returns, as well as their cross-country correlation coefficients. Real equity returns are obtained by subtracting three-month realized inflation to nominal equity returns. Real interest rates correspond to nominal interest rates minus 12-month inflation. Panel B reports the slope coefficients (β_{pd} or β_{cay}) and the R^2 in predictability tests of equity excess returns over 5 years on dividend yields or, for the U.S., the consumption-wealth ratio of Lettau and Ludvigson (2001). Panel C report the mean, standard deviation, and autocorrelation of the expected U.S. equity excess returns. Expected excess returns over the next quarter are obtained using either the dividend yield or the wealth-consumption ratio. Standard errors are reported in parentheses; they are obtained by block-boostrapping. Data are quarterly, from the Datastream (equity indices and dividend yields) and IMF (money market rates) databases. All variables are reported in percentage points, except for the autocorrelation and cross-country correlation coefficients. Equity returns and excess returns as well as risk-free rates are annualized (i.e., average obtained on quarterly returns are multiplied by 4 and the standard deviations are multiplied by 2). U.S. equity returns series are from the CRSP database, while ROW series are built from MSCI data. Predictability tests are run on MSCI returns and dividend yields. The sample period is 1973.1–2010.4. The simulated moments correspond to samples without disasters.

Predictability regressions show that equity excess returns are predictable over long horizons. Panel B of Table 2 reports the slope coefficients (β_{pd} or β_{cay}) and the R^2 obtained in predictability tests of equity excess returns over 5 years on dividend yields or, for the U.S., the consumption-wealth ratio of Lettau and Ludvigson (2001). The slope coefficients are statistically significant and the R^2 range from 10% to 30%. The model matches particularly well the amount of predictability implied by the wealth-consumption ratio. Panel C of Table 2 reports the mean, standard deviations, and autocorrelations of expected equity excess returns in the U.S. obtained using either the price-dividend ratio or the wealth consumption ratio as predictors. Expected equity excess returns, i.e. risk premia, are clearly time-varying.

Table 3 focuses on exchange rates. The real exchange rate between the U.S. and the ROW has an annualized volatility of 8.9% and a small and insignificant autocorrelation. Carry trade excess returns are obtained by building three portfolios of currencies sorted by their interest rates: carry trades then correspond to strategies long the last portfolio of high interest rate currencies and short the first portfolio of low interest rate currencies. The carry trade offers an average excess return of 2.5% in the sample and a Sharpe ratio of 0.28, higher than the Sharpe ratios on U.S. and ROW aggregate equity markets. Carry trade excess returns tend to be low when global equity volatility surges: the correlation between the two is significantly negative. The exchange rate of low interest rate countries tend to appreciate while the exchange rate of high interest rate countries tend to depreciate when global volatility increases, leading in both cases to carry trade losses. This pattern is at the root of a risk-based explanation of the large average carry trade excess returns. Risk-averse investors expecting losses in bad times require a risk premium as a compensation for bearing the exchange rate risk.

5.3 Parameters

We use data on macroeconomic variables and asset returns to calibrate our model, starting with the endowment processes. Table 4 reports all the parameters of the model.

In the simulation, the two countries differ in their risk-aversion (3.3 for the U.S. vs. 4 for the ROW) and their IES (2.4 for the U.S. vs 1.4 for the ROW). The other preference parameters are the same in both countries. The subjective discount factor is 0.99. The domestic consumption share is 0.95, and the elasticity of substitution between the domestic and foreign goods is 0.885.

We follow Rouhenworst (1995) to calibrate the Markov processes such that they replicate the GDP series. The dividend share of total endowment is assumed to be ten times more volatile than the labor

Table 3: Exchange Rates

	Data				Model			
	Panel A: Exchange Rates and Currency Excess Returns							
	Mean	Std	AC(1)	Corr(ER, WV.)	Mean	Std	AC(1)	Corr(ER, WV)
ROW Real FX chge	-3.99	8.94	0.07		-0.00	4.09	-0.04	-0.72
	(14.50)	(0.50)	(0.08)					
Carry ER	3.67	8.95	0.11	-0.45	4.61	4.76	0.16	-0.37
	(1.42)	(1.31)	(0.08)	(0.11)				
	Panel B: Backus-Smith Correlations							
	$\frac{C_I^{us}}{C_I^{RoW}}, Q$	$\frac{C_W^{us}}{C_W^{RoW}}, Q$	$\frac{C^{us}}{C^{RoW}}, Q$		$\frac{C_I^{us}}{C_I^{RoW}}, Q$	$\frac{C_W^{us}}{C_W^{RoW}}, Q$	$\frac{C^{us}}{C^{RoW}}, Q$	
Growth			-0.11		-0.44	0.13	-0.11	
			(0.09)					
HP filter			0.01		-0.40	0.19	-0.06	
			(0.10)					

Notes: Panel A of the table reports the mean, standard deviation, and autocorrelation of the real exchange rate change between the U.S. and the ROW, as well as the same moments for the currency carry trade excess returns, along with its correlation with world equity volatility. Currencies are sorted by the level other short-term interest rates into three portfolios as in Lustig and Verdelhan (2007). Carry trade excess returns correspond to the returns on the high interest rate portfolios minus the returns on the low interest rate portfolio. Panel B of the table reports the Backus-Smith correlation between exchange rates and the relative consumption in the U.S. and ROW. Consumption and exchange rates are either measured on growth rates or H.P.-filtered. Consumption corresponds to workers' (denoted C_W) or investors' (denoted C_I) or aggregate (C) consumption. Standard errors are reported in parentheses; they are obtained by block-boostrapping. Data are quarterly, from the Datastream (exchange rates) and IMF (money market rates) databases. All variables are reported in percentage points, except for the autocorrelation and cross-country correlation coefficients. Exchange rate changes and currency excess returns are annualized (i.e., average obtained on quarterly returns are multiplied by 4 and the standard deviations are multiplied by 2). The sample period is 1973.1–2010.4. The simulated moments correspond to samples without disasters.

Table 4: Parameters

Parameter (Quarterly)	Symbol	Home/Foreign
	Panel A: Preferences	
Subjective discount rate	β	0.99
Relative risk aversion	γ_1/γ_2	3.8/4
EIS coefficient	ψ_1/ψ_2	2.4/1.1
Consumption ES coefficient	ϵ	0.885
Consumption share coefficient	s	0.93
	Panel B: Endowment	
Country-spec. volatility	σ_c	2%
Global volatility	σ_g	0.6%
Average growth	μ_g	0.675%
	Panel C: Dividend and Wage Income	
Wage income share of investors	W_I	10%
Dividend share of output	$\overline{d}$	5%
Dividend leverage on country-spec. shock	s_d	0.19
Dividend leverage on global. shock	s_g	0.19
Dividend leverage on disaster shock	s_{gd}	0.9
	Panel D: Disasters	
Disaster size	φ_d	9.7%
Disaster escaping prob.	$1 - p_d$	11.1%
Average log prob.	$\log(\overline{p})$	$\log(0.314\%)$
Std. log prob.	σ_p	4.9%
Autocorr. log prob.	ρ_p	0.9

Notes: This table reports the parameters used in the benchmark simulation of the model. The two countries share the same parameters, except for their risk-aversion and elasticity of substitution.

income share.

The average probability of a disaster is low, equal to 0.3%, but the disaster size is large: when it occurs, it entails a GDP decrease of 9.7%. The probability of leaving the disaster state the next period is 11.1%. The log probability of a disaster is persistent, with an autocorrelation of 0.9, and volatile, with a standard deviation of 4.9%. As the disaster probability is not directly observed, its parameters are subject to a large uncertainty. The model parameters are in line with those suggested by Barro (2006) and Gourio (2012).

Going back to Tables 1, 2, and 3, we check that the model reproduces the basic features of GDP, consumption, interest rates, equity prices and returns, and exchange rates. The attentive reader can compare moment by moment, series by series, the actual to the simulated data. The main discrepancy is the volatility of net exports and trade openness, which are more volatile in the data than in the model.

The model delivers a large equity premium. It also delivers time-variation in equity returns that is in line with the data. In the data, price-dividend and wealth-consumption ratios predict future equity returns. The model reproduces these findings. The volatility of the expected excess return obtained using the price-dividend ratio is higher in the model than in the data, but the volatility of the expected excess return obtained using the wealth-consumption ratio is the same in the model and the data. The current calibration, however, implies dividend yields that are more correlated than in the data. Likewise, the realized returns are more correlated in the model than in the data. As a result, the simulated cross-country correlation of realized and expected returns is counterfactually high. The model also misses the level of the ROW risk-free rate, calling for an adjustment in the EIS parameter.

The model delivers exchange rates that are less volatile than in the data, but the currency risk premium is the same in the model and the data. While frictionless complete markets where agents are characterized by constant relative risk-aversion imply a perfect correlation between the exchange rate changes and relative consumption growth (Backus and Smith, 1993), our model implies a negative correlation, closer to its empirical counterpart.

Overall, the model delivers its premises: large and time-varying risk premia with reasonable endowment and preference assumptions. We turn now to the simulation results obtained with this calibration.

6 Benchmark Simulation

We start by describing the policy functions and then turn to the key result of the paper: the comparison between the volatility of foreign assets and capital flows in the model and in the data.

6.1 Policy Functions

Symmetric Countries To build intuition on the model, let us start with the case of symmetric countries: both countries share the same preference parameters ($\gamma_1 = \gamma_2 = 4$ and $\psi_1 = \psi_2 = 2$), and all the other parameters are the same. Figure 3 reports the distribution of relative wealth along with policy functions that describe the asset holdings.

The upper left panel shows that the distribution of relative wealth, defined as $w_t \equiv W_{1,t}/[W_{1,t} + W_{2,t}]$, is symmetric, centered around 0.5 as expected. The lower right panel shows the amount of lending and borrowing chosen by country 1 (the U.S.). When the U.S. is relatively poor, the U.S. borrows from the ROW; when the U.S. is relatively rich, the U.S. lends to the ROW. The policy function is perfectly symmetric around the 0.5 relative wealth. On average, the U.S. does not have any debt. The role of the borrowing constraint appears when one country is much poorer than the other. For example, when the ROW is relatively poor (on the right hand side of the graph) and the U.S. holds more than 70% of total wealth, then any additional increase in the U.S. wealth decreases its lending to the ROW. The ROW would like to borrow but is not rich enough to post collateral. The borrowing constraint becomes binding. The distribution of relative wealth shows that this state of the world happens rarely in the model.

The upper right panel describes the U.S. holdings of U.S. equity. The home bias in consumption implies that the U.S. holds more than half of U.S. equity even when the two countries share the same wealth level. When the U.S. become relatively richer, they invest more in their own equity. The increase in their equity holdings is not monotone. At high wealth level, the binding borrowing constraint of the ROW impacts the U.S. equity choice. Because the U.S. cannot lend as much as they would like, they adjust their equity position downwards. This mechanism is particularly strong when the disaster probability is high, and thus equity prices are low: in that case, the ROW has less collateral and borrows less, thus affecting more the equity holdings of the U.S. At the other extreme, when the U.S. is relatively very poor, the U.S. would like to short their own equity, but the short-selling constraint on equity binds, and the U.S. simply stop holding equity. The lower left panel describes the U.S. holdings of the ROW equity. Since equity is either held by the U.S. or the ROW, the set of policy functions in that panel mirrors the previous one.

Asymmetric Countries We turn now to the asymmetric case. Figure 4 reports the distribution of relative wealth and the policy functions in that model. As Panel A shows, the simulation delivers again a stationary distribution of relative wealth. The U.S., which is less risk-averse, tends to be wealthier on average than

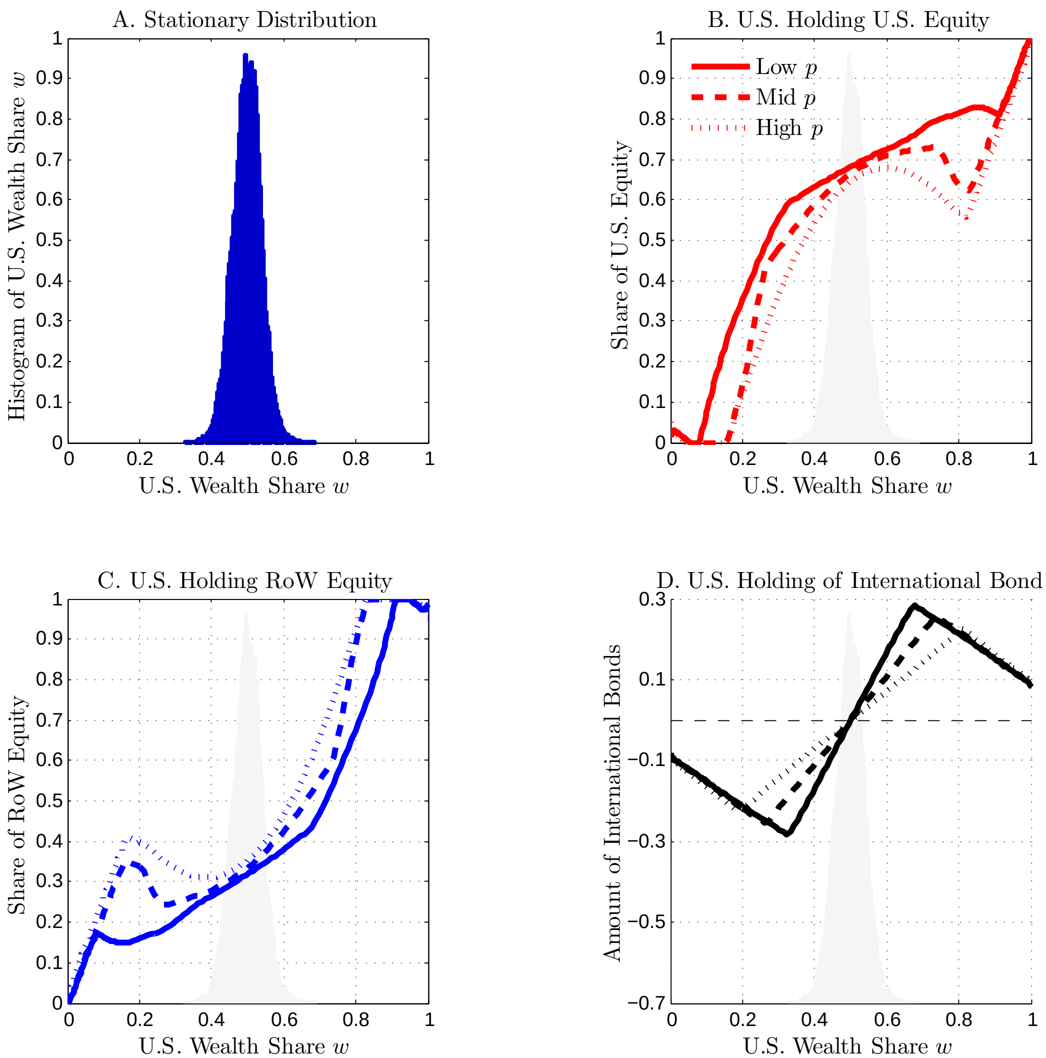

Figure 3: Relative Wealth and Asset Holdings in the Symmetric Case

This is the symmetric case with $\gamma_1 = \gamma_2 = 4$ and $\psi_1 = \psi_2 = 2$. The Panel A of this figure reports the stationary distribution of relative wealth, defined as $w_t \equiv W_{1,t}/[W_{1,t} + W_{2,t}]$, where the country 1 corresponds to the U.S. and country 2 the ROW. A large value for w_t therefore corresponds to a state of the world where the U.S. is rich compared to the ROW. Panel B reports the U.S. holdings of the U.S. equity; Panel C reports the U.S. holdings of the ROW equity; and Panel D reports the U.S. holdings of the international bond. All these holdings are reported as a function of the relative wealth w_t. In these three graphs, the plain line corresponds to the average growth rate, while the thing dotted line corresponds to low growth and the large dotted done corresponds to the disaster state.

the ROW.

The three other panels describe the U.S. holdings of the U.S. equity, ROW equity, and international bonds. At the mode of relative wealth, the U.S. holds a large share of U.S. equity (Panel B of Figure 4), again in line with the well-known home equity bias, but also a large share of foreign equity (Panel C of Figure 4). To do so, the U.S. tends to borrow from the ROW (Panel D of Figure 4) and thus exhibits a levered position in equity markets: borrowing on average from the ROW in order to buy U.S. and ROW equity. Only when the U.S. is much much wealthier than the ROW does the U.S. lend to the ROW. As in the symmetric case, the U.S. lending increases with U.S. relative wealth up to a point, where the borrowing constraint binds for the ROW: the ROW is then so poor that it can no longer collateralize its borrowing. After that point, the U.S. lending decreases with the U.S. relative wealth.

6.2 International Capital Stocks and Flows

We turn now to the comparison between actual and simulated foreign capital stocks and flows.

Stocks Over the last forty years, the total stocks of U.S. foreign assets and liabilities (even scaled by U.S. GDP) has increased tremendously from less than 10% to more than 160%. The large increase in international positions occurs across all four categories of investments reported in the balance of payments and international investments statistics: debt, equity, FDI, and other investments. It follows an increase in the financial openness of the US and ROW, as encoded for example from the restrictions on cross-border financial transactions reported in the IMF's Annual Report on Exchange Arrangements and Exchange Restrictions. To parallel the model, we report statistics on two asset categories, equity vs debt, both built from the the Lane and Milesi-Ferretti dataset (2007). All "equity" stocks correspond to the sum of equity, foreign direct investment, and other investments. For debt, we focus on net debt holdings because the model features only one international bond. Net debt assets correspond to the difference between debt portfolio assets and liabilities.

Table 5 reports basic summary statistics on U.S. international stocks. Because of the trend in foreign holdings, we report statistics on raw data as well as on HP-filtered series.

While the average level of debt is slightly higher in the data than in the model, the average equity position is much higher in the model than in the data. The current calibration offers expected equity excess returns that are too large, inducing large foreign holdings. The foreign capital holdings are also

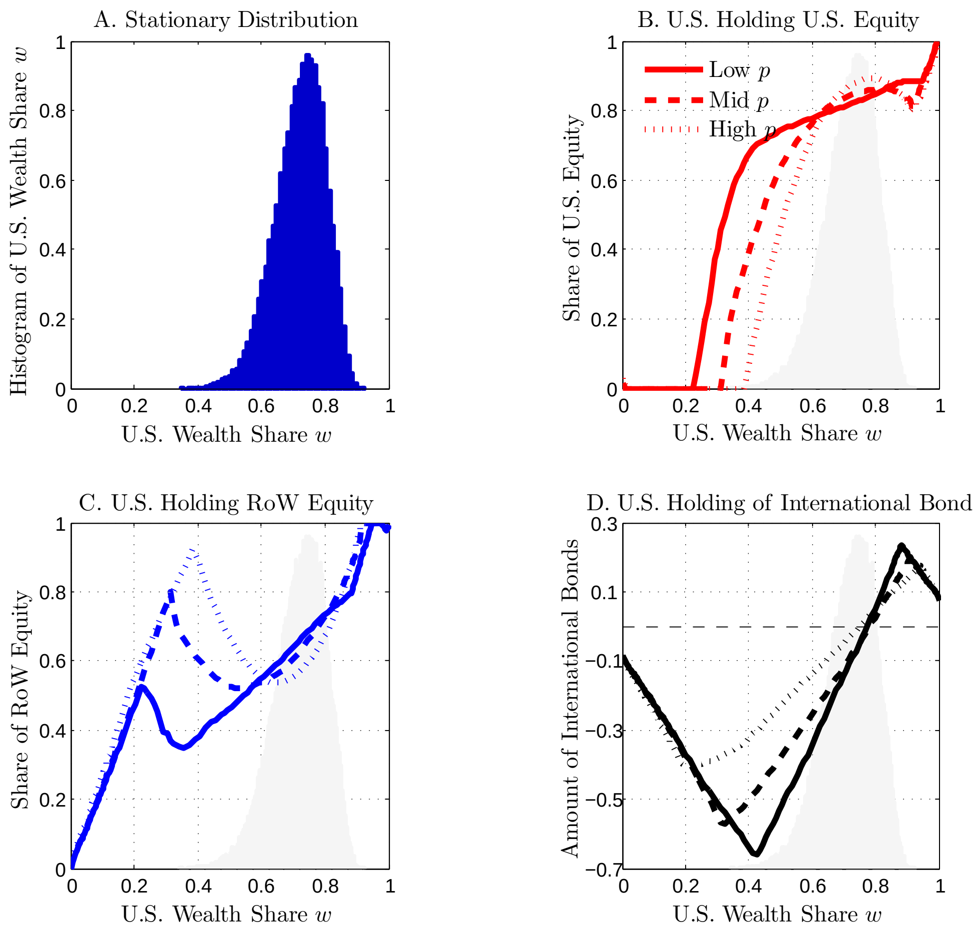

Figure 4: Relative Wealth and Asset Holdings

The Panel A of this figure reports the stationary distribution of relative wealth, defined as $w_t \equiv W_{1,t}/[W_{1,t} + W_{2,t}]$, where the country 1 corresponds to the U.S. and country 2 the ROW. A large value for w_t therefore corresponds to a state of the world where the U.S. is rich compared to the ROW. Panel B reports the U.S. holdings of the U.S. equity; Panel C reports the U.S. holdings of the ROW equity; and Panel D reports the U.S. holdings of the international bond. All these holdings are reported as a function of the relative wealth w_t. In these three graphs, the plain line corresponds to the average growth rate, while the thing dotted line corresponds to low growth and the large dotted done corresponds to the disaster state.

