In all of the examples above, the bidders incur the cost of information. But, the seller in fact indirectly bears these costs. The intuition is that these costs will affect the bidders' (ex-ante) willingness to pay, and this in turn impacts their participation and bidding. Therefore, the seller should ensure that bidders do not over-invest or under-invest in information acquisition. On one hand, to avoid under-investment, the seller should motivate some of the bidders to obtain information. On the other hand, to avoid over-investment, the seller may would like to restrict some of the bidders from accessing to costly information if he is able to control access to information.

In many applications, the seller can exert such a control. For instance, in the sale of complex financial assets, the seller may control how much detail is disclosed to the potential buyers. In timber (oil and gas) auctions, the bidders can only obtain the additional information if the auctioneer allows them to examine the tracts (fields). In the context of online advertising, the seller (online publisher) can release the identity of the user to only a subset of the advertisers. The identity of the user can be revealed to advertisers by disclosing HTTP cookies (Kristol 2001). If the publisher releases these cookies to an advertiser, the advertiser can subsequently take the cookies to the aforementioned information provider and obtain (purchase) information. On the flip side, if the publisher does not disclose these cookies to some advertisers, then these advertisers cannot acquire any information about the user.[2]

In this work, we answer the following question: If the seller controls access to (additional) information, how can he incentivize a right set of bidders to invest in information acquisition?

Our Contributions

We present a model to study costly information acquisition in auctions. Our model consists of an auctioneer that sells an item to a set of agents. The agents have an initial private valuation for the item and can obtain additional information at a cost. However, "access"

a website that a user may not have visited. HTTP cookies are strings of characters that can only be interpreted by the party that has created them (Kristol 2001). In general, some of the cookies could have been created by the advertisers themselves. In this work, we focus on the third-party cookies created by the information providers. Using this technology, it is possible to track a user across different websites in order to identify the users' interests and intentions.

[2] Major ad exchanges, such as Google AdX, allow publishers to run private auctions (Google DoubleClick Documentation 2016) where they can control the information that is provided to the advertisers.

to this additional information is controlled by the auctioneer, and the mechanism may grant access only to a subset (or none) of the agents.

We present a two-stage **efficient mechanism** in our setting in Section 4. A mechanism is efficient if it maximizes the sum of the social welfare of the auctioneer and the agents, taking into account the cost incurred to obtain the additional information. When there is no such cost, the efficient mechanism allows all agents to obtain the additional information. However, when the information is costly, the efficient mechanism grants access to the additional information only to a subset of the agents.

The efficient mechanism works as follows: In the first stage, agents bid in an initial round of bidding. Then, based on their initial bids, the auctioneer *selects* a subset of agents and grants them access to obtain additional information. Each *selected* agent can acquire the additional information by incurring a cost. The selected agents then update their valuations and bid in the second round of bidding. The second stage corresponds to the second-price auction with no reserve.[3]

In order to increase the revenue, the seller may want to set a reserve price in the second stage. In Section 5.1, we extend our analysis to mechanisms where the item is allocated via a second-price auction with a reserve. These mechanisms are appealing from a practical perspective because they allocate the item using the second-price auctions which is the prevalent mechanism used in ad exchanges (Muthukrishnan 2009).

We further present a **revenue-optimal mechanism** in our setting. It turns out that the allocation stage of the revenue-optimal mechanism is a bit more complicated than a second-price auction. In order to optimize the revenue, the mechanism selects a set of agents so that it maximizes the "virtual revenue" minus the cost of information. The item is allocated via a weighted second-price auction where the weights favor the agents with higher initial bids.

In the above mechanisms, the auctioneer controls access to information. To study the impacts of such a control, in Sections 4.1 and 5.3, we investigate a mechanism called **All-Access**, where the auctioneer does not control access to the additional information. In this mechanism, agents obtain additional information if they choose to incur the cost. That

[3] In this paper, we consider two stage mechanisms where all the agents are selected at the same time (in the first stage) and the item is allocated in the second stage. See the discussion on adaptive selection rules at the end of Section 4.

is, the mechanism leaves the decision on obtaining the costly information to the agents. The item is allocated via a standard second-price auction (with a reserve). We show that the mechanism always admits a pure strategy Nash equilibrium. The equilibria, however, might not be efficient/revenue-optimal. We observe that under the All-Access mechanism, the agents tend to over-invest in information acquisition when the cost is low. Similarly, when the cost is high, the agents tend to under-invest in information acquisition.