Table 5: U.S. International Capital Stocks

	Raw Data			HP-Filtered Series			
	Min	Mean	Max	Std	AC(1)	Corr US GDP	Corr ROW GDP
				Panel I: Data			
US All ”Equity” assets	13.62	45.63	108.86	6.25	0.23	0.40	0.04
US All ”Equity” liabilities	8.93	39.12	84.24	4.20	0.39	0.44	0.04
US Net All ”Equity” assets	-3.90	6.51	24.61	3.29	0.36	0.20	0.03
US Net Debt assets	-41.82	-14.02	-2.36	1.09	0.54	-0.32	-0.03
US Net Foreign assets	-29.54	-6.73	4.56	3.20	0.20	0.08	-0.03
				Panel II: Model			
US All ”Equity” assets	118.41	356.45	742.15	39.64	0.69	0.19	-0.04
US All ”Equity” liabilities	0.00	69.12	444.46	13.70	0.69	0.27	0.00
US Net All ”Equity” assets	77.05	287.33	484.13	29.03	0.69	0.13	0.05
US Net Debt assets	-71.43	-9.72	20.17	2.23	0.69	-0.11	-0.04
US Net Foreign assets	43.74	277.61	500.19	27.41	0.69	0.13	0.04

Notes: This table reports the min, mean, max, standard deviation, autocorrelation, and cross-country correlation of U.S. international capital stocks in different asset classes. All ”equity” stocks correspond to the sum of equity, foreign direct investment, and other investments. Net all ”equity” assets correspond to the difference between all “equity” assets and liabilities. Net debt assets correspond to the difference between debt portfolio assets and liabilities. The last two columns correspond to the cross-country correlation coefficients between international capital flows and U.S. or rest-of-the-world (ROW) HP-filtered GDP series. All series are scaled by GDP. The min, mean, and max statistics are computed on raw data, while the standard deviation, autocorrelation, and correlations are computed on HP-filtered series. Standard errors are reported in parentheses; they are obtained by block-boostrapping. Data are annual, from the Lane and Milesi-Ferretti dataset and the OECD. All variables are reported in percentage points, except for the autocorrelation and cross-country correlation coefficients. The sample period is 1973–2010.

more volatile in the model than in the data, particularly for equity assets. They are also too persistent compared to their actual counterparts. The model, however, captures the cyclicality of U.S. equity assets and liabilities with respect to the U.S. GDP, as well as the counter-cyclicality of the net U.S. debt position.

Flows In the data, the large increase in total assets and liabilities is accompanied by a large increase in the size and volatility of all categories of international capital flows. Balance of payments record international capital flows at the quarterly frequency, distinguishing between foreign direct investment, portfolio flows, and the remainder, denoted “other flows.”[6] To quantify the volatility of the capital flows, Table 6 reports some simple summary statistics. Total U.S. equity outflows attain more than 13% of GDP,

[6]Gross outflows are defined as net purchases of foreign financial instruments by domestic residents. Gross inflows are defined as net sales of domestic financial instruments to foreign residents. By convention, negative outflows mean that residents are buying more foreign assets than they are selling, contributing positively to negatively to net inflows. Intuitively, a negative outflow means than money is leaving the home country and flowing to the foreign country. Positive inflows means that foreigners are purchasing more domestic assets than they are selling, contributing positively to net inflows. Intuitively, a positive inflow means that money is flowing into the home country. Up to accounting errors, net inflows are then the sum of gross outflows and gross inflows.

Table 6: U.S. International Capital Flows

	Raw Data			HP-Filtered Series				
	Min	Mean	Max	Std	AC(1)	Corr US GDP	Corr ROW GDP	Corr World Vol.
				Panel I: Data				
US All ”Equity” Outflows	-13.43	-2.94	5.82	2.55	0.21	-0.22	-0.28	-0.06
	(1.22)	(0.33)	(1.02)	(0.29)	(0.08)	(0.09)	(0.10)	(0.12)
US All ”Equity” Inflows	-5.89	3.16	12.43	2.42	0.18	0.26	0.24	0.12
	(1.47)	(0.33)	(1.00)	(0.29)	(0.09)	(0.09)	(0.09)	(0.10)
US All ”Equity” Net Inflows	-3.56	0.22	4.70	1.40	0.11	0.05	-0.10	0.10
	(0.19)	(0.15)	(0.27)	(0.10)	(0.07)	(0.09)	(0.09)	(0.10)
US Net Debt Inflows	-3.55	1.73	8.56	1.23	0.30	0.24	0.36	0.05
	(1.35)	(0.25)	(0.82)	(0.18)	(0.06)	(0.11)	(0.12)	(0.07)
US Net Capital Inflows	-2.31	1.96	9.04	1.31	0.03	0.25	0.20	0.14
	(0.26)	(0.27)	(0.88)	(0.16)	(0.06)	(0.10)	(0.10)	(0.07)
				Panel II: Model				
US All ”Equity” Outflows	-348.34	-0.15	415.95	17.38	-0.09	-0.08	0.08	-0.04
US All ”Equity” Inflows	-280.42	-0.01	237.17	10.52	-0.09	0.08	-0.08	0.03
US All ”Equity” Net Inflows	-116.75	-0.17	146.91	7.54	-0.09	-0.07	0.07	-0.06
US Net Debt Inflows	-34.58	-0.00	32.18	1.79	-0.09	-0.01	-0.04	0.14
US Net Capital Inflows	-128.24	-0.17	157.13	6.91	-0.09	-0.07	0.07	-0.03

Notes: This table reports the min, mean, max, standard deviation, autocorrelation, and cross-country correlation of U.S. international capital flows in different asset classes. All ”equity” flows correspond to the sum of equity, foreign direct investment, and other investments. Net debt flows correspond to the sum of debt portfolio inflows and outflows. The next two columns correspond to the cross-country correlation coefficients between international capital flows and U.S. or rest-of-the-world (ROW) HP-filtered GDP series. The last column corresponds to the cross-country correlation coefficients between international capital flows and the change in world equity volatility. All series are scaled by GDP. The min, mean, and max statistics are computed on raw data, while the standard deviation, autocorrelation, and correlations are computed on HP-filtered series. Standard errors are reported in parentheses; they are obtained by block-boostrapping. Data are quarterly, from the Bluedorn et al. (2013) dataset, Datastream, and the OECD. All variables are reported in percentage points, except for the autocorrelation and cross-country correlation coefficients. The sample period is 1973.4–2010.4.

and sometimes even reverse sign. The total equity inflows amount to close to 12% GDP at their maximum.

Turning to HP-filtered series to eliminate the trends, both equity inflows and outflows exhibit a low but significant autocorrelation of around 0.2. The autocorrelation of net equity flows is only 0.1, much lower than the autocorrelation of net debt inflows (0.3). The total net inflows (debt and equity) are essentially uncorrelated. Total gross inflows and outflows tend to increase (more capital flowing abroad and in the U.S.) when US and ROW GDP are high, delivering significant correlation coefficients between capital flows and GDP series.

Table 6 also shows that capital flows tend to shrink in times of high aggregate volatility. We measure aggregate volatility as the cross-country average of the realized standard deviations of daily equity returns over each quarter. When aggregate volatility increases, capital outflows out of the U.S. become less negative, i.e. shrink in magnitude. likewise, capital inflows in the U.S. decrease. Such correlations appear clearly for equity and debt portfolio flows, as well as for the "other" flows and the total inflows and outflows. Foreign direct investment and net capital flows, however, do not exhibit any significant correlation with aggregate volatility. These correlation are best exemplified during the Great Recession. As already noted by several authors, the Great Recession is characterized by retrenchment: foreigners pull out their wealth out of U.S. equity and equity-like assets (equity inflows turn negative), while U.S. residents repatriate part of their foreign equity-like holdings (outflows turn positive). These unusual patterns coincide with large increases in world volatility, from pre-crisis levels of 20% to close to 60% (in annualized terms). Net debt inflows remain positive during the spike in volatility but turn negative when volatility recesses.

As in the data, our model produces volatile stock holdings because the value of the stock holdings move a lot. Recall that in the model as in the data, stock returns exhibit a 16% annualized volatility. The volatility of equity flows is much higher in the model, as it is in the data. Turning to bonds, the model reproduces the volatility of net debt flows, with little valuation effects.

Overall the model thus reproduces the start contrast between the volatility of the U.S. foreign assets and liabilities. Changes in debt liabilities are mostly due to changes in the amount of borrowing and thus international debt flows. To the contrary, changes in equity assets are mostly due to valuation changes. In the model, changes in equity prices and thus returns are either expected and unexpected. The large expected returns on ROW equity help the U.S. finance its negative trade balance and reimburse its debt. The model does not feature sovereign default and the negative trade balance is sustainable.

7 Conclusion

This paper presents a two-good, two-country real model that replicates basic stylized facts on equity excess returns and real interest rates. In the model, the U.S. borrows from the ROW and invests in ROW equity. The gross foreign asset positions are large and volatile. The changes in asset positions reflect both capital flows and changes in the value of the existing assets. The returns on existing assets feature an expected component that compensate investors for the risk of losing money in times of high marginal utility. Valuation effects appear key to understand the volatility of international asset holdings and the sustainability of current account imbalances.

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The Volatility of International Capital Flows and Foreign Assets - Supplementary Online Appendix - NOT FOR PUBLICATION

This appendix presents additional theoretical results on the expectation correspondence in Section A, the proof of Proposition 1 in Section B, the proof of Theorem 1 in Section C, two simple cases with analytical solutions in Section D, a description of the numerical algorithm in Section E, and some additional empirical results on the data in Section F and the model in Section G.

A Equations of Expectation Correspondence Φ

The optimization problem of each country $i \in \{1,2\}$ can be re-formulated using Lagrangian multipliers, for each $s \in \mathcal{S}$ and $e \in \overline{\mathbb{R}}_+$,

$$U_i(W_i) = \min_{\mu_i^j \geq 0, \mu_i^b \geq 0, \mu_{i,\tilde{s}}^b \geq 0} \max_{c_i^j, \vartheta_i^j, b_i^j} \frac{C_i^{1-\psi_i^{-1}}}{1-\psi_i^{-1}} + \beta \mathbb{E}\left[U_i(\tilde{W}_i)^{\theta_i} | s, e\right]^{1/\theta_i} + \sum_{j=1}^{2} \mu_i^j \vartheta_i^j - \mu_i^b b_i^i + \sum_{\tilde{s} \in \mathcal{S}} \mu_{i,\tilde{s}}^b \mathbb{C}_i(s^t, \tilde{s})$$

subject to the intra-termporal "budget constraint"

$$C_i = G_i(c_i^1, c_i^2),$$

the inter-temporal budget constraint

$$W_i = \sum_{j=1}^{2} p_j c_i^j + \sum_{j=1}^{2} \vartheta_i^j q_j + \sum_{j=1}^{2} b_i^j q_j^b,$$

$$\tilde{W}_i = \tilde{p}_i \tilde{\omega}_i + \sum_{j=1}^{2} \vartheta_i^j \left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right) + \sum_{j=1}^{2} b_i^j \tilde{p}_j.$$

Using textbook arguments, we can show that each value function $U_i(W_i)$ is concave, continuous, and increasing. Then, the standard variational argument leads to the envelop condition:

$$U_{i,W}(W_i) = C_i^{-\psi_i^{-1}} G_{i,c^2}\left(c_i^1, c_i^2\right), \tag{13}$$

where $U_{i,W}$ is the partial derivative of U_i w.r.t. W and G_{i,c^2} is the partial derivative of G_i w.r.t. c^2.

For notational simplicity, we denote $\mathbb{E}[\cdot|s,e] \equiv \mathbb{E}_{s,e}[\cdot]$. The first-order condition for ϑ_i^j, with $i, j \in \{1,2\}$, gives

$$\begin{aligned} q_j C_1^{-\psi_1^{-1}} G_{1,c^2}\left(c_1^1, c_1^2\right) =& \beta \mathbb{E}_{s,e}\left[\left(\tilde{U}_1\right)^{\theta_1}\right]^{1/\theta_1 - 1} \mathbb{E}_{s,e}\left[\tilde{U}_{1,W}\left(\tilde{U}_1\right)^{\theta_1 - 1}\left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right)\right] \\ &+ \mu_1^j + \sum_{\tilde{s} \in \mathcal{S}} \mu_{1,\tilde{s}}^b \eta_j \left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right), \end{aligned} \tag{14}$$

and

$$q_j C_2^{-\psi_2^{-1}} G_{2,c^2}\left(c_2^1, c_2^2\right) = \beta \mathbb{E}_{s,e}\left[\left(\tilde{U}_2\right)^{\theta_2}\right]^{1/\theta_2 - 1} \mathbb{E}_{s,e}\left[\tilde{U}_{2,W}\left(\tilde{U}_2\right)^{\theta_2 - 1}\left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right)\right] + \mu_2^j + \sum_{\tilde{s}\in\mathcal{S}} \mu_{2,\tilde{s}}^b\left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right). \tag{15}$$

If we plug the Envelop condition (13) into the FOC in (15), we can get

$$q_j C_i^{-\psi_i^{-1}} G_{i,c^2}\left(c_i^1, c_i^2\right) = \beta \mathbb{E}_{s,e}\left[\left(\tilde{U}_i\right)^{\theta_i}\right]^{1/\theta_i - 1} \mathbb{E}_{s,e}\left[\left(\tilde{C}_i\right)^{-\psi_i^{-1}} G_{i,c^2}\left(\tilde{c}_i^1, \tilde{c}_i^2\right)\left(\tilde{U}_i\right)^{\theta_i - 1}\left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right)\right] + \mu_i^j + \sum_{\tilde{s}\in\mathcal{S}} \mu_{i,\tilde{s}}^b\left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right). \tag{16}$$

Note that when $\theta_i = 1$, the condition above is simplified as

$$q_j C_i^{-\psi_i^{-1}} G_{i,c^2}\left(c_{i,1}, c_{i,2}\right) = \beta \mathbb{E}_{s,e}\left[\left(\tilde{C}_i\right)^{-\psi_i^{-1}} G_{i,c^2}\left(\tilde{c}_i^1, \tilde{c}_i^2\right)\left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right)\right] + \mu_i^j + \sum_{\tilde{s}\in\mathcal{S}} \mu_{i,\tilde{s}}^b\left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right).$$

This is simply the first-order condition for CRRA utility functions. Similar equation is derived in Stepanchuk and Tsyrennikov (2015).

Similarly, the first-order condition for bond holdings of agent 1 (i.e. b_1^j) and Envelop condition together lead to

$$q_1^b C_1^{-\psi_1^{-1}} G_{1,c^2}\left(c_1^1, c_1^2\right) = \beta \mathbb{E}_{s,e}\left[\left(\tilde{U}_1\right)^{\theta_1}\right]^{1/\theta_1 - 1} \mathbb{E}_{s,e}\left[\left(\tilde{C}_1\right)^{-\psi_1^{-1}} G_{1,c^2}\left(\tilde{c}_1^1, \tilde{c}_1^2\right)\left(\tilde{U}_1\right)^{\theta_1 - 1} \tilde{p}_1\right] - \mu_1^b + \sum_{\tilde{s}\in\mathcal{S}} \mu_{1,\tilde{s}}^b \tilde{p}_1. \tag{17}$$

and

$$q_2^b C_1^{-\psi_1^{-1}} G_{1,c^2}\left(c_1^1, c_1^2\right) = \beta \mathbb{E}_{s,e}\left[\left(\tilde{U}_1\right)^{\theta_1}\right]^{1/\theta_1 - 1} \mathbb{E}_{s,e}\left[\left(\tilde{C}_1\right)^{-\psi_1^{-1}} G_{1,c^2}\left(\tilde{c}_1^1, \tilde{c}_1^2\right)\left(\tilde{U}_1\right)^{\theta_1 - 1} \tilde{p}_2\right]. \tag{18}$$

Also, the first-order condition for bond holdings of agent 2 (i.e. b_2^j) and Envelop condition together lead to, for $j = 1, 2$,

$$q_1^b C_2^{-\psi_i^{-1}} G_{2,c^2}\left(c_2^1, c_2^2\right) = \beta \mathbb{E}_{s,e}\left[\left(\tilde{U}_i\right)^{\theta_2}\right]^{1/\theta_2 - 1} \mathbb{E}_{s,e}\left[\left(\tilde{C}_2\right)^{-\psi_2^{-1}} G_{2,c^2}\left(\tilde{c}_2^1, \tilde{c}_2^2\right)\left(\tilde{U}_2\right)^{\theta_2 - 1} \tilde{p}_1\right] + \sum_{\tilde{s}\in\mathcal{S}} \mu_{2,\tilde{s}}^b \tilde{p}_1. \tag{19}$$

and

$$q_2^b C_2^{-\psi_i^{-1}} G_{2,c^2}\left(c_2^1, c_2^2\right) = \beta \mathbb{E}_{s,e}\left[\left(\tilde{U}_i\right)^{\theta_2}\right]^{1/\theta_2 - 1} \mathbb{E}_{s,e}\left[\left(\tilde{C}_2\right)^{-\psi_2^{-1}} G_{2,c^2}\left(\tilde{c}_2^1, \tilde{c}_2^2\right)\left(\tilde{U}_2\right)^{\theta_2 - 1} \tilde{p}_2\right] + \sum_{\tilde{s}\in\mathcal{S}} \mu_{2,\tilde{s}}^b \tilde{p}_2 - \mu_2^b. \tag{20}$$

The intra-temporal Euler conditions for country $i \in \{1, 2\}$ is

$$p_1 G_{i,c^2}\left(c_i^1, c_i^2\right) = p_2 G_{i,c^1}\left(c_i^1, c_i^2\right). \tag{21}$$

Therefore, the expectation correspondence $\widehat{\Phi}$ consists of the following five groups of conditions for all $i, j \in \{1, 2\}$:

(1) The intra-temporal Euler equations in (21);

(2) The inter-temporal Euler equations about equity holdings in (14) and (15);

(3) The inter-temporal Euler equations about bond holdings in (17), (18), (19) and (20) and the feasibility conditions, for all $\tilde{s} \in \mathcal{S}$,

$$\sum_{j=1}^{2} \eta_j \vartheta_1^j \left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right) + b_1^1 \tilde{p}_1 \geq 0 \quad \text{and} \quad \tilde{p}_2 \tilde{\omega}_2 + \sum_{j=1}^{2} \vartheta_2^j \left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right) + \sum_{j=1}^{2} b_2^j \tilde{p}_j \geq 0, \tag{22}$$

and slackness conditions, for all $\tilde{s} \in \mathcal{S}$,

$$\mu_{1,\tilde{s}}^b \left[\sum_{j=1}^{2} \vartheta_1^j \eta_j \left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right) + b_1^1 \tilde{p}_1\right] = 0 \quad \text{and} \quad \mu_{2,\tilde{s}}^b \left[\tilde{p}_2 \tilde{\omega}_2 + \sum_{j=1}^{2} \vartheta_2^j \left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right) + \sum_{j=1}^{2} b_2^j \tilde{p}_j\right] = 0; \tag{23}$$

(4) The inter-temporal budget constraints, for all $\tilde{s} \in \mathcal{S}$,

$$\tilde{w} \left(\sum_{i=1}^{2} \tilde{p}_i \tilde{e}_i + \sum_{j=1}^{2} \tilde{q}_j\right) = \tilde{p}_1 \tilde{\omega}_1 + \sum_{j=1}^{2} \vartheta_1^j \left(\tilde{q}_j + \tilde{p}_j \tilde{d}_j\right) + \sum_{j=1}^{2} b_1^j \tilde{p}_j \tag{24}$$

and

$$\tilde{w} \left(\sum_{i=1}^{2} \tilde{p}_i \tilde{e}_i + \sum_{j=1}^{2} \tilde{q}_j\right) = \sum_{j=1}^{2} \tilde{p}_j \tilde{c}_1^j + \sum_{j=1}^{2} \tilde{\vartheta}_1^j \tilde{q}_j + \sum_{j=1}^{2} \tilde{b}_1^j \tilde{q}_j^b; \tag{25}$$

(5) The commodity market clearing conditions, for all $\tilde{s} \in \mathcal{S}$,

$$\tilde{c}_1^j + \tilde{c}_2^j = (c_1^j + c_2^j)\zeta(\tilde{s}). \tag{26}$$

B Proof of Proposition 1

Suppose that when $e = 1$, a wealth-recursive equilibrium exists and has the policy functions with the following form

$$\Pi(w, s, 1) \equiv \left\{c_i^j(w,s), \vartheta_i^j(w,s), b_i^j(w,s), p_i(w,s), q_i(w,s), q_i^b(w,s), \mu_i^j(w,s), \mu_{i,\tilde{s}}^b(w,s)\right\}. \tag{27}$$

More precisely, the policy functions in $\Pi(w,s)$, the transition map $\Omega(w,s)$ and the value function $U_i(w,s)$ satisfy the following conditions, for all $(w,s) \in [0,1] \times \mathcal{S}$:

(0) The vectors of endogenous variables lie in $\mathcal{Y}$ defined in (9) - (10), i.e. $\Pi(w,s,1), \Pi(\tilde{w}, \tilde{s}, \zeta(\tilde{s})) \in \mathcal{Y}$.

(1) The intra-temporal Euler equations are held:

$$p_1(w,s) G_{i,c^2}\left(c_i^1(w,s), c_i^2(w,s)\right) = p_2(w,s) G_{i,c^1}\left(c_i^1(w,s), c_i^2(w,s)\right). \tag{28}$$

(2) The inter-temporal Euler equations about equity positions are held:

$$\begin{aligned} q_j(w,s) C_i(w,s)^{-\psi_i^{-1}} G_{i,c^2}\left(c_i^1(w,s), c_i^2(w,s)\right) &= \beta \mathbb{E}_{w,s}\left[U_i(\tilde{w}, \tilde{s}, \zeta(\tilde{s}))^{\theta_i}\right]^{1/\theta_i - 1} \\ &\times \mathbb{E}_{w,s}\Big[C_i(\tilde{w}, \tilde{s}, \zeta(\tilde{s}))^{-\psi_i^{-1}} G_{i,c^2}\left(c_i^1(\tilde{w}, \tilde{s}, \zeta(\tilde{s})), c_i^2(\tilde{w}, \tilde{s}, \zeta(\tilde{s}))\right) \\ &\times U_i(\tilde{w}, \tilde{s}, \zeta(\tilde{s}))^{\theta_i - 1}(q_j(\tilde{w}, \tilde{s}, \zeta(\tilde{s})) + p_j(\tilde{w}, \tilde{s}, \zeta(\tilde{s})) d_j(\tilde{s}) \zeta(\tilde{s}))\Big] \\ &+ \mu_i^j(w,s) + \sum_{\tilde{s} \in \mathcal{S}} \mu_{i,\tilde{s}}^b(w,s)\left(q_j(\tilde{w}, \tilde{s}, \zeta(\tilde{s})) + p_j(\tilde{w}, \tilde{s}, \zeta(\tilde{s})) d_j(\tilde{s}) \zeta(\tilde{s})\right). \end{aligned} \tag{29}$$

(3) The inter-temporal Euler equations about bond are held

$$
\begin{aligned}
q_j^b(w,s)C_i(w,s)^{-\psi_i^{-1}}G_{i,c^2}\left(c_i^1(w,s),c_i^2(w,s)\right) &= \beta\mathbb{E}_{w,s}\left[U_i(\tilde{w},\tilde{s},\zeta(\tilde{s}))^{\theta_i}\right]^{1/\theta_i-1} \\
&\times \mathbb{E}_{w,s}\left[\tilde{C}_i(\tilde{w},\tilde{s},\zeta(\tilde{s}))^{-\psi_i^{-1}}G_{i,c^2}\left(c_i^1(\tilde{w},\tilde{s},\zeta(\tilde{s})),c_i^2(\tilde{w},\tilde{s},\zeta(\tilde{s}))\right)\left(U_i(\tilde{w},\tilde{s},\zeta(\tilde{s}))\right)^{\theta_i-1}p_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))\right] \\
&+ \sum_{\tilde{s}\in\mathcal{S}}\mu_{i,\tilde{s}}^b p_j(\tilde{w},\tilde{s},\zeta(\tilde{s})).
\end{aligned}
$$

and the feasibility conditions, for all $\tilde{s}\in\mathcal{S}$,

$$
\begin{aligned}
p_i(\tilde{w},\tilde{s},\zeta(\tilde{s}))\omega_i(\tilde{s})\zeta(\tilde{s}) + \sum_{j=1}^{2}\vartheta_i^j(w,s)\left[q_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))+p_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))d_j(\tilde{s})\zeta(\tilde{s})\right] \\
+ \sum_{j=1}^{2}b_i^j(w,s)p_j(\tilde{w},\tilde{s},\zeta(\tilde{s})) \geq 0,
\end{aligned}
$$

and slackness conditions, for all $\tilde{s}\in\mathcal{S}$,

$$
\begin{aligned}
\mu_{i,\tilde{s}}^j(w,s)\Bigg[p_i(\tilde{w},\tilde{s},\zeta(\tilde{s}))\omega_i(\tilde{s})\zeta(\tilde{s}) + \sum_{j=1}^{2}\vartheta_i^j(w,s)\left[q_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))+p_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))d_j(\tilde{s})\zeta(\tilde{s})\right] \\
+ \sum_{j=1}^{2}b_i^j(w,s)p_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))\Bigg] = 0,
\end{aligned}
$$

(4) The inter-temporal budget constraints, for all $\tilde{s}\in\mathcal{S}$, the wealth share corresponding to exogenous shock in the period $\tilde{s}$ is $\tilde{w}=\Omega(w,s;\tilde{s})$. More precisely,

$$
\begin{aligned}
&\tilde{w}\left(\sum_{i=1}^{2}p_i(\tilde{w},\tilde{s},\zeta(\tilde{s}))e_i(\tilde{s})\zeta(\tilde{s}) + \sum_{j=1}^{2}q_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))\right) \\
&\quad = p_1(\tilde{w},\tilde{s},\zeta(\tilde{s}))\omega_1(\tilde{s})\zeta(\tilde{s}) + \sum_{j=1}^{2}\vartheta_1^j(w,s)\left[q_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))+p_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))d_j(\tilde{s})\zeta(\tilde{s})\right] \\
&\quad + \sum_{j=1}^{2}b_1^j(w,s)p_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
&\tilde{w}\left(\sum_{i=1}^{2}p_i(\tilde{w},\tilde{s},\zeta(\tilde{s}))e_i(\tilde{s})\zeta(\tilde{s}) + \sum_{j=1}^{2}q_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))\right) \\
&\quad = \sum_{j=1}^{2}p_j(\tilde{w},\tilde{s},\zeta(\tilde{s}))c_1^j(\tilde{w},\tilde{s},\zeta(\tilde{s})) + \sum_{j=1}^{2}\vartheta_1^j(\tilde{w},\tilde{s},\zeta(\tilde{s}))q_j(\tilde{w},\tilde{s},\zeta(\tilde{s})) \\
&\quad + \sum_{j=1}^{2}b_1^j(\tilde{w},\tilde{s},\zeta(\tilde{s}))q_j^b(\tilde{w},\tilde{s},\zeta(\tilde{s}));
\end{aligned}
\tag{31}
$$

(5) The commodity market clearing conditions, for all $\tilde{s}\in\mathcal{S}$,

$$
c_1^j(\tilde{w},\tilde{s},\zeta(\tilde{s})) + c_2^j(\tilde{w},\tilde{s},\zeta(\tilde{s})) = (c_1^j(w,s)+c_2^j(w,s))\zeta(\tilde{s}). \tag{32}
$$

For general value of e, we plug the expressions of (10) - (12) into the Bellman equation and conditions (0) - (5) above, and then we can see that the current size e is perfectly canceled out. By assumption for the case of $e = 1$, we know that they are policy functions, transition map, and value functions for wealth-recursive Markov equilibrium for any current size e.

C Proof of Theorem 1

We prove the existence by construction which combines important ideas of the proofs in Duffie et al. (1994), Kubler and Schmedders (2003) and Geanakoplos and Zame (2013). The existence results of equilibria are standard for finite-horizon economy even with incomplete market, while for the infinite-horizon economy the proofs are much more involving. The key idea of the proofs in the literature[7] is basically backward induction and is based on the existence of competitive equilibria on all finitely-truncated economy whose equilibrium variables are uniformly bounded. We extend the proof in Kubler and Schmedders (2003) to allow for Epstein-Zin preferences including those which are not bounded below (i.e. EIS parameter is bigger than one).

The T-truncated economy is defined to be a finite-horizon economy built on an event tree, denoted by $\mathbb{S}^T$, which consists of all the nodes and edges along the path $s^T = (s_0, s_1, \cdots, s_T)$ in the original event tree $\mathbb{S}$. The endowments and asset payoffs at the nodes of the truncated tree, as well as agents' preferences and portfolio constraints at these nodes, are the identical to the original infinite-horizon economy. The sequential budget constraint of agent i in the T-truncated economy is $\mathbb{B}_{\mathbb{S}^T}(\mathcal{P}^{\mathbb{S}^T})$ which is a collection of consumption plans $\mathcal{C}_i^{\mathbb{S}^T} = \left\{c_i^1(s^t), c_i^2(s^t)\right\}_{s^t \in \mathbb{S}^T}$ and portfolio choice plans $\mathcal{A}_i^{\mathbb{S}^T} = \left\{\vartheta_i^1(s^t), \vartheta_i^2(s^t), b_i^1(s^t), b_i^2(s^t)\right\}_{s^t \in \mathbb{S}^T}$ such that, at each node of the event tree $s^t \in \mathbb{S}^T$, the portfolio positions satisfy the short-selling constraint (4) and borrowing constraint (5) and at each node s^t on the event tree $\mathbb{S}^T$,

$$\begin{aligned}\sum_{j=1}^{2} p_j(s^t) c_i^j(s^t) + \sum_{j=1}^{2} q_j(s^t) \vartheta_i^j(s^t) + \sum_{j=1}^{2} q_j^b(s^t) b_i^j(s^t) \\ = p_i(s^t)\omega_i(s^t) + \sum_{j=1}^{2} \left[q_j(s^t) + p_j(s^t) d_j(s^t)\right] \vartheta_i^j(s^{t-1}) + \sum_{j=1}^{2} p_j(s^t) b_j(s^{t-1}) \end{aligned} \tag{33}$$

where s^{t-1} is the ancestor node of the node s^t on the event tree and $s^0 = s_0$ is the initial node.

Inspired by the result in Proposition 1, we take off the scaling effect of the economy by assuming the world tree always has size one, i.e. $e(s^t) \equiv 1$ for all $s^t \in \mathbb{S}$. We first show that the competitive equilibria exist and the equilibrium variables are uniformly bounded over $T \geq 1$. We first formally introduce the following lemma and leave its proof to Appendix C.1.

Lemma 1. *For all $T \geq 1$, there exists a competitive equilibrium for the T-truncated economy in which all equilibrium variables, including consumptions, portfolio holdings and prices, all lie in a compact set $\mathcal{Y}^* \subset \mathcal{Y}$.*

For any compact set $\mathcal{K} \subset \mathcal{Y}$, and a policy correspondence $\Upsilon : \mathcal{S} \times [0,1] \rightrightarrows \mathcal{K}$, we define an operator $O_{\mathcal{K}}$, that maps the policy correspondence $\Upsilon : \mathcal{S} \times \Delta \rightrightarrows \mathcal{K}$ to another policy correspondence $O_{\mathcal{K}}(\Upsilon)$ such that for all $s \in \mathcal{S}$ and $w \in [0,1]$

$$O_{\mathcal{K}}(\Upsilon)(s,w) = \left\{ y \in \mathcal{K} \;:\; \exists \; \Big((\tilde{w}_1, \tilde{y}_1), \cdots, (\tilde{w}_{|\mathcal{S}|}, \tilde{y}_{|\mathcal{S}|})\Big) \in \Phi(w, y, s, 1) \text{ s.t. } \tilde{y}_{\tilde{s}} \in \Upsilon(\tilde{s}, \tilde{w}_{\tilde{s}}),\; \forall\, \tilde{s} \in \mathcal{S} \right\}.$$

[7] Examples for existence of competitive equilibria in infinite-horizon incomplete market economy with heterogeneous agents include Levine and Zame (1996), Magill and Quinzii (1996), and Hernandez and Santos (1996), among others. Examples for existence of recursive Markov equilibria include Duffie et al. (1994) and Kubler and Schmedders (2003), among others.

The correspondence $O_{\mathcal{K}}(\mathrm{Y})(w,s)$ is basically computing the endogenous variables $y \in \mathcal{Y}$ given the state variables (w,s) in the current period and the next period's equilibrium endogenous variables $(w_1, y_1), \cdots, (w_{|\mathbb{S}|}, y_{|\mathbb{S}|})$.

Define constant correspondence Y^0 by $\mathrm{Y}^0(y,w) \equiv \mathcal{Y}^*$ for all $w \in [0,1]$ and all $y \in \mathcal{Y}$. Given a correspondence Y^n, we define recursively $\mathrm{Y}^{n+1} = O_{\mathcal{Y}^*}(\mathrm{Y}^n)$. First, for each n, the set Y^n is nonempty. This is because of Lemma 1, which implies that for all n there exists a n-horizon competitive equilibrium whose endogenous variables lie in the compact set $\mathcal{Y}^*$. Second, we show that Y^n is closed for each n. We prove it by induction. It is obvious that $\mathrm{Y}^0 \equiv \mathcal{Y}^*$ is closed. Suppose Y^n is closed, then $\mathrm{Y}^{n+1} = O_{\mathcal{Y}^*}(\mathrm{Y}^n)$ is also closed because the graph of Φ is closed and the graph of Y^n is closed. Third, for each n, $\mathrm{Y}^{n+1} \subset \mathrm{Y}^n$. By definition, it is obvious that $\mathrm{Y}^1 \subset \mathrm{Y}^0 \equiv \mathcal{Y}^*$. Suppose that $\mathrm{Y}^n \subset \mathrm{Y}^{n+1}$, then we have $\mathrm{Y}^{n+1} \subset \mathrm{Y}^{n+2}$. This is because $O_{\mathcal{Y}^*}(\mathrm{Y}^n) \subset O_{\mathcal{Y}^*}\left(\mathrm{Y}^{n+1}\right)$ by definition.

We define a correspondence Y^* such that for all $(w,s) \in [0,1] \times \mathcal{S}$

$$\mathrm{Y}^*(w,s) \equiv \cap_{n=0}^{\infty} \mathrm{Y}^n(w,s). \tag{34}$$

Because for each $(w,s) \in [0,1] \times \mathcal{S}$, the sequence of sets $\{\mathrm{Y}^n(w,s)\}$ are compact, nested, and nonempty, thus $\mathrm{Y}^*(w,s)$ is a closed and nonempty set. $\mathrm{Y}^*(w,s)$ is policy correspondence in recursive Markov equilibria and the definition of operator $O_{\mathcal{Y}^*}$ implies the existence of a transition for the recursive Markov equilibrium.

C.1 Proof of Lemma 1

We first show that the policy functions in equilibria are uniformly bounded for all $T \geq 1$ if equilibria exist. The agent i's budget constraint $\mathbb{B}_{i,\mathbb{S}^T}(\mathcal{P}^{\mathbb{S}^T})$ contains the portfolio constraints including:

$$\begin{aligned}
&\vartheta_i^j(s^t) \geq 0, \ -\overline{D}e_i(s^t) \leq b_i^j(s^t) \leq \overline{D}e_i(s^t), \ \text{ and} \\
&p_i(s^t)\omega_i(s^t) + \sum_{j=1}^{2} \vartheta_i^j(s^{t-1})\left[q_j(s^t) + p_j(s^t)d_j(s^t)\right] + \sum_{j=1}^{2} p_j(s^t) b_i^j(s^{t-1}) \geq 0, \quad \text{for} \quad s^{t-1}, s^t \in \mathbb{S}^T.
\end{aligned}$$

For all $T \geq 1$, in equilibria, we know that the consumptions lie in the interval $[0, \bar{e}]$ where $\bar{e} \equiv \max_{s \in \mathbb{S}} \{e_1(s) + e_2(s)\}$ and we know that by nonnegativity and commodity market clearing

$$0 \leq c_i^j(s^t) \leq e_j(s^t) \leq \bar{e}, \quad \text{for} \ \ s^t \in \mathbb{S}^T, \tag{35}$$

and also by short-selling constraint and equity market clearing

$$0 \leq \vartheta_j(s^t) \leq 1, \quad \text{for} \ \ s^t \in \mathbb{S}^T. \tag{36}$$

By the debt ceiling and bond market clearing, we know that

$$-\overline{D}\bar{e} \leq b_i^j(s^t) \leq \overline{D}\bar{e} \quad \text{for} \ \ s^t \in \mathbb{S}^T. \tag{37}$$

The intra-temporal Euler equations must hold in equilibria,

$$\frac{c_1^1(s^t)}{c_1^2(s^t)} = \left[p_1(s^t)\frac{1-s}{s}\right]^{\frac{1}{\rho-1}}, \tag{38}$$

and

$$\frac{e_1(s^t) - c_1^1(s^t)}{e_2(s^t) - c_1^2(s^t)} = \left[p_1(s^t)\frac{s}{1-s}\right]^{\frac{1}{\rho-1}}. \tag{39}$$

Thus, combining (38) and (39), we have

$$e_1(s^t) - e_2(s^t)\left[p_1(s^t)\frac{s}{1-s}\right]^{\frac{1}{\rho-1}} = c_1^2(s^t)p_1(s^t)^{\frac{1}{\rho-1}}\left\{\left(\frac{1-s}{s}\right)^{\frac{1}{\rho-1}} - \left(\frac{s}{1-s}\right)^{\frac{1}{\rho-1}}\right\}. \tag{40}$$

On the one hand, because $c_1^1(s^t) \geq 0$ and $\rho \leq 1$, then

$$p_1(s^t) \geq \frac{1-s}{s}\left[\frac{e_1(s^t)}{e_2(s^t)}\right]^{\frac{1}{1-\rho}} \geq \frac{1-s}{s}\left(\frac{1}{\kappa}\right)^{\frac{1}{1-\rho}} \equiv \underline{P}_1, \tag{41}$$

with $\kappa \equiv \max_{s\in\mathbb{S},i_1,i_2=1,2}\frac{e_{i_1}(s)}{e_{i_2}(s)} > 1$. And, on the other hand, because $c_1^1(s^t) \leq e_1(s^t)$, then

$$p_1(s^t) \leq \frac{s}{1-s}\kappa^{\frac{1}{\rho-1}} \equiv \overline{P}_1. \tag{42}$$