We numerically compare the above mechanisms. Interestingly, on average, the revenue-optimal mechanism allows fewer agents to obtain the additional information compared to the efficient mechanism. It is well established that the revenue-optimal mechanism distorts the allocation and creates inefficiencies in order to extract more revenue from agents with higher valuations. We observe that the revenue-optimal mechanism distorts the revelation of information in addition to the allocation; see Section 6. In addition, we observe that the revenue-optimal mechanism can significantly increase the revenue compared with the aforementioned All-Access mechanism; see Section 5.3.

Our proposed mechanisms are flexible and can be generalized to a setting where multiple units of the item are sold; see Section 7.2. Furthermore, they can be extended to environments where the cost of information can be seen as an entry cost, and all agents who may receive the item with non-zero probability must invest in obtaining information; see Section 7.1.

Related Work

In this section, we briefly discuss the literature related to our work.

Dynamic Mechanism Design: Our work belongs to the growing body of research on mechanism design; see Bergemann and Said (2011) for a survey. In particular, our work is closely related to that of Ëso and Szentes (2007) and generalizes their model to a setting where information acquisition is costly. In the absence of this cost, Ëso and Szentes (2007) show that the revenue-optimal mechanism grants all agents access to the additional information. In contrast, we show that when obtaining the additional information is costly, the auctioneer, even in the efficient mechanism, may not allow all bidders to acquire the additional information. The selection rule of our mechanism determines the set of agents who could access (and obtain) the additional information. From a technical perspective, as we discuss later in Appendix A.1, this makes our proof a bit challenging. See Kakade et al.

(2013), Pavan et al. (2014), Battaglini and Lamba (2012), Boleslavsky and Said (2013), and Lobel and Xiao (2013) for recent results on designing optimal dynamic mechanisms.

Costly Information Acquisition: Most previous work on information acquisition considers settings where the bidders do not have any private information prior to entering the auction. In such a setting, where the auctioneer controls the bidder's access to information, Crémer et al. (2009) show that the auctioneer can extract all the surplus by imposing an admission fee; see also Pancs (2013). Information acquisition has also been studied in the principle-agent context (Cremer and Khalil 1992, Szalay 2009) and reverse auctions (Beil et al. 2015).

Shi (2012) studies costly information acquisition in a setting where bidders do not have any private information prior to entry and can decide on how much to invest in order to obtain information. He shows that the optimal mechanism takes the form of standard auctions (e.g., second-price) with a reserve price. In contract, in our setting the bidders are privately informed before they decide on the obtaining additional information.

Ye (2007) and Quint and Hendricks (2012) study *indicative bidding* auctions that are commonly used in selling financial assets. The auction works as follows: bidders submit non-binding bids to indicate their interest in the assets. The auctioneer then selects some of the bidders that have higher valuations to proceed to the second round, which involves a costly due diligence process and final bidding. Ye (2007) shows that in the indicative bidding, efficient entry of the bidders is not guaranteed and the most qualified bidders might not be selected by the auctioneer. Note that in contrast to our work, the number of selected bidders is predetermined. In addition, only selected bidders who invest in obtaining the additional information may participate in the allocation stage of the mechanism.

One of the closest works in the literature to ours is that by Lu and Ye (2014), who study the design of a two-stage revenue-maximizing mechanism when acquiring information is costly. Similar to indicative bidding auctions, they assume that obtaining the costly additional information is necessary for agents to participate in the second stage. Under this assumption, as the initial valuations of the agents increase, fewer agents will be allowed to acquire information. Specifically, the selection rule of the mechanism is "monotone" in initial valuations.

In contrast, we observe in Section 6 that the selection rule of our proposed mechanisms is non-monotone. The reason is that in our model all agents—including those who do not update their valuations—participate in the second stage and have a chance to receive the item; see Section 7 for details. The non-monotonicity of the allocation rule makes our proofs more complicated. Furthermore, they assume that the seller can observe who obtains the additional information, which may not be a realistic assumption in many practical contexts. In contrast, in our setting the seller does not observe who obtains the additional information. Therefore, our mechanisms should be designed in a way that all selected agents willingly acquire the costly information. This requirement makes the mechanism design problem more challenging.