Now, we consider the prices of bonds. According to Santos and Woodford (1997), if aggregate endowment is bounded away from zero, then any stationary and recursive preference ordering does satisfy the form of impatience that for each agent $i \in I$, there exist $K > 0$ and $0 \leq \delta < 1$ such that for every $s^t \in \mathbb{S}$,

$$\Big((c_i^1(s^t), c_i^2(s^t) + Ke_1(s^t)), (\delta c_{i,+}^1(s^t), \delta c_{i,+}^2(s^t))\Big) \succ^i \Big((c_i^1(s^t), c_i^2(s^t)), (c_{i,+}^1(s^t), c_{i,+}^2(s^t))\Big)$$

for all consumption plans satisfying $c_i^j(s^t) \leq e_j(s^t)$ for all $s^t \in \mathbb{S}$ and $c_{i,+}^j(s^t)$ represents the consumption of agent i for goods j over all remaining nodes; i.e. $s^r \in \mathbb{S}$ such that $s^r \succcurlyeq s^t$ and $r > t$. It is obvious that in our economy for a given price system $\mathcal{P}^T$, if a consumption plan c can be supported by an initial wealth W, then the consumption plan δc can be supported by the initial wealth δW for any constant $\delta \in (0,1)$. Thus, we know that for each agent $i \in I$, there exist a $K > 0$ and $0 \leq \delta < 1$ such that for every $s^t \in \mathbb{S}$,

$$\Big((c_i^1(s^t), c_i^2(s^t) + Ke_1(s^t)), (\delta W_i(s^{t+1}) \,:\, s^{t+1} \succcurlyeq s^t)\Big) \succ^i \Big((c_i^1(s^t), c_i^2(s^t)), (W_i(s^{t+1}) \,:\, s^{t+1} \succcurlyeq s^t)\Big)$$

for all current consumption satisfying $c_i^j(s^t) \leq e_j(s^t)$ for all $s^t \in \mathbb{S}$ and wealth in the beginning of the next period satisfying $W_i(s^{t+1}) \leq \sum_{i=1}^2 p_i(s^{t+1})e_i(s^{t+1}) + q_i(s^{t+1})$ for all $s^t \in \mathbb{S}$. Let $Q_2^b \equiv \frac{K\overline{e}}{w_m(1-\delta)}$, and are going to show that the bond price $q_2^b(s^t)$ cannot be higher than Q_2^b by contradiction. Suppose in an equilibrium, there is a note s^t such that $q_2^b(s^t) > Q_2^b$ in an equilibrium of the T-truncated economy. At this node s^t, there must be one agent who is not borrowing in net position, and hence her wealth in the state $s^{t+1} \succcurlyeq s^t$ in the next period is at least w_m. Let's just assume she is agent 1, without loss of any generosity. Suppose her current consumption and next period's wealth plan is $\big((c_i^1(s^t), c_i^2(s^t)), (W_i(s^{t+1}) \,:\, s^{t+1} \succcurlyeq s^t)\big)$. If the agent 1 sells $w_m(1-\delta)$ unit of bond 2 at the node s^t (i.e. borrow $w_m(1-\delta)$ unit more bond 2), she could gain at least K amount of proceeds and then use all of the proceeds to buy at least $Ke_2(s^t)$ units of commodity 2 which are consumed at s^t. However, this selling of bond 2 makes her wealth plan in the next period is not lower than $\big(\delta W_i(s^{t+1}) \,:\, s^{t+1} \succcurlyeq s^t\big)$. Therefore, the new plan strictly preferred relative to the original plan and at the same time the new plan is in the budget constraint given the price system, which contradicts with the agent optimization condition for general equilibrium. Similarly, we can show that there is a large constant $Q_1^b < +\infty$ such that the equilibrium price of bond 1 satisfies $q_1^b(s^t) \leq Q_1^b$ for all $s^t \in \mathbb{S}^T$ in the T-truncated economy and all $T \geq 1$.

Now, let's consider the equity prices. For all $T \geq 1$, the agent i's value function at each node $s^t \in \mathbb{S}^T$ is upper bounded by $\frac{\overline{e}^{1-\psi_i^{-1}}}{1-\psi_i^{-1}} + U_i(\overline{e}, \overline{e}, \cdots)$, where $U_i(\overline{e}, \overline{e}, \cdots)$ denotes the value function for the consumption plan of consuming constant $\bar{e}$ of both commodities over the infinite-horizon tree $\mathbb{S}$. It is easy to get $U_i(\bar{e},\bar{e},\cdots) = \frac{1}{1-\beta}\frac{\bar{e}^{1-\psi_i^{-1}}}{1-\psi_i^{-1}}$. Therefore, for any consumption plan $\left((c_i^1(s^t),c_i^2(s^t)),(c_{i,+}^1(s^t),c_{i,+}^2(s^t))\right)$ such that $c_i^j(s^r) \leq e_j(s^r)$ for all $i,j=1,2$ and $s^r \succcurlyeq s^t$, we have

$$U_i\left((c_i^1(s^t),c_i^2(s^t)),(c_{i,+}^1(s^t),c_{i,+}^2(s^t))\right) \leq \frac{2}{1-\beta}\frac{\bar{e}^{1-\psi_i^{-1}}}{1-\psi^{-1}}, \text{ for } i=1,2.$$

Due to the assumption that $\psi_i \geq 1$, there exists large constant K such that for $i=1,2$,

$$\frac{K^{1-\psi_i^{-1}}}{1-\psi_i^{-1}} \geq \frac{2}{1-\beta}\frac{\bar{e}^{1-\psi_i^{-1}}}{1-\psi^{-1}}.$$

Thus, we know that for each agent $i \in I$, there exist a $K>0$ such that for every $s^t \in \mathbb{S}$,

$$\left((c_i^1(s^t)+K,c_i^2(s^t)+K),(\underbrace{0,0,\cdots,0}_{|\mathbb{S}|})\right) \succ^i \left((c_i^1(s^t),c_i^2(s^t)),(W_i(s^{t+1}) \;:\; s^{t+1} \succcurlyeq s^t)\right)$$

for all current consumption satisfying $c_i^j(s^t) \leq e_j(s^t)$ for all $s^t \in \mathbb{S}$ and wealth in the beginning of the next period satisfying $W_i(s^{t+1}) \leq \sum_{i=1}^2 p_i(s^{t+1})e_i(s^{t+1}) + q_i(s^{t+1})$ for all $s^t \in \mathbb{S}$. We define a constant $Q_2 \equiv 4\max\left\{K(1+\overline{P}_1),(Q_1^b+Q_2^b)\overline{D}\bar{e}\right\}$ and show that the equity prices are uniformly bounded from above by this large constant by contradiction. Suppose that there exists a node $s^t \in \mathbb{S}^T$ such that $q_2(s^t) > Q_2$ in a T-truncated equilibrium. There must one agent whose position on equity 2 is no less than $1/2$ in a equilibrium. Without loss of generosity, we assume that the agent 1 holds no less than $1/2$ of equity 2. If agent 1 sells $1/4$ shares of equity 2 and consumes the proceeds for K units of goods 1 and K units of goods 2, then the new plan strictly preferred relative to the original plan and at the same time the new plan is in the budget constraint given the price system, which contradicts with the agent optimization condition for general equilibrium. Similarly, we can show that there is a large constant $Q_1 < +\infty$ such that the equilibrium price of equity 1 satisfies $q_1(s^t) \leq Q_1$ for all $s^t \in \mathbb{S}^T$ in an equilibrium of the T-truncated economy and all $T \geq 1$.

Therefore, we have shown that in equilibria of all T-truncated economies with $T \geq 1$ uniformly lie within a bounded rectangular area, denoted as $\mathcal{Y}^*$.

Now, we show that competitive equilibria exist for all $T \geq 1$. For the purpose of showing equilibrium existence, we change the price normalization following Kubler and Schmedders (2003). That is, instead of setting the price of consumption commodity 2 at every node $s^t \in \mathbb{S}^T$ to be one, we assume the prices $\mathcal{P}^{\mathbb{S}^T}(s^t) := \left\{p_i(s^t),q_i(s^t),q_i^b(s^t)\right\}_{i=1,2}$ at each node s^t to lie in the unit simplex Δ, i.e. $\sum_{i=1}^2 p_i(s^t) + \sum_{i=1}^2 q_i(s^t) + \sum_{i=1}^2 q_i^b(s^t) = 1$ and every price is nonnegative. We define the truncated budget constraint to imposing the uniform bounds for the equilibria if they exist. We construct truncated budget sets in this economy by adding extra bounds on the allocations and holdings, where the truncation will not affect the equilibria under portfolio constraints. More precisely, we define the truncated budget set by, for $i=1,2$,

$$\overline{\mathbb{B}}_{i,\mathbb{S}^T}(\mathcal{P}^{\mathbb{S}^T}) = \mathbb{A}_{i,\mathbb{S}^T} \cap \mathbb{B}_{i,\mathbb{S}^T}(\mathcal{P}^{\mathbb{S}^T}),$$

where $\mathbb{A}_{i,\mathbb{S}^T}(\mathcal{P}^{\mathbb{S}^T})$ imposes the uniform bounds on allocation and portfolio defined as

$$\mathbb{A}_{i,\mathbb{S}^T} \equiv \left\{0 \leq \vartheta_i^j(s^t) \leq 1,\ -\overline{D}\bar{e} \leq b_i^j(s^t) \leq \overline{D}\bar{e}\ ,\ 0 \leq c_i^j(s^t) \leq \bar{e}, \text{ for } s^{t-1},s^t \in \mathbb{S}^T \text{ and } j=1,2\right\}.$$

Based on the truncated budget constraint, we define the truncated demand correspondences which would be enough for our analysis. More precisely, we denote

$$\sigma_{i,T}(\mathcal{P}^{\mathbb{S}^T}) \equiv \underset{(\mathcal{C}_i^{\mathbb{S}^T}, \mathcal{A}_i^{\mathbb{S}^T}) \in \overline{\mathbb{B}}_{i,\mathbb{S}^T}(\mathcal{P}^{\mathbb{S}^T})}{\operatorname{Arg\,max}} U_i(\mathcal{C}_i^{\mathbb{S}^T}). \tag{43}$$

Note that truncated demand exists at every price system $\mathcal{P}^{\mathbb{S}^T}$ because the equity holdings are lower bounded and the bond holdings are bounded. Absent such bounds, demand correspondence could be empty at some prices.

Denote the demand correspondence component at the node s^t to be $\sigma_{i,T}(\mathcal{P}^{\mathbb{S}^T}; s^t)$ and the aggregate excess demand at the node s^t is

$$\Sigma_T(\mathcal{P}^{\mathbb{S}^T}) \equiv \sum_{i=1}^{2} \sigma_{i,T}(\mathcal{P}^{\mathbb{S}^T}; s^t) - \left(e_1(s^t), e_2(s^t), 1, 1, 0, 0\right), \tag{44}$$

and define the excess demand of the T-truncated economy to be

$$\Sigma_T(\mathcal{P}^{\mathbb{S}^T}) \equiv \Pi_{s^t \in \mathbb{S}^T} \Sigma_T(\mathcal{P}^{\mathbb{S}^T}; s^t). \tag{45}$$

It's easy to check that $\Sigma_T(\mathcal{P}^{\mathbb{S}^T})$ is nonempty (because $\sigma_{i,T}(\mathcal{P}^{\mathbb{S}^T})$ is nonempty), compact-valued (because U_i is continuous), convex-valued (because U_i is quasi-concave) and upper hemi-continuous. Also, it is obvious that $\Sigma_T(\mathcal{P}^{\mathbb{S}^T})$ is uniformly bounded, because consumptions and asset holdings are all uniformly bounded in the truncated budget sets $\overline{\mathbb{B}}_{i,\mathbb{S}^T}(\mathcal{P}^{\mathbb{S}^T})$. That is, there exists $R > 0$ such that for all $\mathcal{P}^{\mathbb{S}^T} \in \Delta^{|\mathbb{S}^T|}$ it holds that

$$\Sigma_T(\mathcal{P}^{\mathbb{S}^T}) \subset [-R, R]^{|\mathbb{S}^T| \times I(J+E+B)}. \tag{46}$$

We further define the truncated space of endogenous variables $\left\{ (c_i^1, c_i^2)_{i=1,2}, (\vartheta_i^1, \vartheta_i^2, b_i^1, b_i^2)_{i=1,2} \right\}$

$$\mathcal{Y}(s^t) \equiv \left\{ y \in \mathbb{R}^{I(J+E+B)} \ : \ ||y|| \leq R \right\}. \tag{47}$$

We first define the correspondence

$$P_T(\cdot; s^t) \ : \ \mathcal{Y}(s^t) \rightrightarrows \Delta \tag{48}$$

such that

$$P_T(y; s^t) \equiv \underset{\mathcal{P} \in \Delta}{\operatorname{Arg\,max}}\, \mathcal{P} \cdot y. \tag{49}$$

It's obvious that $P_T(\cdot; s^t)$ is nonempty, compact-valued, convex-valued, and upper hemi-continuous correspondence. Now, we define the correspondence

$$F_T(\cdot, \cdot; s^t) \ : \ \Delta \times \mathcal{Y}(s^t) \rightrightarrows \Delta \times \mathcal{Y}(s^t) \tag{50}$$

such that

$$F_T(\mathcal{P}, y; s^t) = P_T(y; s^t) \times \Sigma_T(\mathcal{P}; s^t) \tag{51}$$

The product correspondence $F_T \ : \ \Delta^{|\mathbb{S}^T|} \times \Pi_{s^t \in \mathbb{S}^T} \mathcal{Y}(s^t)$ is defined as

$$F_T(\mathcal{P}^{\mathbb{S}^T}, y) = \Pi_{s^t \in \mathbb{S}^T} F_T\left(\mathcal{P}^{\mathbb{S}^T}(s^t), y(s^t); s^t\right). \tag{52}$$

It is obvious that F_T is nonempty, compact-valued, convex-valued, and upper hemi-continuous correspondence. Therefore, by Kakutani Theorem, we know that F_T has fixed point. We denote the collection of fixed points to be G_T.

We shall show that every fixed point $(\mathcal{P}^{\mathbb{S}^T}, y^{\mathbb{S}^T}) \in G_T$ constitutes an equilibrium for the T-truncated economy.

Equivalently, we shall show that $\forall\ (\mathcal{P}^{\mathbb{S}^T}, y^{\mathbb{S}^T}) \in G_T$,

$$y^{\mathbb{S}^T} \equiv 0 \quad \text{, and} \quad \mathcal{P}^{\mathbb{S}^T} >> 0^8. \tag{53}$$

Our plan is to prove $y^{\mathbb{S}^T}(s^t) \equiv 0$ for all $s^t \in \mathbb{S}^T$ by induction first, and then show the positiveness of prices. At the initial node s^0, because of local non-satiation, we know that agent i's budget equation at node s^t is then

$$\begin{aligned}\sum_{j=1}^{2} p_j(s^0)c_i^j(s^0) + \sum_{j=1}^{2} q_j(s^0)\vartheta_i^j(s^0) + \sum_{j=1}^{2} q_j^b(s^0)b_i^j(s^0)\\ - p_i(s^0)w_i(s^0) - \sum_{j=1}^{2} \vartheta_i^j(s^{-1})(q_j(s^0) + p_j(s^0)d_j(s^0)) - \sum_{j=1}^{2} b_i^j(s^{-1})p_j(s^0) = 0,\end{aligned} \tag{54}$$

which is due to the assumption that $\sum_{i=1}^{2} \vartheta_i^j(s^{-1}) = 1$ and $\sum_{i=1}^{2} b_i^j(s^{-1}) = 0$ for $j = 1, 2$. We sum over all agents and get

$$\sum_{j=1}^{2} p_j(s^0)\left[\sum_{i=1}^{2} c_i^j(s^0) - e_j(s^0)\right] + \sum_{j=1}^{2} q_j(s^0)\left[\sum_{i=1}^{2} \vartheta_i^j(s^0) - 1\right] + \sum_{j=1}^{2} q_j^b(s^0)\left[\sum_{i=1}^{2} b_i^j(s^0)\right] = 0.$$

This implies that

$$0 = \max_{\mathcal{P}\in\Delta} \mathcal{P}\cdot y^{\mathbb{S}^T}(s^0). \tag{55}$$

Suppose that there is positive excess demand in some market at the node s^0. Without loss of generality, we assume the largest excess demand is in the market of commodity 1. Then, the optimal solution for maximization problem (55) at the node s^0 would be to set $p_1(s^0) = 1$ and $p_2(s^0) = q_1(s^0) = q_2(s^0) = q_1^b(s^0) = q_2^b(s^0) = 0$. However, this leads to a positive value which contradicts with (55). On the other hand, suppose that there is negative excess demand in some market. Without loss of generality, we assume that the most negative excess demand is in the market of commodity 1. In this case, the price of commodity 1 must be zero, i.e. $p_1(s^0) = 0$, in order to make $\mathcal{P}^{\mathbb{S}^T}(s^0)$ to be the solution to (55). With zero price of commodity 1, the excess demand of commodity 1 should be positive for two agents because of monotonicity of preference, which is contradictory.

Suppose that $y^{\mathbb{S}^T}(s^t) \equiv 0$. For any node $s^{t+1} \succcurlyeq s^t$, because of local non-satiation, we know that agent i's budget equation at node s^{t+1} is then

$$\begin{aligned}\sum_{j=1}^{2} p_j(s^{t+1})c_i^j(s^{t+1}) + \sum_{j=1}^{2} q_j(s^{t+1})\vartheta_i^j(s^{t+1}) + \sum_{j=1}^{2} q_j^b(s^{t+1})b_i^j(s^{t+1})\\ - p_i(s^{t+1})w_i(s^{t+1}) - \sum_{j=1}^{2} \vartheta_i^j(s^t)(q_j(s^{t+1}) + p_j(s^{t+1})d_j(s^{t+1})) - \sum_{j=1}^{2} b_i^j(s^t)p_j(s^{t+1}) = 0,\end{aligned} \tag{56}$$

which is due to the assumption that $\sum_{i=1}^{2} \vartheta_i^j(s^t) = 1$ and $\sum_{i=1}^{2} b_i^j(s^t) = 0$ for $j = 1, 2$. We sum over all agents and get

$$\sum_{j=1}^{2} p_j(s^{t+1})\left[\sum_{i=1}^{2} c_i^j(s^{t+1}) - e_j(s^{t+1})\right] + \sum_{j=1}^{2} q_j(s^{t+1})\left[\sum_{i=1}^{2} \vartheta_i^j(s^{t+1}) - 1\right] + \sum_{j=1}^{2} q_j^b(s^{t+1})\left[\sum_{i=1}^{2} b_i^j(s^{t+1})\right] = 0.$$

This implies that

$$0 = \max_{\mathcal{P}\in\Delta} \mathcal{P}\cdot y^{\mathbb{S}^T}(s^{t+1}). \tag{57}$$

[8] This means that every element of $\mathcal{P}^{\mathbb{S}^T}$ is positive.

and

$$\pi_t W_{2,t}^* = \mathbb{E}_t\left[\sum_{\tau\geq t}\pi_\tau\left(p_{1,\tau}c_{2,\tau}^1 + c_{2,\tau}^2\right)\right] \tag{64}$$

From (60), (61) and (63), we have

$$\pi_t W_{1,t}^* = \lambda\mathbb{E}_t\left[\sum_{\tau\geq t}\beta^\tau\frac{1}{p_{1,t}}\frac{s\rho(c_{1,\tau}^1)^{\rho-1}}{s(c_{1,\tau}^1)^\rho + (1-s)(c_{1,\tau}^2)^\rho}(p_{1,\tau}c_{1,\tau}^1 + c_{1,\tau}^2)\right] = \frac{\lambda\rho}{1-\beta}. \tag{65}$$

And, similarly, we have

$$\pi_t W_{2,t}^* = (1-\lambda)\mathbb{E}_t\left[\sum_{\tau\geq t}\beta^\tau\frac{1}{p_{1,t}}\frac{(1-s)\rho(c_{2,\tau}^1)^{\rho-1}}{(1-s)(c_{2,\tau}^1)^\rho + s(c_{2,\tau}^2)^\rho}(p_{1,\tau}c_{2,\tau}^1 + c_{2,\tau}^2)\right] = \frac{(1-\lambda)\rho}{1-\beta}. \tag{66}$$

Thus, the wealth ratio is equal to the Pareto weight

$$\lambda = \frac{W_{1,t}^*}{W_{1,t}^* + W_{2,t}^*}. \tag{67}$$

The coincides above has a strong implication that the total wealth share λ is constant over time in the equilibrium. Also, we have

$$W_{1,t}^* = \frac{1}{1-\beta}\left(p_{1,t}c_{1,t}^1 + c_{1,t}^2\right), \tag{68}$$

and

$$W_{2,t}^* = \frac{1}{1-\beta}\left(p_{1,t}c_{2,t}^1 + c_{2,t}^2\right). \tag{69}$$

So far, we have only assumed log utility and complete market. The consumption policies are characterized by the consumption shares $\nu_{i,t}$ with $i = 1, 2$ where $c_1^i = \nu_{i,t}e_i$

$$\left(\frac{s}{1-s}\right)^{\frac{2}{\rho-1}} = \frac{1/\nu_{1,t} - 1}{1/\nu_{2,t} - 1} \tag{70}$$

and

$$\lambda = \frac{\nu_{1,t}e_{1,t}\frac{s}{1-s}\left(\frac{\nu_{1,t}}{\nu_{2,t}}\right)^{\rho-1}\left(\frac{e_{1,t}}{e_{2,t}}\right)^{\rho-1} + \nu_{2,t}e_{2,t}}{e_{1,t}\frac{s}{1-s}\left(\frac{\nu_{1,t}}{\nu_{2,t}}\right)^{\rho-1}\left(\frac{e_{1,t}}{e_{2,t}}\right)^{\rho-1} + e_{2,t}} = \frac{\nu_{1,t}\frac{s}{1-s}\left(\frac{\nu_{1,t}}{\nu_{2,t}}\right)^{\rho-1}\left(\frac{e_{1,t}}{e_{2,t}}\right)^{\rho} + \nu_{2,t}}{\frac{s}{1-s}\left(\frac{\nu_{1,t}}{\nu_{2,t}}\right)^{\rho-1}\left(\frac{e_{1,t}}{e_{2,t}}\right)^{\rho} + 1}. \tag{71}$$

Thus, in general, the first-best consumption plans (i.e. $\nu_{i,t}$) depend on the parameters ρ and s, as well as the total wealth share λ and the output ratio $e_{2,t}/e_{1,t}$. We consider two special cases where the consumption shares $\nu_{i,t}$ are constant over time and hence facilitates analytical solutions. One example is the well-known Cole-Obstfeld Economy[9], and the other is the Symmetric Economy.

[9] In their classic analysis of the irrelevance of asset markets for international risk sharing, Cole and Obstfeld (1991) show that in an open economy with two differentiated goods, agents with logarithmic preferences and Cobb-Douglas aggregator, and no trade costs, the central-planners allocation can be achieved even without trade in asset markets. This occurs because the endogenous response of the Term of Trade to supply shocks to the two goods is sufficient to implement the international wealth transfers that support the central planners consumption allocation. As is well known, the Cole and Obstfeld equilibrium features: perfectly correlated Home and Foreign stock markets, symmetric aggregate stock market portfolio holdings, zero holdings of risk-free bonds, equal consumption state by state, zero NX, and indeterminate NFA and CA. The exchange rate is either constant ($s = 0.5$) or positively related to the Term of Trade ($s > 0.5$).

D.1 Cole and Obstfeld Economy

Based on the two assumptions in the beginning of Appendix D, we further assume that the aggregator is Cobb-Douglas (i.e. $\rho = 0$) and the equity leverage ratio coefficient is zero (i.e. $\varrho = 0$) in our model. This economy is effectively the Cole-Obstfeld economy. The solution to equations (70) and (71) are simply

$$\nu_{1,t} \equiv \nu_1 \equiv \frac{1}{1 + \frac{1-s}{s}\frac{1-\lambda}{\lambda}} \quad \text{and} \quad \nu_{2,t} \equiv \nu_2 \equiv \frac{1}{1 + \frac{s}{1-s}\frac{1-\lambda}{\lambda}}. \tag{72}$$

Thus, the optimal consumptions are

$$c^1_{1,t} = \nu_1 e_{1,t},\ c^2_{1,t} = \nu_2 e_{2,t}, \tag{73}$$

$$c^1_{2,t} = (1-\nu_1)e_{1,t},\ c^2_{2,t} = (1-\nu_2)e_{2,t}. \tag{74}$$

The Term of Trade is

$$p_{1,t} = A\frac{e_{2,t}}{e_{1,t}}, \quad \text{with} \quad A \equiv \frac{s}{1-s}\frac{\nu_2}{\nu_1}, \tag{75}$$

and the real exchange rate is

$$Q_t \equiv p_{1,t}^{2s-1}. \tag{76}$$

Because the wealth share is constant, based on the Proposition 1, the Euler equation for equity prices are

$$\begin{aligned} e_2(s^t)^{-1}\left[q_1(s_t)e(s^t)\right] &= \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t,s_{t+1})e_2(s^{t+1})^{-1}\left[q_1(s_{t+1})e(s^{t+1}) + p_1(s_{t+1})\bar{d}e_1(s^{t+1})\right] \\ &= \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t,s_{t+1})e_2(s^{t+1})^{-1}\left[q_1(s_{t+1})e(s^{t+1}) + A\bar{d}e_2(s^{t+1})\right] \end{aligned} \tag{77}$$

and

$$e_2(s^t)^{-1}\left[q_2(s_t)e(s^t)\right] = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t,s_{t+1})e_2(s^{t+1})^{-1}\left[q_2(s_{t+1})e(s^{t+1}) + \bar{d}e_2(s^{t+1})\right] \tag{78}$$

Because two equities have perfectly correlated dividend flows $\{A\bar{d}e_{2,t}\}_{t\geq 0}$ and $\{\bar{d}e_{2,t}\}_{t\geq 0}$, then it is straightforward to know that $q_1(s_t) \equiv Aq_2(s_t)$. Therefore, the US and ROW equities are perfectly correlated, and hence the equity holdings are indeterminate. Thus, one possible set of portfolio holdings are

$$\vartheta^1_{1,t} \equiv \frac{\nu_1 - (1-\bar{d})}{\bar{d}}, \quad \vartheta^2_{1,t} \equiv \frac{\nu_2}{\bar{d}}, \quad b^1_{1,t} \equiv b^2_{1,t} \equiv 0, \tag{79}$$

$$\vartheta^1_{2,t} \equiv \frac{1-\nu_1}{\bar{d}}, \quad \vartheta^2_{2,t} \equiv \frac{1-\nu_2-(1-\bar{d})}{\bar{d}}, \quad b^1_{2,t} \equiv b^2_{2,t} \equiv 0. \tag{80}$$

The bonds' prices are, for $i = 1, 2$,

$$e_2(s^t)^{-1}\left[q_i^b(s_t)e(s^t)\right] = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t,s_{t+1})e_2(s^{t+1})^{-1}\left[p_i(s_{t+1})e(s^{t+1})\right]. \tag{81}$$

Based on the structures of endowment processes specified in Section 3.1, the Euler equations for asset prices in (77), (78) and (81) can re-written as

$$x_{2,t}^{-1}q_1(s_t) = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t,s_{t+1})\left[x_{2,t+1}^{-1}q_1(s_{t+1}) + A\bar{d}\right] \tag{82}$$

$$x_{2,t}^{-1}q_2(s_t) = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \left[x_{2,t+1}^{-1}q_2(s_{t+1}) + \bar{d}\right] \tag{83}$$

and

$$x_{2,t}^{-1}q_i^b(s_t) = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) x_{2,t+1}^{-1} p_i(s_{t+1}). \tag{84}$$

The equity prices are solution of the S by S linear equations, while the bond prices can be directly calculated. We can see that the normalized asset prices are independent of global component and the disaster probability.

D.2 Symmetric Economy

Based on the two assumptions in the beginning of Appendix D, we further assume that the consumption share coefficient (i.e. $s = 0.5$) and the equity leverage ratio coefficient is zero (i.e. $\varrho = 0$) in our model. The solution to equations (70) and (71) are simply

$$\nu_{1,t} \equiv \nu_{2,t} \equiv \lambda. \tag{85}$$

Thus, the optimal consumptions are

$$c_{1,t}^1 = \lambda e_{1,t},\ c_{1,t}^2 = \lambda e_{2,t}, \tag{86}$$

$$c_{2,t}^1 = (1-\lambda)e_{1,t},\ c_{2,t}^2 = (1-\lambda)e_{2,t}. \tag{87}$$

The Term of Trade is

$$p_{1,t} = \left(\frac{e_{1,t}}{e_{2,t}}\right)^{\rho-1}, \tag{88}$$

and the real exchange rate is

$$Q_t \equiv 1. \tag{89}$$

Because the wealth share is constant, the Euler equation for equity prices are

$$\frac{e_2(s^t)^{\rho-1}}{e_1(s^t)^\rho + e_2(s^t)^\rho}\left[q_1(s_t)e(s^t)\right] = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{e_2(s^{t+1})^{\rho-1}}{e_1(s^{t+1})^\rho + e_2(s^{t+1})^\rho}\left\{\left[q_1(s_{t+1})e(s^{t+1})\right] + p_1(s_{t+1})\bar{d}e_1(s^{t+1})\right\}$$

and

$$\frac{e_2(s^t)^{\rho-1}}{e_1(s^t)^\rho + e_2(s^t)^\rho}\left[q_2(s_t)e(s^t)\right] = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{e_2(s^{t+1})^{\rho-1}}{e_1(s^{t+1})^\rho + e^2(s^{t+1})^\rho}\left\{\left[q_2(s_{t+1})e(s^{t+1})\right] + \bar{d}e_2(s^{t+1})\right\}$$

The Inter-temporal Euler equations above can be re-written as

$$\frac{q_1(s_t)e(s^t)}{p_1(s_t)e_1(s^t) + e_2(s^t)} = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{q_1(s_{t+1})e(s^{t+1}) + p_1(s_{t+1})\bar{d}e_1(s^{t+1})}{p_1(s_{t+1})e_1(s^{t+1}) + e_2(s^{t+1})} \tag{90}$$

and

$$\frac{q_2(s_t)e(s^t)}{p_1(s_t)e_1(s^t) + e_2(s^t)} = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{q_2(s_{t+1})e(s^{t+1}) + \bar{d}e_2(s^{t+1})}{p_1(s_{t+1})e_1(s^{t+1}) + e_2(s^{t+1})}. \tag{91}$$

The equilibrium portfolio holdings are

$$\vartheta_{1,t}^1 \equiv \frac{\lambda - (1-\bar{d})}{\bar{d}},\quad \vartheta_{1,t}^2 \equiv \frac{\lambda}{\bar{d}},\quad b_{1,t}^1 \equiv b_{1,t}^2 \equiv 0, \tag{92}$$

$$\vartheta^1_{2,t} \equiv \frac{1-\lambda}{\bar{d}}, \quad \vartheta^2_{2,t} \equiv \frac{1-\lambda-(1-\bar{d})}{\bar{d}}, \quad b^1_{2,t} \equiv b^2_{2,t} \equiv 0. \tag{93}$$

The bonds' prices Euler equations are, for $i = 1, 2$,

$$\frac{q^b_i(s_t)e(s^t)}{p_1(s_t)e_1(s^t)+e_2(s^t)} = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{p_i(s_{t+1})e(s^t)}{p_1(s_{t+1})e_1(s^{t+1})+e_2(s^{t+1})}. \tag{94}$$

Based on the structures of endowment processes specified in Section 3.1, the Euler equations for asset prices in (90), (91) and (94) can re-written as

$$\frac{q_1(s_t)}{p_1(s_t)x_{1,t}+x_{2,t}} = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{q_1(s_{t+1})+p_1(s_{t+1})\bar{d}x_{1,t+1}}{p_1(s_{t+1})x_{1,t+1}+x_{2,t+1}}, \tag{95}$$

$$\frac{q_2(s_t)}{p_1(s_t)x_{1,t}+x_{2,t}} = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{q_2(s_{t+1})+p_2(s_{t+1})\bar{d}x_{2,t+1}}{p_1(s_{t+1})x_{1,t+1}+x_{2,t+1}}, \tag{96}$$

and for $i = 1, 2$

$$\frac{q^b_i(s_t)}{p_1(s_t)x_{1,t}+x_{2,t}} = \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \frac{p_i(s_{t+1})\zeta(s_{t+1})^{-1}}{p_1(s_{t+1})x_{1,t+1}+x_{2,t+1}}, \tag{97}$$

where

$$p_1(s_t) = \left(\frac{x_{1,t}}{x_{2,t}}\right)^{\rho-1}. \tag{98}$$

So, the equity prices can be solved from the S by S linear system, and the bond prices can be directly calculated. We can see that the normalized asset prices are independent of global component and the disaster probability.

How about the value function $U_i(\lambda, s_t, e_t)$? From the definition, we know that

$$\begin{aligned} U_1(\lambda, s_t, e_t) &= \mathbb{E}_t \left\{ \sum_{\tau\geq t} \beta^{\tau-t} \frac{1}{\rho} \log\left[s(c^1_1(\lambda, s_\tau, e_\tau))^\rho + (1-s)(c^2_1(\lambda, s_\tau, e_\tau))^\rho \right] \right\} \\ &= \frac{1}{1-\beta}\log(\lambda) + \frac{1}{1-\beta}\log(e_t) + F_1(s_t), \end{aligned}$$

where

$$F_1(s_t) = \sum_{\tau>t} \beta^{\tau-t} \mathbb{E}_t \left[\log\left(\frac{e_\tau}{e_t}\right)\right] + \frac{1}{\rho}\sum_{\tau\geq t} \beta^{\tau-t}\mathbb{E}_t \left[\log(sx^\rho_{1,t} + (1-s)x^\rho_{2,t}) \right].$$

Plugging back into the recursive formulation of the value function, we have for $s_t, s_{t+1} \in \mathcal{S}$,

$$F_1(s_t) = \frac{1}{\rho}\log(sx^\rho_{1,t} + (1-s)x^\rho_{2,t}) + \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1}) \left[F_1(s_{t+1}) + \frac{1}{1-\beta}\log(\zeta_{t+1}) \right]. \tag{99}$$

Thus, the function $F_1(s)$ can be solved out from the S by S linear system.

$$\begin{aligned} U_2(\lambda, s_t, e_t) &= \mathbb{E}_t \left\{ \sum_{\tau\geq t} \beta^{\tau-t} \frac{1}{\rho} \log\left[(1-s)(c^1_2(\lambda, s_\tau, e_\tau))^\rho + s(c^2_2(\lambda, s_\tau, e_\tau))^\rho \right] \right\} \\ &= \frac{1}{1-\beta}\log(1-\lambda) + \frac{1}{1-\beta}\log(e_t) + F_2(s_t), \end{aligned}$$

where

$$F_2(s_t) = \sum_{\tau>t} \beta^{\tau-t}\mathbb{E}_t\left[\log\left(\frac{e_\tau}{e_t}\right)\right] + \frac{1}{\rho}\sum_{\tau\geq t}\beta^{\tau-t}\mathbb{E}_t\left[\log((1-s)x_{1,t}^{\rho} + s x_{2,t}^{\rho})\right].$$

Plugging back into the recursive formulation of the value function, we have for $s_t, s_{t+1} \in \mathcal{S}$,

$$F_2(s_t) = \frac{1}{\rho}\log((1-s)x_{1,t}^{\rho} + s x_{2,t}^{\rho}) + \beta \sum_{s_{t+1}\in\mathcal{S}} P(s_t, s_{t+1})\left[F_2(s_{t+1}) + \frac{1}{1-\beta}\log(\zeta_{t+1})\right]. \tag{100}$$

Thus, the function $F_2(s)$ can be solved out from the S by S linear system.

D.3 Financial Wealth Share as Endogenous State Variable

The total wealth (including the present value of labor income) distribution serves as a natural state variable in complete market to characterize the equilibrium, as we have shown above where the first-best allocation can be achieved even in an incomplete market and more generally described in standard textbooks of complete market, such as Magill and Quinzii (2002).

However, when the market is incomplete, a natural endogenous state variable would be the financial wealth (excluding the present value of non-tradable cash flows) distribution, instead of the total wealth distribution. In our simple examples above, the financial wealth share can be expressed in terms of total wealth share, asset prices and endowments.

D.3.1 Cole-Obstfeld Economy

D.3.2 Symmetric Economy

The financial wealth share is

$$\begin{aligned}
w_t \equiv W_{1,t}/(W_{1,t}+W_{2,t}) &= \frac{\vartheta_{1,t}^1\left[q_1(s_t) + p_1(s_t)\bar{d}x_{1,t}\right] + \vartheta_{1,t}^2\left[q_2(s_t) + \bar{d}x_{2,t}\right] + p_1(s_t)(1-\bar{d})x_{1,t}}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}} \\
&= \frac{q_1(s_t)\frac{\lambda-(1-\bar{d})}{\bar{d}} + q_2(s_t)\frac{\lambda}{\bar{d}} + \lambda p_1(s_t)x_{1,t} + \lambda x_{2,t}}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}} \\
&= \frac{\lambda}{\bar{d}} - \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}} - \frac{1-\bar{d}}{\bar{d}}\frac{\lambda\left[p_1(s_t)x_{1,t}+x_{2,t}\right]}{p_1(s_t)x_{1,t}+x_{2,t}+q_1(s_t)+q_2(s_t)} \\
&= \frac{\lambda}{\bar{d}} - \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}} - \frac{1-\bar{d}}{\bar{d}}\frac{\lambda\left[p_1(s_t)x_{1,t}+x_{2,t}\right]}{p_1(s_t)x_{1,t}+x_{2,t}+\frac{\beta}{1-\beta}\bar{d}\left[p_1(s_t)x_{1,t}+x_{2,t}\right]} \\
&= \frac{\lambda}{\bar{d}} - \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}} - \frac{1-\bar{d}}{\bar{d}}\frac{\lambda}{1+\frac{\beta}{1-\beta}\bar{d}} \\
&= \frac{\lambda}{1-\beta+\beta\bar{d}} - \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}},
\end{aligned} \tag{101}$$

where (101) is due to the fact that in the Symmetric economy

$$\begin{aligned}
q_1(s_t)+q_2(s_t) &= \left[p_1(s_t)x_{1,t}+x_{2,t}\right]\mathbb{E}_t\left[\sum_{\tau>t}\beta^{\tau-t}\frac{\bar{d}p_1(s_\tau)x_{1,\tau}+\bar{d}x_{2,\tau}}{p_1(s_\tau)x_{1,\tau}+x_{2,\tau}}\right] \\
&= \left[p_1(s_t)x_{1,t}+x_{2,t}\right]\bar{d}\sum_{\tau>t}\beta^{\tau-t} = \bar{d}\frac{\beta}{1-\beta}\left[p_1(s_t)x_{1,t}+x_{2,t}\right]
\end{aligned}$$

Thus, we have

$$\lambda = (1-\beta+\beta\bar{d})\left[w_t + \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}}\right]. \tag{102}$$

And hence, we can see that the financial wealth share varies over time, though the total wealth share is constant. However, the "transition map" Ω is independent of current period state variables (w,s) and only depends on the next period's exogenous state $\tilde{s}\in\mathcal{S}$. More precisely, for any $w\in[0,1]$ and $s,\tilde{s}\in\mathcal{S}$, it holds that

$$\begin{aligned}\Omega(w,s;\tilde{s}) &= \frac{\lambda}{1-\beta+\beta\bar{d}} - \frac{1-\bar{d}}{\bar{d}}\frac{q_1(\tilde{s})}{q_1(\tilde{s})+q_2(\tilde{s})+p_1(\tilde{s})\tilde{x}_1+\tilde{x}_2}\\ &= w + \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s)}{q_1(s)+q_2(s)+p_1(s)x_1+x_2} - \frac{1-\bar{d}}{\bar{d}}\frac{q_1(\tilde{s})}{q_1(\tilde{s})+q_2(\tilde{s})+p_1(\tilde{s})\tilde{x}_1+\tilde{x}_2}\end{aligned}$$

where $p_1(s) = (x_1/x_2)^{\rho-1}$ and $q_1(s)$ and $q_2(s)$ are solutions to the S by S linear equations in (95) and (96), respectively. Therefore, by plugging (102) into the equilibrium results in Appendix D.2, we can re-express the equilibrium results in terms of the new endogenous state variable w_t. The equity holdings are

$$\vartheta^1_{1,t} = \frac{(1-\beta+\beta\bar{d})}{\bar{d}}\left[w_t + \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}}\right] - \frac{1-\bar{d}}{\bar{d}} \tag{103}$$

$$\vartheta^2_{1,t} = \frac{(1-\beta+\beta\bar{d})}{\bar{d}}\left[w_t + \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}}\right] \tag{104}$$

and

$$\vartheta^1_{2,t} = 1-\vartheta^1_{1,t}, \quad \vartheta^2_{2,t} = 1-\vartheta^2_{1,t}. \tag{105}$$

The debt holdings are

$$b^1_{1,t} \equiv b^2_{1,t} \equiv b^1_{2,t} \equiv b^2_{2,t} \equiv 0. \tag{106}$$

The normalized consumptions are

$$c^1_1(w_t,s_t) = (1-\beta+\beta\bar{d})\left[w_t + \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}}\right]x_{1,t} \tag{107}$$

$$c^2_1(w_t,s_t) = (1-\beta+\beta\bar{d})\left[w_t + \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}}\right]x_{2,t} \tag{108}$$

and

$$c^1_2(w_t,s_t) = x_{1,t} - c^1_1(w_t,s_t), \quad c^2_2(w_t,s_t) = x_{2,t} - c^2_1(w_t,s_t). \tag{109}$$

The value functions are then

$$\begin{aligned}U_1(w_t,s_t,e_t) = \frac{1}{1-\beta}\log(1-\beta+\beta\bar{d}) + \frac{1}{1-\beta}\log\left[w_t + \frac{1-\bar{d}}{\bar{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}}\right]\\ + \frac{1}{1-\beta}\log(e_t) + F_1(s_t).\end{aligned} \tag{110}$$

and

$$U_2(w_t,s_t,e_t) = \frac{1}{1-\beta}\log\left\{1-(1-\beta+\beta\overline{d})\left[w_t+\frac{1-\overline{d}}{\overline{d}}\frac{q_1(s_t)}{q_1(s_t)+q_2(s_t)+p_1(s_t)x_{1,t}+x_{2,t}}\right]\right\} \tag{111}$$
$$+\frac{1}{1-\beta}\log(e_t)+F_2(s_t).$$

E Numerical Solution

The algorithm is a time iteration algorithm.