In addition, we observe that the seller earns (significantly) higher revenue in our setting. We provide a numerical example in Section 7.1. This is quite intuitive; When the cost of information is high as in our setting, the seller can allocate the item even when no agent invests in information. Another reason that the seller can obtain higher revenue in our setting is that he can allocate the item to one of the agents that did not obtain information when the updated valuations were all low.

In Section 7.1, we discussed how our mechanisms can be extended to the setting studied by Lu and Ye (2014). In particular, we extend their revenue-maximizing mechanism to a setting where the cost of information is not the same across agents. Note that in Lu and Ye's paper, all agents incur the same cost when they obtain information. In addition, we presented a two-stage efficient mechanism in their setting.

Another related work is that by Hatfield et al. (2015). They focus on efficient mechanisms where bidders can invest in costly information acquisition to determine their valuations. They show that bidders make efficient investment choices when the utility of an agent is equal to his marginal contribution to the social welfare; see also Bergemann and Välimäki (2002). In contrast, we observer that for the All-Access mechanism, there might exist an equilibrium where agents do not make efficient investment decisions; see Example 1 in Section 4.1.

For a discussion on settings where the computation of valuation is costly, see Larson and Sandholm (2001).

Information Disclosure: Our work also contributes to the vast literature on information disclosure. In the following, we briefly discuss this literature, focusing on the works motivated by applications in online advertising.

Recently, several papers have studied the effect of sharing cookies and targeting in advertising. Abraham et al. (2011) show that in a common value setting when some advertisers are able to better utilize information obtained from cookies, asymmetry of information can sometimes lead to low revenue in this market; see also Syrgkanis et al. (2013). Several recent papers, such as Ghosh et al. (2007), Rayo and Segal (2010), Bergemann and Bonatti (2011), Emek et al. (2012), Hummel and McAfee (2012), Bergemann and Bonatti (2013), and Bhawalkar et al. (2014), analyze the role of providing more (targeting) information in the context of online advertising and show that more information may reduce the revenue. Our proposed mechanisms can control the access to information in order to maximize the revenue.

Information disclosure has been studied in other applications. For instance, Jing (2011) studies customer learning for new durable goods. In his model, the seller invests in informing customers before releasing the goods to the market. In addition, see Lewis (2011).

The remainder of this paper is organized as follows: In Section 2, we formally define our model. Direct mechanisms are defined in Section 3. We present our efficient mechanism followed by the All-Access mechanism in Section 4. Section 5 discusses revenue maximization and presents the revenue-optimal mechanism. We discuss the selection rule of our mechanisms in Section 6. Finally, Section 7 explores some of the extensions of our mechanisms.

2. Setting

We consider a setting with a seller of one (indivisible) item and n agents. The initial valuation of each agent i for the item is denoted by $v_{i,0} \in [\underline{v}, \overline{v}]$, which is drawn independently from distribution F, with probability distribution function (p.d.f.) f. Distribution F is known to the seller and all the agents. However, $v_{i,0}$ is known only to agent i.

The seller may allow some of the agents to obtain (additional) information about their valuations. Suppose agent i is one of the agents to whom the seller has "granted" access to the additional information. In this case, agent i may decide to incur cost c_i and obtain signal δ_i about his valuation where δ_i is drawn independently (of $v_{i,0}$ and other agents' second signals) from distribution G_i. The distributions G_i, $i = 1, 2, \ldots, n$, are publicly known, but the second signals are private information. Without loss of generality, we assume $\mathrm{E}[\delta_i] = 0$. Note that if $\mathrm{E}[\delta_i] = \Delta > 0$, we can add Δ to $v_{i,0}$ and then subtract Δ from δ_i.

If agent i obtains second signal δ_i, his updated final valuation, denoted by $v_{i,1}$, would be equal to $v_{i,0} + \delta_i$. For the agents who did not learn their second signals, either because the seller denied them the access or by their own choice, let $v_{i,1} = v_{i,0}$.

As an example, suppose an advertiser values male users at \$0 and female users at \$6. Assume that each user has the same chance of being male as of being female. Thus, when the user's gender is unknown, his expected value, that is, his initial valuation, is $\frac{6+0}{2} = 3$. By revealing the gender, the valuation of the advertiser will change; with probability $\frac{1}{2}$, his valuation is increased by \$3, and with probability $\frac{1}{2}$, it is decreased by the same amount. That is, the second signal is either 3 or -3 with equal probability.