Step 0: Select an error tolerance ϵ for the stopping criterion and a grids discretizing the financial wealth ratio endogenous state variable w on $[0,1]$. Denote the grids as $0<\overline{w}_1<\cdots<\overline{w}_N<1$. We also choose the initial guess for Π and Ω as $\hat{\Pi}_0$ and $\hat{\Omega}_0$.

Step 1: For $k=1,\cdots,K$, given piecewise-linear (or more general interpolation methods) functions $\hat{\Pi}_{k-1}$ and $\hat{\Omega}_{k-1}$, we solve out the policy functions and transition map $\hat{\Pi}_k$ and $\hat{\Omega}_k$. This is key part of time iteration algorithm.

(1) Solve out $\hat{\Pi}_k$ given $\hat{\Pi}_{k-1}$ and $\hat{\Omega}_{k-1}$:

(a) Given a grid point $\overline{w}_i$, $s\in\mathcal{S}$ and $\tilde{s}\in\mathcal{S}$, calculate $\tilde{w}$ based on $\hat{\Omega}_{k-1}$:

$$\tilde{w}=\hat{\Omega}_{k-1}(\overline{w}_i,s,\tilde{s})$$

(b) Interpolate/Extroplate the policy function $\hat{\Pi}_{k-1}$ at $\tilde{w}$ or you can also say evaluating the interpolated function $\hat{\Pi}_{k-1}$ at $\tilde{w}$.

(c) Solve out $\hat{\Pi}_k$ as current period's policies based on taking the next period's policies as those interpolated above.

(2) Update $\hat{\Omega}_k$ given $\hat{\Pi}_k$ and $\hat{\Omega}_{k-1}$:

(a) Given a grid point $\overline{w}_i$, $s\in\mathcal{S}$ and $\tilde{s}\in\mathcal{S}$, calculate $\tilde{w}$ based on $\hat{\Omega}_{k-1}$:

$$\tilde{w}=\hat{\Omega}_{k-1}(\overline{w}_i,s,\tilde{s})$$

(b) Calculate $\hat{\Omega}_k(\overline{w}_i,s,\tilde{s})$:

$$\hat{\Omega}_k(\overline{w}_i,s,\tilde{s})=\frac{p_1(\tilde{w},\tilde{s})w_1(\tilde{s})+\sum_{j=1}^{2}\vartheta_1^j(\overline{w}_i,s)(q_j(\tilde{w},\tilde{s})+p_j(\tilde{w},\tilde{s})d_j(\tilde{w},\tilde{s}))+\sum_{j=1}^{2}b_1^j(\overline{w}_i,s)p_j(\tilde{w},\tilde{s})}{\sum_{j=1}^{2}p_j(\tilde{w},\tilde{s})e_j(\tilde{s})+q_j(\tilde{w},\tilde{s})}$$

Step 2: Check stropping criterion. If

$$\max_{\overline{w}\in\overline{\mathcal{W}},z,z'\in\mathcal{Z}}\left\{|\hat{\rho}_k(w,z)-\hat{\rho}_{k-1}(w,z)|,|\hat{\Omega}_k(w,z,z')-\hat{\Omega}_{k-1}(w,z,z')|\right\}<\epsilon$$

then go to Step 3. Otherwise, $k=k+1$ and go to Step 1.

Step 3: The algorithm terminates. Set

$$\hat{\rho}=\hat{\rho}_k \text{ and } \hat{\Omega}=\hat{\Omega}_k.$$

F Additional Empirical Results

F.1 U.S. Capital Stocks

Figure 5 reports the total stocks of U.S. foreign assets and liabilities (scaled by U.S. GDP), along with the Chinn-Ito openness index.

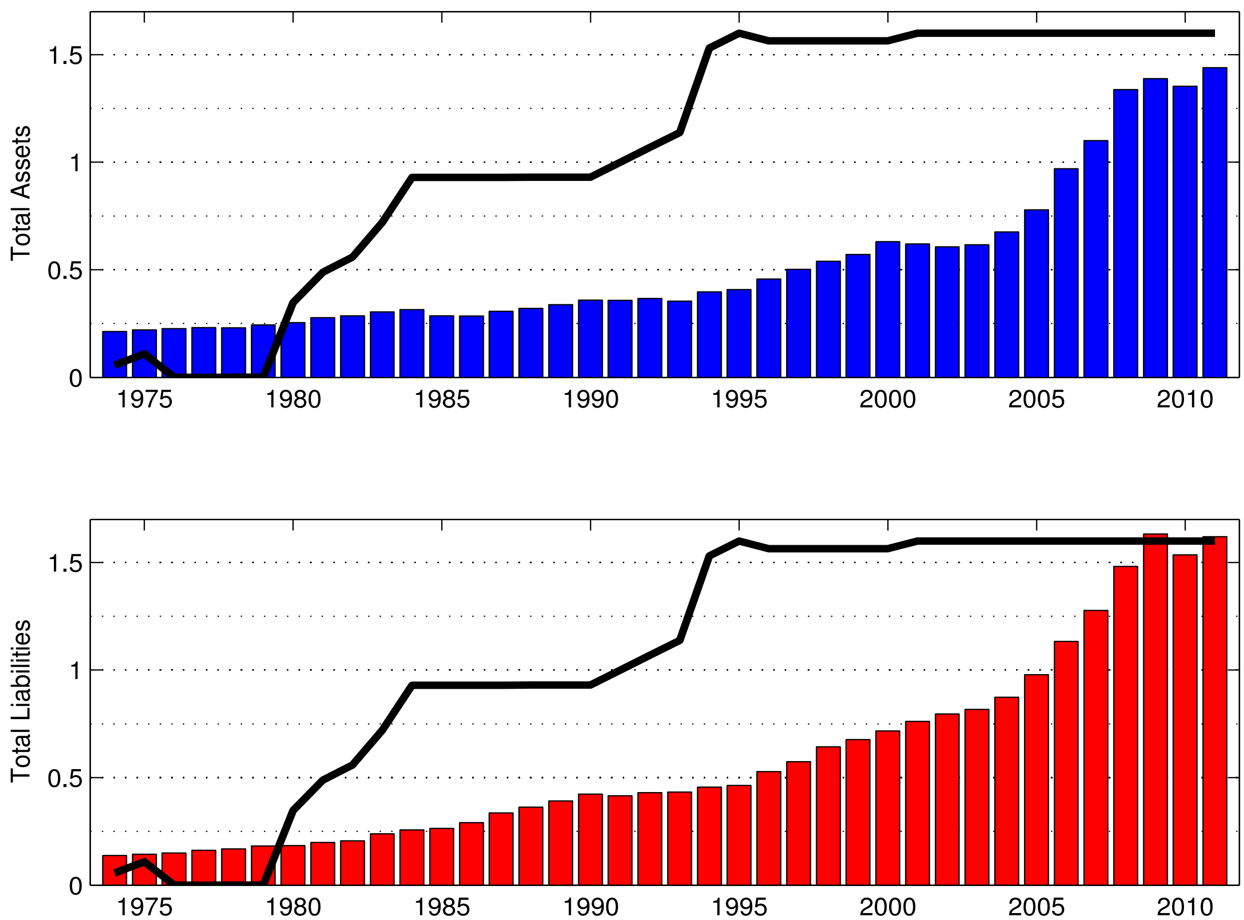

Figure 5: U.S. International Assets and Liabilities

The figure presents the total stocks of U.S. foreign assets and liabilities (scaled by U.S. GDP), along with the Chinn-Ito openness index. Data are annual, from an updated and extended version of the Lane and Milesi-Ferretti (2007) and Chinn and Ito (2006) datasets. The openness index is based on the binary dummy variables that codify the tabulation of restrictions on cross-border financial transactions reported in the IMF's Annual Report on Exchange Arrangements and Exchange Restrictions. The index is recalled to the [0–1.6] range. The sample is 1973–2010.

Using the quarterly datasets of Bertaut and Tryon (2007) and Bertaut and Judson (2014), Figure 6 reports the dynamics of the equity and bond assets and liabilities over the last twenty years. The figure presents assets on the right and liabilities on the left. The scale is the same for every subplot. The leverage position of the U.S. appears again clearly. The U.S. borrows abroad up to $ 7 trillions at the end of the sample, investing close to $ 5 trillions in foreign equity. Foreign buy U.S. equity, but only for around $ 3 trillions. The large drop in the value of the U.S. equity asset holdings during the recent crisis suggests that the U.S. holdings are risky.

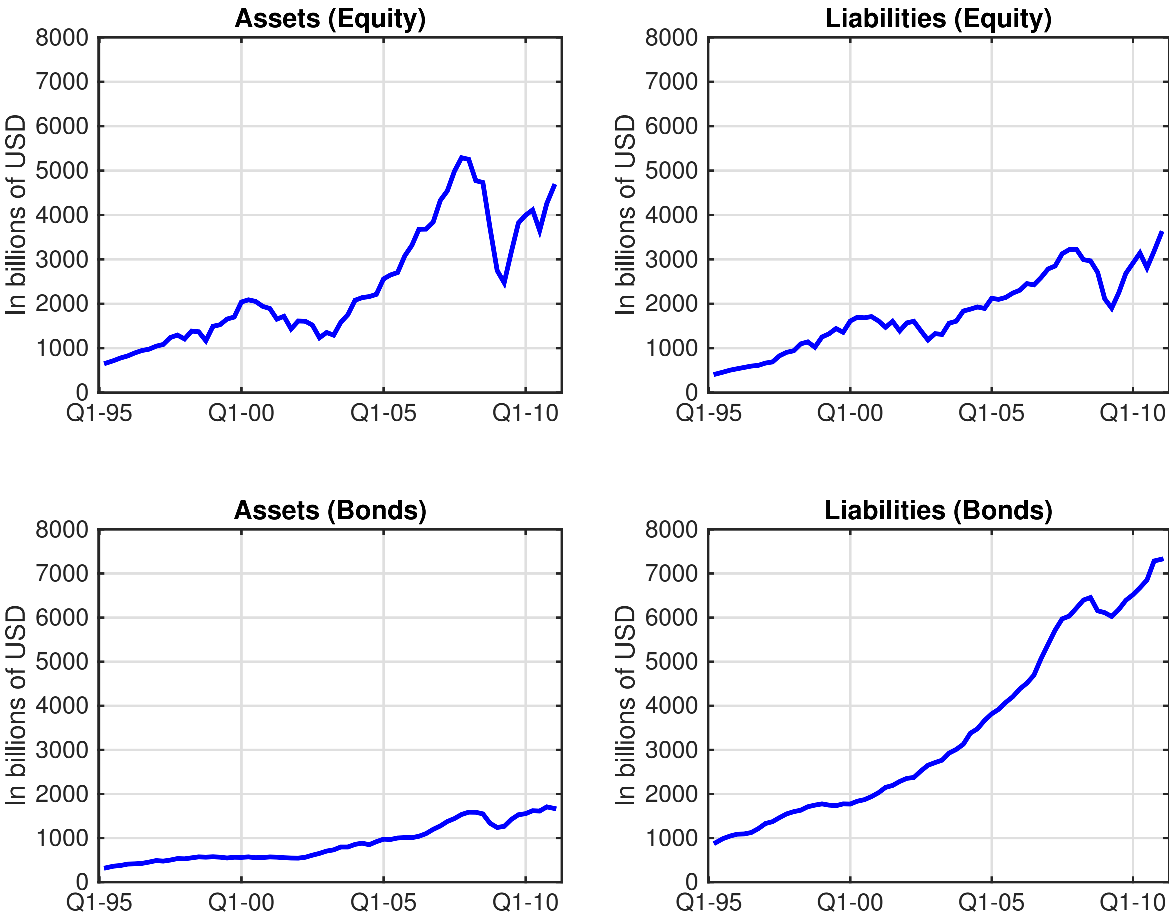

Figure 6: U.S. Assets and Liabilities

The figure presents the net foreign asset position and the sum of past current accounts (both scaled by U.S. GDP). Data are quarterly, from the Bertaut and Tryon (2007) and Bertaut and Judson (2014) datasets. The sample is 1995.I–2010.IV

F.2 U.S. Capital Flows

Figure 7 presents the quarterly dynamics of four categories of U.S. international inflows and outflows (scaled by U.S. GDP): again, debt portfolio investments, equity portfolio investments, foreign direct investments (FDI), and other investments. Positive inflows correspond to capital entering the U.S., while negative outflows correspond to capital exiting the U.S. At the end of the sample, capital inflows and outflows vary from 0 to 10 times GDP on a quarterly basis. The inflows of debt and the inflows and outflows of ”Other Investments” are the most volatile.

Table 7 reports summary statistics on disaggregated categories of capital flows: equity, foreign direct investment (FDI), debt, and other investments.

Figure 8 compares the dynamics of U.S. international capital flows (scaled by U.S. GDP) to U.S. or rest-of-the-world (ROW) GDP. The upper panel reports U.S. total inflows (blue bars) and U.S. GDP (red line), while the bottom panel reports U.S. total outflows (with a minus sign, i.e., a positive number means that capital is going out of the U.S.) and ROW GDP.

Figure 9 reports inflows and outflows over the 2007.4–2010.4 period, along with world volatility.

Table 7: U.S. International Capital Flows: Data

	Raw Data			HP-Filtered Series				
	Min	Mean	Max	Std	AC(1)	Corr US GDP	Corr ROW GDP	Corr World Vol
US Equity Portfolio Outflows	-3.55	-0.45	1.25	0.50	-0.05	-0.12	0.01	0.19
	(0.51)	(0.07)	(0.41)	(0.07)	(0.12)	(0.09)	(0.10)	(0.10)
US Equity Portfolio Inflows	-0.60	0.45	3.00	0.47	0.26	0.18	0.14	-0.17
	(0.11)	(0.08)	(0.17)	(0.07)	(0.06)	(0.10)	(0.11)	(0.08)
US FDI Outflows	-3.45	-1.11	2.06	0.60	0.19	-0.12	-0.11	0.05
	(0.10)	(0.11)	(0.70)	(0.07)	(0.06)	(0.11)	(0.11)	(0.08)
US FDI Inflows	-0.18	0.98	6.21	0.65	0.06	0.36	0.22	0.06
	(0.06)	(0.11)	(0.96)	(0.13)	(0.09)	(0.10)	(0.10)	(0.09)
US Debt Portfolio Outflows	-4.24	-0.40	3.19	0.78	0.41	0.09	0.20	0.35
	(0.55)	(0.11)	(1.12)	(0.14)	(0.10)	(0.11)	(0.14)	(0.13)
US Debt Portfolio Inflows	-0.66	2.13	10.10	1.20	0.31	0.18	0.24	-0.18
	(0.15)	(0.29)	(1.12)	(0.24)	(0.07)	(0.11)	(0.11)	(0.06)
US Other Investment Outflows	-8.61	-1.38	7.96	2.33	0.16	-0.19	-0.28	-0.12
	(0.58)	(0.27)	(1.37)	(0.27)	(0.10)	(0.09)	(0.09)	(0.13)
US Other Investment Inflows	-8.61	1.72	9.08	2.12	0.09	0.15	0.17	0.15
	(2.04)	(0.24)	(0.77)	(0.28)	(0.09)	(0.08)	(0.08)	(0.12)
US Total Capital Outflows	-16.32	-3.34	8.74	2.85	0.28	-0.17	-0.20	0.04
	(2.03)	(0.40)	(2.71)	(0.40)	(0.09)	(0.09)	(0.10)	(0.12)
US Total Capital Inflows	-3.45	5.29	21.96	2.97	0.36	0.29	0.29	0.03
	(0.89)	(0.57)	(2.60)	(0.50)	(0.07)	(0.10)	(0.11)	(0.09)

Notes: This table reports the min, mean, max, standard deviation, autocorrelation, and cross-country correlation of U.S. international capital flows in different asset classes: equity, foreign direct investment (FDI), debt, and other investments. For each asset class, the table reports the outflows and inflows from the perspective of the U.S. The last two rows correspond to the total capital outflows and inflows. The next two columns correspond to the cross-country correlation coefficients between international capital flows and U.S. or rest-of-the-world (ROW) HP-filtered GDP series. The last column corresponds to the cross-country correlation coefficients between international capital flows and the change in world equity volatility. All series are scaled by GDP. The min, mean, and max statistics are computed on raw data, while the standard deviation, autocorrelation, and cross-country correlations are computed on HP-filtered series. Standard errors are reported in parentheses; they are obtained by block-boostrapping. Data are quarterly, from the Bluedorn et al. (2013) dataset, Datastream, and the OECD. All variables are reported in percentage points, except for the autocorrelation and cross-country correlation coefficients. The sample period is 1973.1–2010.4.