Throughout the paper, we denote the vector of the initial and final valuations of all agents by v_0 and v_1. Also, $v_{-i,0}$ and $v_{-i,1}$, respectively denote the vector of the initial and final valuations of all agents except for agent i.

The agents are risk neutral. The utility of an agent i who has received the item is equal to his valuation, $v_{i,1}$, minus his payment to the mechanism and the (possible) cost of information acquisition. We will specify utility of the agents more precisely in the next section.

3. Direct Mechanisms

In this section, we consider direct revelation mechanisms (Myerson 1986) where agents report their valuations in two rounds. First, they report their initial valuations to the mechanism. Then the mechanism decides on the set of agents that will have access to information. Those agents report their updated valuations to the mechanism in the second round and finally, the mechanism allocates the item.

More precisely, any direct mechanism $\mathcal{M}$ is defined by a tuple (s, q, p), which respectively represents its selection, allocation, and payment rules. The seller announces the mechanism to the agents and commits to (s, q, p). Following are the stages of the mechanism:

1. Initial Bidding: Agents report in the first round. The initial report (bid) of agent i is denoted by $b_{i,0}$. Throughout the paper, we will use "reporting" and "bidding" interchangeably.

2. Selection: Based on the initial reports, the mechanism selects a set of agents that we call *selected agents*. The mechanism grants access to each selected agent i to acquire additional information (signal δ_i) and in return charges them price t_i. More precisely, selection rule $s : \mathbb{R}^n \to (\{0,1\} \times \mathbb{R})^n$ maps the initial bids to a pair (s_i, t_i) for each agent i. If agent i is selected, $s_i(b_0)$ is equal to 1. Otherwise, s_i is equal to 0. Each selected agent pays amount t_i to the mechanism to access his signal (the agent would still need to incur an additional cost c_i to learn the signal). To simplify the presentation, we assume t_i is equal to 0 for non-selected agents.[4]

3. Obtaining Information: Each selected agent i decides on whether to incur cost c_i and learn δ_i. We define e_i to denote the decision variable for agent i for incurring cost c_i and updating his valuation. e_i is equal to 1 if the selected agent i learns δ_i. For non-selected agents, e_i is defined to be 0. Neither the mechanism nor other agents can observe decision e_i of an agent i; it is only known to that agent.

4. Final Bidding: In the final round of reporting, only selected agents get a chance to update their reports. For any selected agent i, $b_{i,1}$ denotes the updated (and final) report of agent i. For all other agents, let $b_{j,1} = b_{j,0}$.

5. Allocation and Payments: Based on the initial and final bids of all agents (both selected and unselected), the seller decides to whom to allocate the item, allocation rule $q : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}^+$, and how much to charge each agent, payment rule $p : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}$. Namely, given all the bids and decision variables, $q_i(b_0, b_1)$ is the allocation probability, and $p_i(b_0, b_1)$ is the payment of agent i.

The utility of agent i participating in the mechanism is equal to $q_i v_{i,1} - p_i - t_i - e_i c_i$ (more precisely, $q_i(b_0, b_1)v_{i,1} - p_i(b_0, b_1) - t_i(b_0) - e_i c_i$). Each agent chooses a best-response strategy to deal with the mechanism and strategies of the other agents in order to maximize his expected utility, where the expectation is taken with respect to the second signals of the (selected) agents. More formally, the best response strategy of each agent i can be described with the following mappings: $b_{i,0} : \mathbb{R} \to \mathbb{R}$, $e_i : \mathbb{R}^3 \to \{0,1\}$, and $b_{i,1} : \mathbb{R}^6 \to \mathbb{R}$. With slight abuse of notation, we denote the decision variables and functions with the same notation. Function $b_{i,0}$ maps $v_{i,0}$, the initial valuation of the agent, to the bid in the first round $b_{i,0}$. e_i is a function of the initial valuation, $v_{i,0}$, and initial bid, $b_{i,0}$, of the agent and his payment t_i. Finally, $b_{i,1}$ is a function of the whole history of agent i (i.e., $< v_{i,0}, b_{i,0}, s_i, t_i, e_i, v_{i,1} >$)

[4] This assumption to a large extent is without loss of generality. In general, a mechanism can charge any agents in the first round independent of being selected or not. We are not putting any restrictions on the payment in the second round; therefore, any such payment in the first round can be added to the payment in the second round.

and determines the final bid. Given the strategy of the other agents, agent i optimizes over tuple $(b_{i,0}, e_i, b_{i,1})$ to obtain his best (utility-maximizing) strategy.

The *truthful strategy* for agent i consists of i) reporting truthfully in the first round ($b_{i,0} = v_{i,0}$); ii) obtaining additional information if selected ($e_i = 1$ if $s_i = 1$); and iii) reporting truthfully in the final round ($b_{i,1} = v_{i,1}$).

A dynamic direct mechanism is *incentive compatible* (IC) if for every agent and every truthful history, truth-telling is a best response given that all other agents report truthfully. More precisely, a mechanism is IC if

$$\mathrm{E}_{\text{TRUTHFUL}}\big[q_i(v_0, v_1)v_{i,1} - p_i(v_0, v_1) - t_i(v_0) - e_i c_i\big]$$
$$= \max_{b_{i,0}, e_i, b_{i,1}} \left\{\mathrm{E}\left[q_i\big((b_{i,0}, v_{-i,0}), (b_{i,1}, v_{-i,1})\big)v_{i,1} - p_i\big((b_{i,0}, v_{-i,0}), (b_{i,1}, v_{-i,1})\big) - t_i\big((b_{i,0}, v_{-i,0})\big) - e_i c_i\right]\right\},$$

where the expectations are taken assuming other agents are truthful; that is, agents report truthfully in both rounds, i.e., $b_{-i,0} = v_{-i,0}$ and $b_{-i,1} = v_{-i,1}$, and obtain information if selected, that is, $e_j = 1$ if $s_j = 1$ for $j \neq i$. In addition, in the l.h.s., the expectation is taken under the truthful strategy of agent i.

We show that our proposed mechanisms satisfy stronger incentive compatibility properties. Namely, selected agents always bid truthfully in the final round even if they have deviated from the truthful strategy in the past. In addition, each selected agent prefers to obtain additional information even if they observe other agents' initial valuations. Currently, we assume that the agent only observes s_i and t_i, which may contain information about other agents' valuations.

We can now define the participation constraints for the mechanism. An IC mechanism is *individually rational* (IR) if for each agent i, the expected utility under the truthful strategy is non-negative, that is,

$$\mathrm{E}_{\text{TRUTHFUL}}\big[q_i(v_0, v_1)v_{i,1} - p_i(v_0, v_1) - t_i(v_0) - e_i c_i\big] \geq 0\ .$$

4. The Efficient Mechanism

The social welfare of a mechanism is defined as the sum of the utility of the agents and the seller minus the cost incurred to obtain additional information. Note that an IC and IR mechanism is efficient if it obtains the maximum social welfare equal to:

$$\max_{S \subseteq \{1,\cdots,n\}} \left\{\mathrm{E}_S\left[\sum_{i=1}^{n} q_i v_{i,1}\right] - \sum_{i \in S} c_i\right\} = \max_{S \subseteq \{1,\cdots,n\}} \{\Omega(v_0, S)\},$$

where E_S denotes the expectation with respect to the realizations of the second signals when all agents in set S (and only those agents) obtain the additional information. In addition, $\Omega(v_0, S)$, defined below, denotes the maximum social welfare that can be obtained when set S of agents acquires information.

$$\Omega(v_0,S)=\mathrm{E}_S\left[\max\Big\{\max_{j\in S}\{v_{j,0}+\delta_j\},\max_{j\notin S}\{v_{j,0}\},0\Big\}\right]-\sum_{j\in S}c_j\ . \tag{1}$$

To gain intuition, let us consider the following scenario. Assume all the agents bid truthfully, agents in set S obtain information, and subsequently, each agent j who updates his valuation incurs cost c_j. The total cost of information is equal to $\sum_{j\in S}c_j$. The mechanism maximizes the social welfare by allocating the item to the agent with the highest non-negative final bid, that is, agent $i^\star \in \arg\max\{\max_{j\in S}\{v_{j,0}+\delta_j\},\max_{j\notin S}\{v_{j,0}\}\}$, where $\max_{j\in S}\{v_{j,0}+\delta_j\}$ and $\max_{j\notin S}\{v_{j,0}\}$ are, respectively, the maximum updated bids of agents who obtain information and the maximum bid of agents who do not. The item is allocated to $i^\star$ if his (final) bid is positive; otherwise, there would be no allocation.