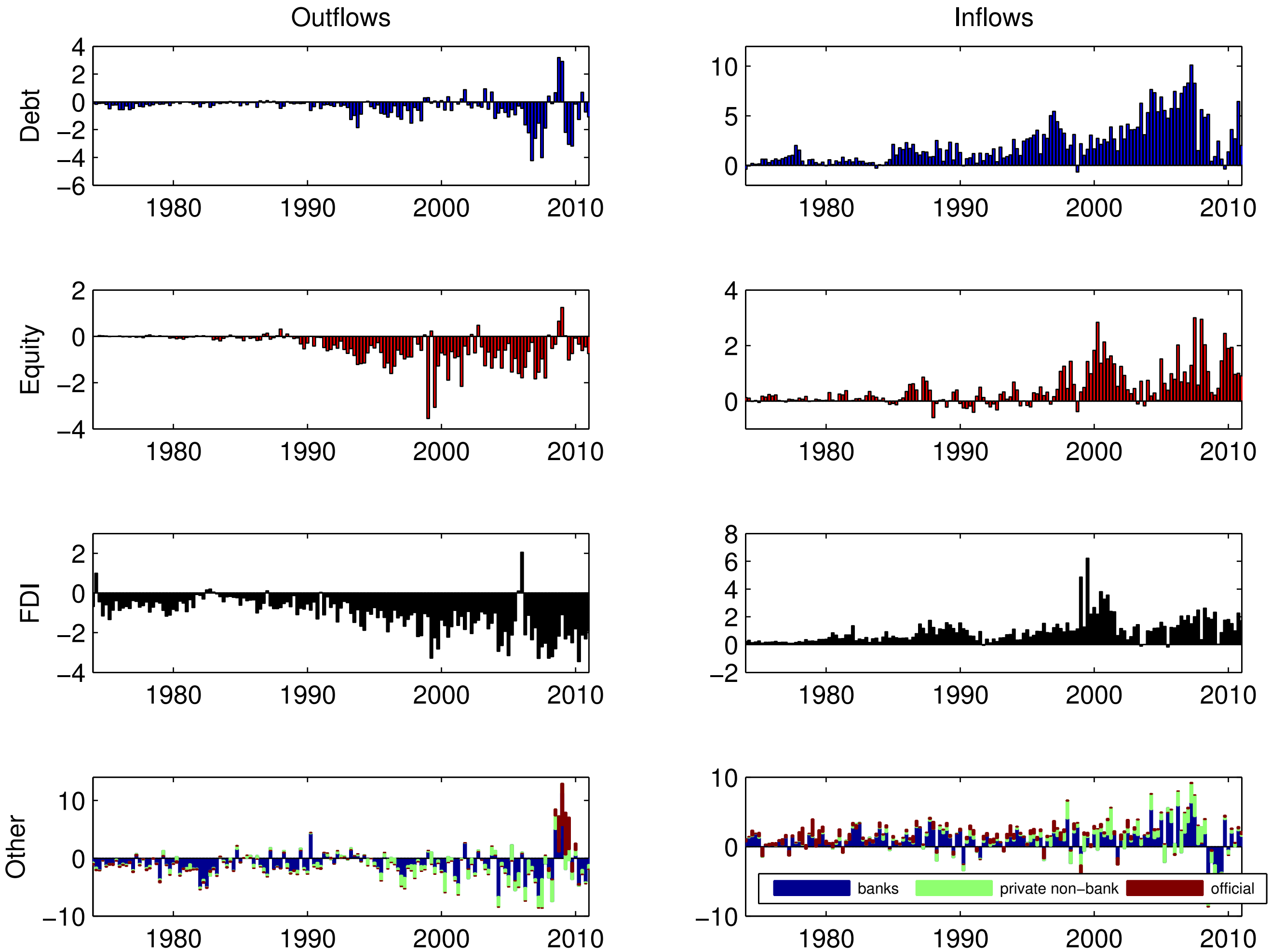

Figure 7: U.S. International Outflows and Inflows

The figure presents the dynamics of four categories of U.S. international outflows and inflows (scaled by U.S. GDP): debt portfolio investments, equity portfolio investments, foreign direct investments (FDI), and other investments. The last group is subdivided into international bank flows, official (i.e., central bank and government) flows, and other private flows. The left panel pertains to outflows while the right panel pertains to inflows. Data are quarterly, from the Bluedorn et al. (2013) dataset. The sample period is 1973.1–2010.4.

F.3 U.S. Foreign Assets and Liabilities

Figure 10 shows the large increase in the stocks of U.S. foreign assets and liabilities (scaled by world GDP) occurs across all four categories of investments reported in the balance of payments and international investments statistics: debt, equity, FDI, and other investments.

G Additional Simulation Results

We simulate an economy for 100 years at a quarterly frequency. This simulation is then repeated 30,000 times. In each simulation the model is hit with a disaster probability shock in the first quarter of year 96. The disaster probability jumps to a large value of $p_t = p_H \equiv 6\%$ and then decays back to average level according to its average

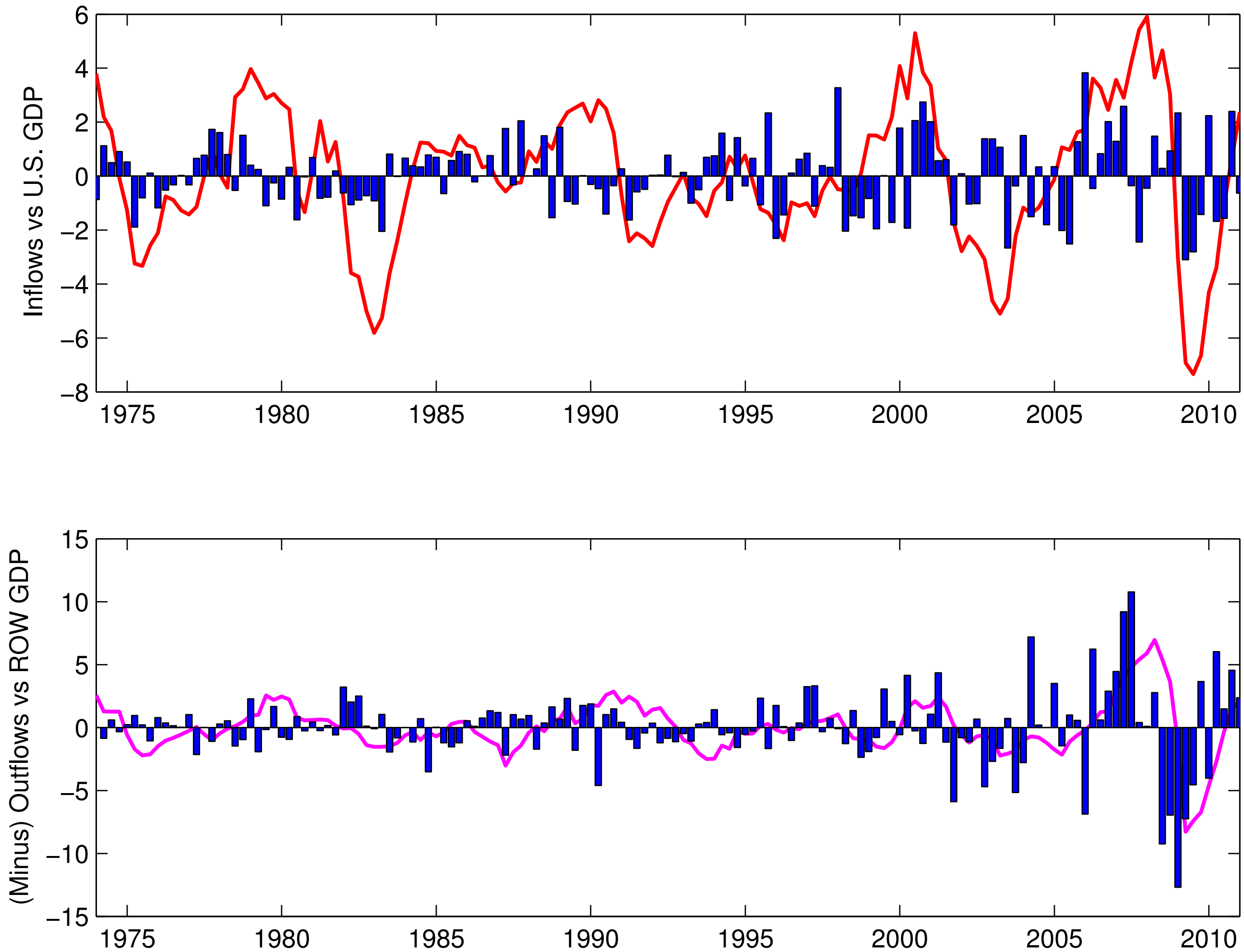

Figure 8: U.S. International Total Capital Outflows and Inflows vs. U.S./ROW GDP

The figure presents the dynamics of U.S. international capital flows (scaled by U.S. GDP) and U.S. or rest-of-the-world (ROW) GDP. All series are HP-filtered. The upper panel reports U.S. total inflows (blue bars) and U.S. GDP (red line), while the bottom panel reports U.S. total outflows (with a minus sign, i.e., a positive number means that capital is going out of the U.S.) and ROW GDP. Data are quarterly, from the Bluedorn et al. (2013) dataset and OECD. The sample period is 1973.4–2010.4.

convergence speed. The half life of the temporary risk shock is 10 quarters according to our calibration. All other shocks are randomly drawn. This experiment generates the average impact of a risk shock, where the average is taken over the distribution of aggregate and country-specific shocks.

Panel A of figure 11 reports the path of the disaster probability p_t. The shock generates a sharp spike in the average disaster probability p_t across the 30,000 simulations, which dies out with a half-life of about 6 months.

Figure 12 reports the impulse response functions of domestic and foreign asset pricing moments to a large buy temporary shock in the disaster probability.

The capital stock, scaled by GDP, is defined as:

$$CS^j_{i,t} \equiv \frac{\vartheta^j_{i,t} q_{j,t}}{Y_{i,t}}.$$

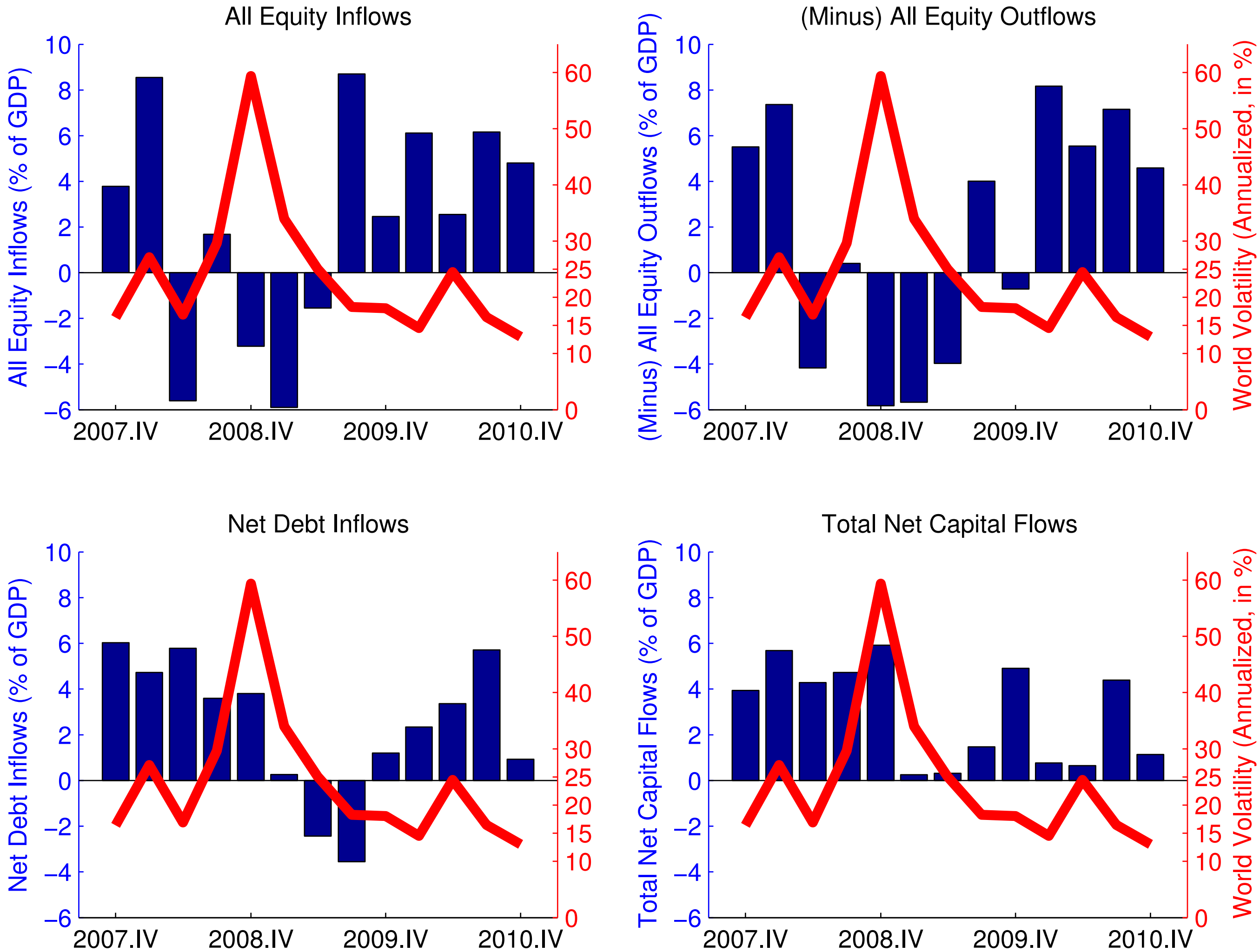

Figure 9: U.S. International Total Capital Flows During the Great Recession

The figure presents the dynamics of U.S. international capital flows (scaled by U.S. GDP) and world volatility during the recent recession. The upper panel reports U.S. total equity-like inflows and (minus) outflows, while the lower panel reports net debt inflows and total capital flows. The minus sign in front of the outflows series implies that a positive number corresponds to capital going out of the U.S. All international capital flows are represented with blue bars. World volatility corresponds to the cross-country average of the stock market volatilities, obtained as the standard deviation of realized daily aggregate equity returns. World volatility is represented by a red line. Data are quarterly, from the Bluedorn et al. (2013) dataset and Datastream. The sample period is 2007.4–2010.4.

Panel B of Figure 13 reports the response of the U.S. holdings of U.S. and ROW equity. The increase in the disaster probability leads to a sharp decreases the price-dividend and consumption-wealth ratios, and a sharp increase in equity risk premia. The U.S. capital stock of U.S. equity decreases sharply while the U.S. capital stock of the ROW equity increases slightly.

The change in capital stocks is due to a change in the value of the existing holdings and a change in the holdings, reflected in the international capital flows. We defined the capital flows, scaled by GDP, as equal to:

$$CF^{j}_{i,t+1} \equiv \frac{\vartheta^{j}_{i,t+1} q_{j,t+1} - \vartheta^{j}_{i,t} q_{j,t+1}}{Y_{i,t+1}}.$$

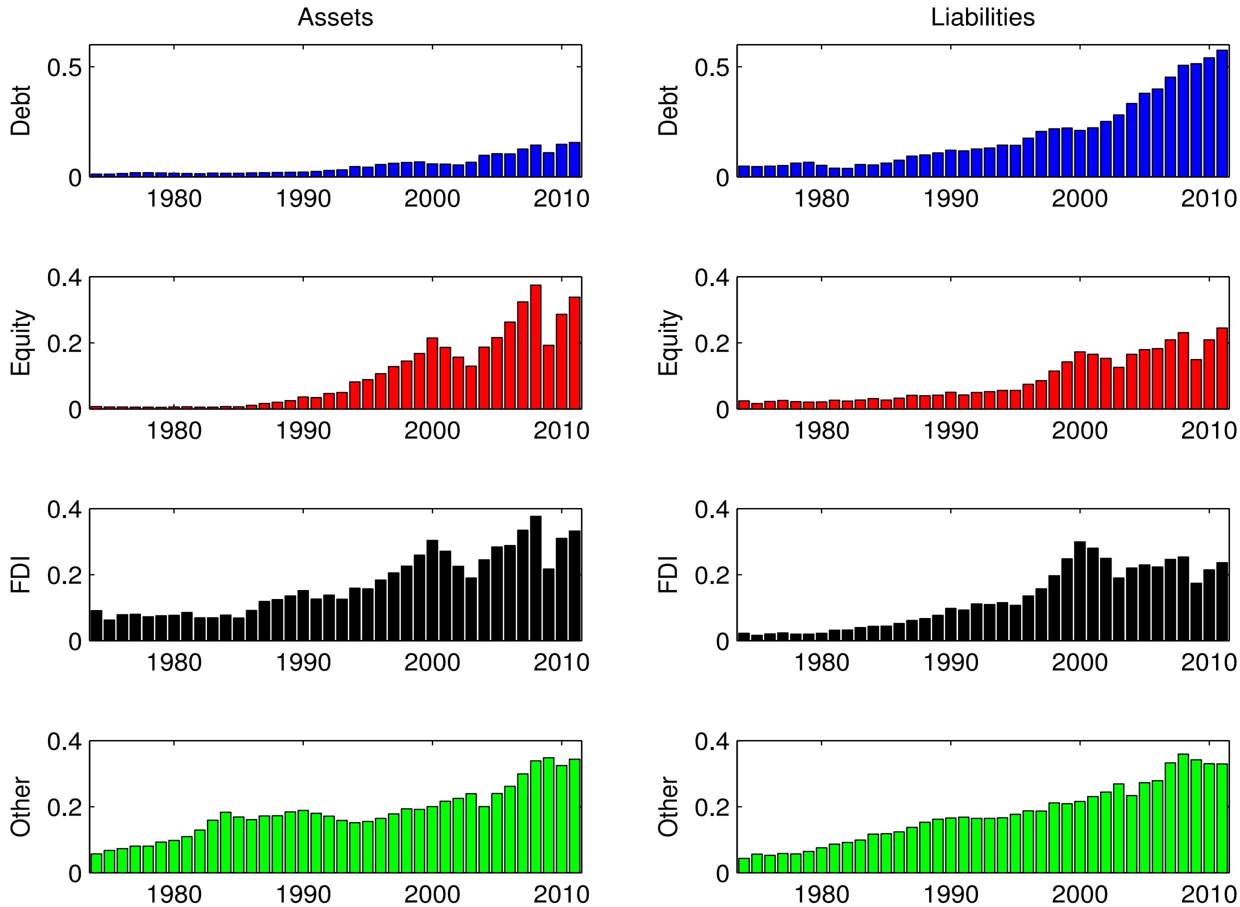

Figure 10: U.S. International Assets and Liabilities

The figure presents the stocks of U.S. foreign assets and liabilities (scaled by U.S. GDP). Data are annual, from an updated and extended version of the Lane and Milesi-Ferretti (2007) dataset. The sample is 1970–2010.

Panel C of Figure 13 reports the gross and net equity flows from the perspective of the U.S. Inflows increase and outflows decrease immediately in response to the disaster probability shock. The net inflows are positive but small in comparison with the gross flows. The inflows and outflows immediately reverse after the shock, but their subsequent size are an order of magnitude smaller than their initial responses. Panel D of Figure **??** shows that the net debt flow is almost immune to the large risk shocks.