Throughout the paper, we assume that the maximum expected social welfare that can be obtained is bounded; that is, $\mathrm{E}\left[\max_{S\subseteq\{1,2,\ldots,n\}}\{\Omega(v_0,S)\}\right]<\infty$, where the expectation is with respect to initial valuations v_0.

We now present an efficient mechanism.

$\mathcal{M}^{\text{Eff}}$ **Mechanism:** The selection, allocation, and payment rules are defined as follows:

- **Selection:** Select a set of agents such that granting them access to information maximizes the social welfare of the seller and agents, taking into account the cost of information. Specifically, select the following set of agents $\mathcal{S}(b_0)=\arg\max_{S\subseteq\{1,\cdots,n\}}\big\{\Omega(b_0,S)\big\}$, where $\Omega(b_0,S)$ is defined in Eq. (1).

The payment of selected agent i is equal to

$$t_i(b_0)=-c_i+\mathrm{E}\left[q_i v_{i,1}-p_i|v_0=b_0\right]-\int_{\underline{v}}^{b_{i,0}}\mathrm{E}\left[q_i|v_{i,0}=z,\ v_{-i,0}=b_{-i,0}\right]dz\ , \tag{2}$$

where the expectations are with respect to the second signals. Notation $\mathrm{E}\left[q_i|v_{i,0}=z,v_{-i,0}=x_{-i}\right]$ denotes the expected probability of the allocation of agent i, where the initial valuations of agent i and other agents are, respectively, equal to z and x_{-i}, assuming all the agents, including agent i, are truthful. Note that the first term in the payment implies that the seller subsidizes the cost of information, c_i, for each selected agent i. In lemmas 5 and 8 and their proofs, we discuss how this payment is calculated using the Envelope Theorem and show that it incentivizes the agents to be truthful. Recall that for non-selected agents, t_i is equal to 0. See Section E.1 in the online appendix for examples that depict initial payments.

- **Allocation and Payments:** Allocate the item to the agent with the highest non-negative bid at a price equal to the second highest bid or a reserve. More precisely, consider an agent $i^\star \in \text{argmax}_i\{b_{i,1}\}$. If $b_{i^\star,1} \geq 0$, agent $i^\star$ receives the item and pays $p_{i^\star} = \max\{\max_{i\neq i^\star}\{b_{i,1}\}, r\}$, where $r : \mathbb{R}^n \to \mathbb{R}$ is a function of initial bids and will be defined later in Eq. (3).

Let agent $\ell \in \arg\max_{j\notin \mathcal{S}(b_0)}\{b_{j,0}\}$ be an unselected agent with the highest initial bid. The reserve price r is simply zero when $b_{\ell,0} < 0$ or all agents are selected; otherwise r is the solution of the equation below

$$\int_{\max\{r,0\}}^{b_{\ell,0}} \Pr\left[z \geq \max_{j\in\mathcal{S}(b_0)}\left\{b_{j,0}+\delta_j\right\}\right]dz = \int_{\underline{v}}^{b_{\ell,0}} \mathrm{E}\left[q_\ell \middle| v_{\ell,0}=z, v_{-\ell,0}=b_{-\ell,0}\right]dz\ . \tag{3}$$

Lemma 3 and Corollary 1 in Online Appendix, Section A, show that there exists an $r \in [0, b_{\ell,0}]$ that satisfies the above equation. If there are multiple solutions to the above equation, we choose the largest one.

Observe that because $r \in [0, b_{\ell,0}]$, the reserve price does not change the allocation or the payment of the selected agents. Specifically, if a selected agent i wins the item, he pays $\max\{\max_{j\neq i}\{b_{j,1}\}, r\}$, which is identical to $\max\{\max_{j\neq i}\{b_{j,1}\}, 0\}$. Then, one can describe the payment of the mechanism as follows: If agent $i^\star$ was a selected agent, then he pays $\max\{\max_{j\neq i^\star}\{b_{j,1}\}, 0\}$. If $i^\star$ was not a selected agent, then he pays $\max\{\max_{j\neq i^\star}\{b_{j,1}\}, r\}$.