The change in capital stocks can be decomposed into a capital flow and a valuation component:

$$\Delta CS^j_{i,t+1} = \frac{\vartheta^j_{i,t+1} q_{j,t+1}}{Y_{i,t+1}} - \frac{\vartheta^j_{i,t} q_{j,t}}{Y_{i,t}},$$

$$\Delta CS^j_{i,t+1} = \underbrace{CS^j_{i,t}\left(\frac{R^j_{i,t+1}}{R^Y_{i,t+1}} - 1\right)}_{\text{Valuation effects}} + \underbrace{CS^j_{i,t+1} - CS^j_{i,t}\frac{R^j_{i,t+1}}{R^Y_{i,t+1}}}_{\text{Flow effects}},$$

where $R^j_{i,t+1} \equiv \frac{q_{j,t+1}}{q_{j,t}}$ and $R^Y_{i,t+1} \equiv \frac{Y_{i,t+1}}{Y_{i,t}}$. The first term above captures the impact of asset prices on the change of

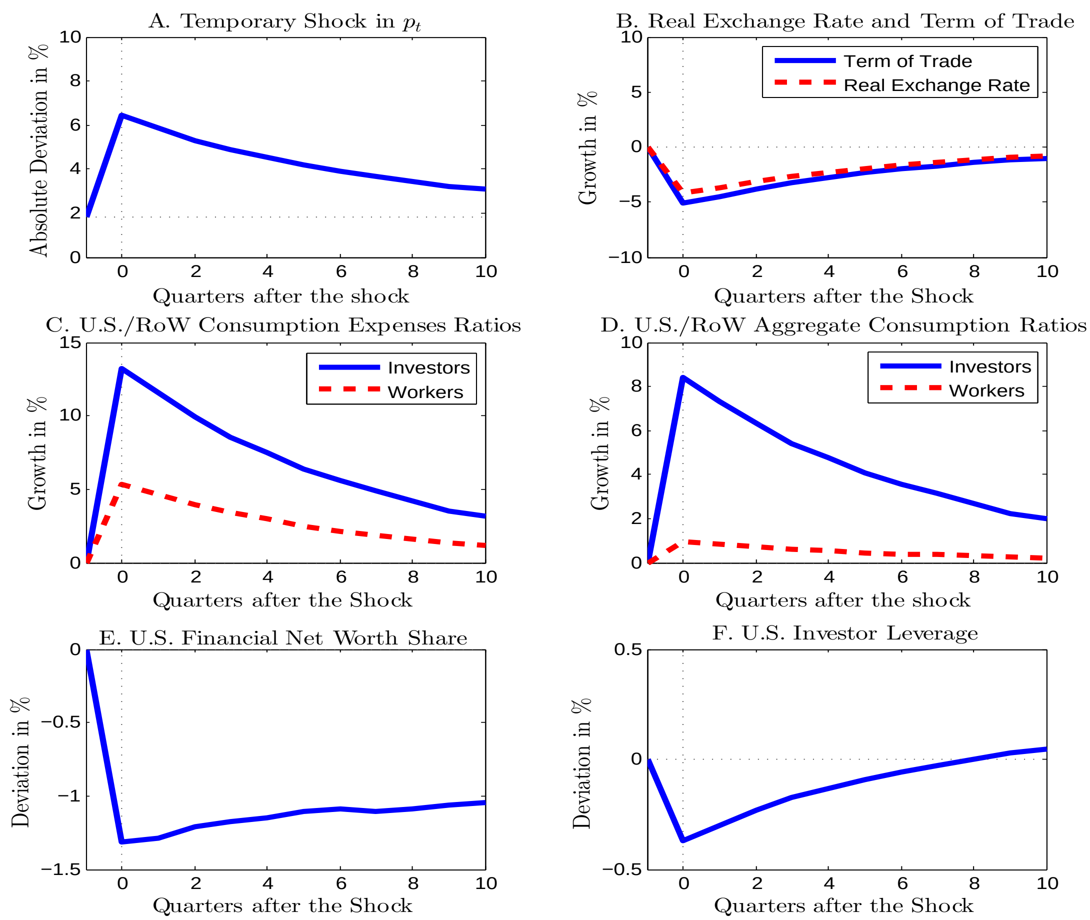

Figure 11: Impulse-Response Functions of Basic Quantities to a Disaster Probability Shock in the Model.

Panel A reports the temporary shock to the probability of disaster. Panel B illustrates how term of trade and real exchange rates respond to the temporary risk shock. The term of trade is $q_t = p_{2,t}/q_{1,t}$ and the real exchange rate is $Q_t = P_{2,t}/P_{1,t}$ where the price indexes P_1 and P_2 for U.S. and RoW, respectively. They are defined in (3). Panel C illustrates the responses of the U.S./RoW consumption expense ratios for investors and workers, respectively. The ratio of U.S. investors consumption expense to that of RoW investors is $(p_{1,t}c^1_{1,t} + p_{2,t}c^2_{1,t})/(p_{1,t}c^1_{2,t} + p_{2,t}c^2_{2,t})$. The ratio of U.S. workers consumption expense to that of RoW workers is $(p_{1,t}c^1_{w,1,t} + p_{2,t}c^2_{w,1,t})/(p_{1,t}c^1_{w,2,t} + p_{2,t}c^2_{w,2,t})$. Panel D illustrates the responses of the U.S./RoW aggregate consumption ratios for investors and workers, respectively. The ratio of U.S. investors aggregate consumption to that of RoW investors is $C_{1,t}/C_{2,t}$ where $C_{i,t}$'s are defined in (2). The ratio of U.S. investors aggregate consumption to that of RoW workers is $C_{w,1,t}/C_{w,2,t}$ where $C_{w,i,t}$'s are defined in (**??**). Panel E and Panel F demonstrate how U.S. financial net worth share $((\vartheta^1_{1,t}q_{1,t} + \vartheta^2_{1,t}q_{2,t} + b_{1,t}q^b_t)/(q_{1,t} + q_{2,t}))$ in the world and U.S. investors' leverage ratio $(\vartheta^1_{1,t}q_{1,t} + \vartheta^2_{1,t}q_{2,t})/(\vartheta^1_{1,t}q_{1,t} + \vartheta^2_{1,t}q_{2,t} + b_{1,t}q^b_t)$ respond to the risk shock.

capital stocks. The second term corresponds to the capital flows, as defined previously:

$$CS^j_{i,t+1} - CS^j_{i,t}\frac{R^j_{i,t+1}}{R^Y_{i,t+1}} = \frac{\vartheta^j_{i,t+1}q_{j,t+1} - \vartheta^j_{i,t}q_{j,t+1}}{Y_{i,t+1}} = CF^j_{i,t+1}.$$

Figure 13 reports the impulse response functions of capital stocks to a large shock in the disaster probability. The figure reports the dynamics of capital stocks levels, capital stock changes and capital flows.

Figure 14 reports the dynamics of U.S. domestic and foreign holdings in response to a shock on the disaster probability. Panel B corresponds to the U.S. holdings of U.S. equity. Panel C corresponds to the U.S. holdings of ROW equity. And, Panel D corresponds to the U.S. holdings of international bond. In each panel, the stock changes are decomposed into their valuation and flow components.

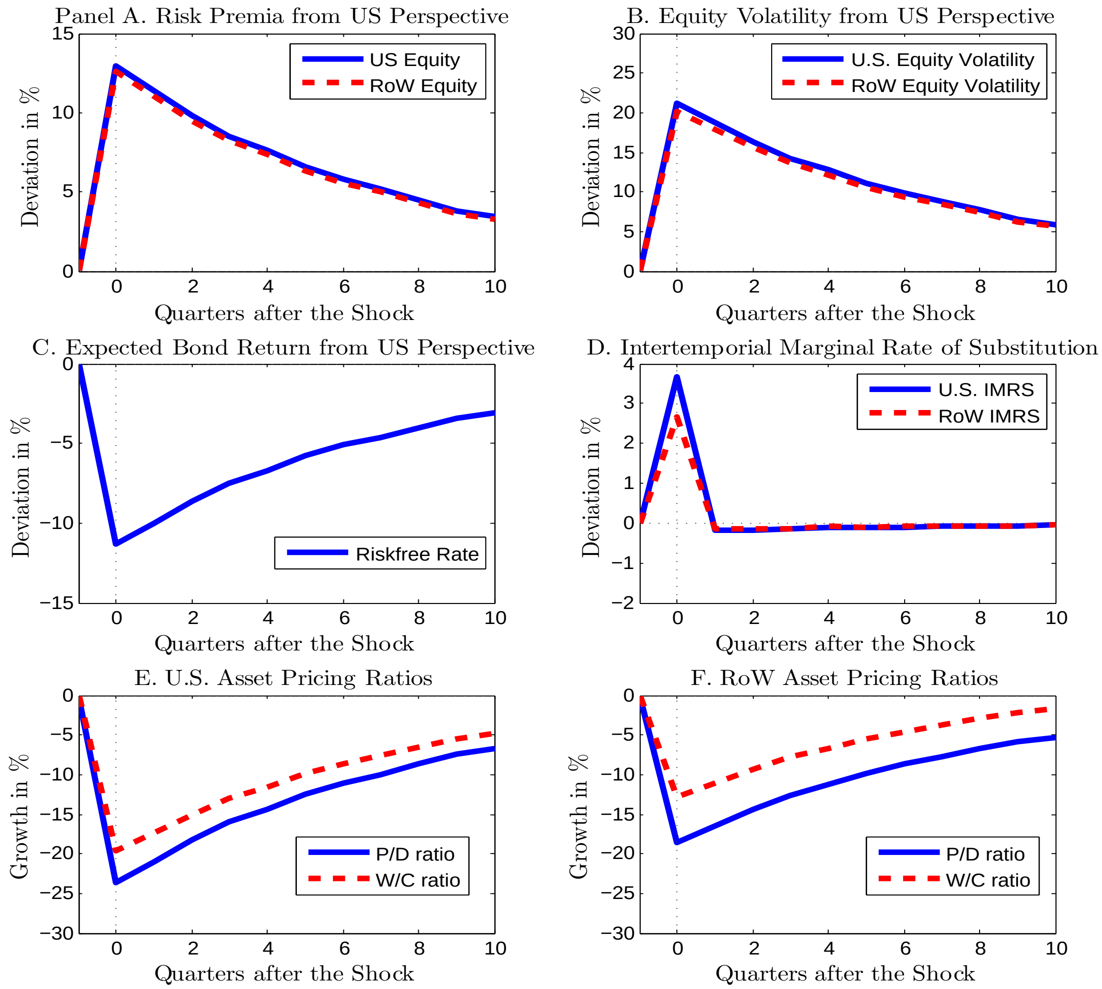

Figure 12: The simulation of pricing moments with a large temporary risk shock.

The U.S. equity return, the RoW equity return and the international bond return from U.S. perspective (i.e. in terms of U.S. goods) are $R^1_{1,t+1} = (q_{1,t+1}/p_{1,t+1} + d_{1,t+1})/(q_{1,t}/p_{1,t})$, $R^2_{1,t+1} = (q_{2,t+1}/p_{1,t+1} + d_{2,t+1}p_{2,t+1}/p_{1,t+1})/(q_{2,t}/p_{1,t})$, and $R^b_{1,t+1} = (p_{\alpha,t+1}/p_{1,t+1})/(q^b_t/p_{1,t})$, respectively. Panel A is about the conditional expectations of $R^1_{1,t+1} - R^b_{t+1}$ and $R^2_{1,t+1} - R^b_{t+1}$. Panel B is about the conditional volatilities of $R^1_{1,t+1}$ and $R^2_{1,t+1}$. Panel C is about the conditional expectations of R^b_{t+1}. Panel D illustrates the response of IMRS of two countries' investors. The IMRS for country i is $\mathcal{M}_{i,t,t+1} = \beta \mathbb{E}_t \left[U^{\theta_i}_{i,t+1}\right]^{1/\theta_i - 1} C^{-\psi^{-1}}_{i,t+1} \left[G_{i,c^2}(c^1_{i,t+1}, c^2_{i,t+1})/G_{i,c^2}(c^1_{i,t}, c^2_{i,t})\right] U^{\theta_i - 1}_{i,t+1}$ with $i = 1,2$. Panel E and Panel F are about the price-dividend ratios $q_{i,t}/(p_{i,t}d_{i,t})$ and wealth-consumption ratios $W_{i,t}/(p_{1,t}c^1_{i,t} + p_{2,t}c^2_{i,t} + p_{1,t}c^1_{w,i,t} + p_{2,t}c^2_{w,i,t})$ for $i = 1,2$.

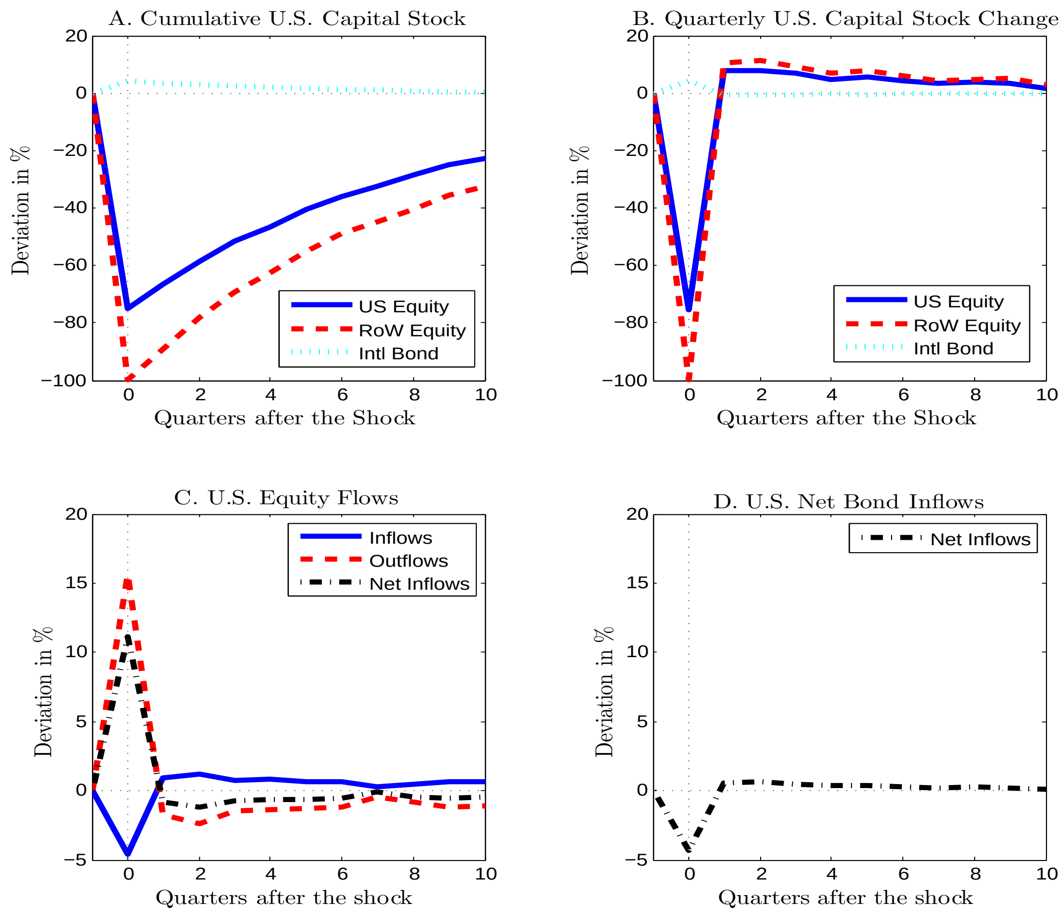

Figure 13: The simulation of capital stocks and on capital stocks with a large risk shock.

Panel A illustrates the response of U.S. capital stocks including U.S equity, RoW equity and international bond. They are defined as $CS^1_{1,t} = (\vartheta^1_{1,t} q_{1,t})/(p_{1,t} e_{1,t})$, $CS^2_{1,t} = (\vartheta^2_{1,t} q_{2,t})/(p_{1,t} e_{1,t})$, and $CS^b_{1,t} = (b_{1,t} q^b_t)/(p_{1,t} e_{1,t})$, respectively. Panel B is about the responses of quarterly capital stock changes. The quarterly U.S. capital stock change of U.S. equity, RoW equity and international bond are defined as $\Delta CS^1_{1,t} \equiv CS^1_{1,t} - CS^1_{1,t-1}$, $\Delta CS^2_{1,t} \equiv CS^2_{1,t} - CS^2_{1,t-1}$ and $\Delta CS^b_{1,t} \equiv CS^b_{1,t} - CS^b_{1,t-1}$. Panel C illustrates the responses of U.S.'s gross capital flows and net capital inflow. The gross equity inflow of U.S. is $CF^1_{1,t} = (\vartheta^1_{2,t} - \vartheta^1_{2,t-1}) q_{1,t}/(p_{1,t} e_{1,t})$, and the (minus) gross equity outflow of U.S. is $CF^2_{1,t} = -(\vartheta^2_{1,t} - \vartheta^2_{1,t-1}) q_{2,t}/(p_{1,t} e_{1,t})$. The net equity inflow is $NCF_{1,t} = CF^1_{1,t} + CF^2_{1,t}$. Panel D illustrates the response of net bond inflow $NCF^b_{1,t} = -b_{1,t} q^b_t/(p_{1,t} e_{1,t})$.

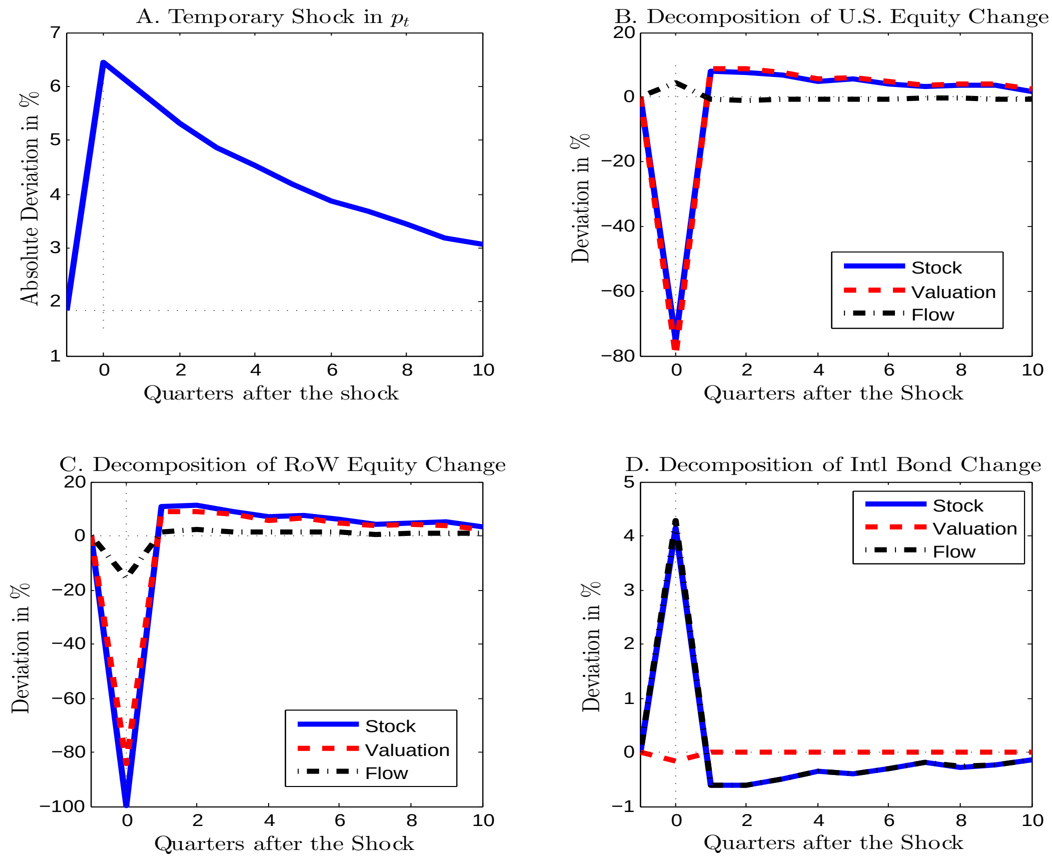

Figure 14: Impulse-Response Functions of U.S. Capital Stocks Decompositions to a Large but Temporary Disaster Probability Shock.

Panel B and Panel C decompose the U.S. capital stock change in U.S. equity and RoW equity into their valuation component and flow component, respectively. The valuation component of U.S. capital stock change in U.S. equity or RoW equity is defined as $VC^i_{1,t} = \vartheta^i_{1,t-1}\left[q_{i,t}/(p_{1,t}e_{1,t}) - q_{i,t-1}/(p_{1,t-1}e_{1,t-1})\right] = CS^i_{1,t}(R_{i,t}/R^Y_{i,t} - 1)$ with $R_{i,t} \equiv q_{i,t}/q_{i,t-1}$ and $R^Y_{i,t} \equiv p_{i,t}e_{i,t}/(p_{i,t-1}e_{i,t-1})$. Panel D illustrates the decomposition of of the U.S. capital stock change in international bond. The valuation and flow components of the bond stock change is $\Delta CS^b_{1,t} = VC^b_{1,t} + NCF^b_{1,t}$ where $VC^b_{1,t} = b_{1,t}\left[q^b_t/(p_{1,t}e_{1,t}) - q^b_{t-1}/(p_{1,t-1}e_{1,t-1})\right]$.