In fact, by introducing the reserve price r, the mechanism charges agent ℓ differently in the second round to incentivize him to bid truthfully in the first round. Note that the initial bid of agent ℓ can change the set of selected agents. In addition, similar to all selected agents, agent ℓ has a chance to win the item. However, unlike the selected agents, agent ℓ was not charged in the first round.

We now present the main result of this section.

THEOREM 1 **(Efficient Mechanism)**. *Mechanism $\mathcal{M}^{\text{EFF}}$ is individually rational, incentive compatible, and efficient.*

From its construction, it is not difficult to see that if mechanism $\mathcal{M}^{\text{EFF}}$ is IC, then it is also efficient. We prove the incentive compatibility of the mechanism in the online appendix, Section A. Observe that selected agents bid truthfully in the second round because the item is allocated using a second-price auction. We then show that any selected agent that bids truthfully in the first round obtains information. To this aim, we show that the mechanism's selection rule aligns with the agent's incentive. Specifically, the marginal change in the utility of a selected agent from not obtaining information is equal to the change in the social welfare. The challenging part of the proof is showing that agents bid truthfully in the first round because agents' bids in the first round determine the set of selected agents endogenously.

The selection rule ensures that a right set of agents invest in information acquisition. Note that as more agents obtain information, there is a higher chance that an agent has a high valuation for the item, which could increase the social welfare. On the other hand, the increase in the highest valuation comes at the cost of information acquisition. Thus, there is a trade-off here, and the selection rule aims to avoid over or under-investment in information acquisition. In fact, as we show in Section 4.1, over or under-investment in information acquisition may not be avoided if the seller cannot control access to information via a selection rule.

We also note that the selection rule of the efficient mechanism chooses a *set* of agents and allows them to update their valuations simultaneously. Alternatively, one can consider an "adaptive" selection rule that discloses information step-by-step (cf. McAfee and McMillan (1988)). During the selection stage, at each step, the mechanism selects one of the agents to obtain information. Then, based on the report of that agent, the mechanism makes a decision on obtaining more information or proceeding to the allocation stage. In this paper, we consider a two-stage information disclosure, which could be more appealing from a practical perspective. The sequential search can be time-consuming and complex. For instance, in the example from online advertising, the mechanism should be executed in milliseconds, and sequential information acquisition may not be feasible.

4.1. What If All Agents Are Allowed to Access the Additional Information?

Intuitively, the ability of the seller to control access to the additional information could impact the social welfare and the utilities of the agents. To formalize this intuition, in this section, we present a mechanism called "All-Access." In this mechanism, the seller allows all agents to obtain information, if they wish, i.e., $s_i = 1$ and $t_i = 0$ for $i = 1, 2, \ldots, n$. We observe that when the seller leaves the decision on acquiring costly information to the agents, the agents' decisions can create inefficiency. Without the seller's control, the agents may over-invest or under-invest in information acquisition. In addition, we observe that the agents tend to invest in information acquisition when their initial valuations are not too high or too low. This is in contrast with the efficient mechanism that incentivizes the agents with higher initial valuations to invest in information acquisition. We provide examples in Section 6.

The All- Access mechanism works as follows: First, agents observe their initial valuations. Next, each agent i makes a decision on incurring cost c_i and obtaining his second signal. The item is allocated via a second-price auction with no reserve price. Similar to our original setting, the investment decision of an agent is only known to him. In addition, initial valuations and second signals are private information and only their distributions are publicly known.

In this mechanism, agents bid only once after they decide on refining their valuations. To be consistence with the notation of our original setting, we denote the bid of an agent i in the second-price auction by $b_{i,1}$; the initial bid of an agent i, $b_{i,0}$, is set to be zero. The item is allocated to an agent with the highest non-negative bid, that is, $i^\star \in \arg\max_i \{b_{i,1}\}$ if $b_{i^\star,1} \geq 0$; in case of ties, the item is allocated at random to one of the agents. Agent $i^\star$ pays the second highest bid $p_{i^\star} = \max\{\max_{i \neq i^\star} \{b_{i,1}\}, 0\}$. If agent $i^\star$ has obtained additional information, his utility will be equal to $(v_{i^\star,0} + \delta_{i^\star}) - p_{i^\star} - c_{i^\star}$. Otherwise, his utility is equal to $v_{i^\star,0} - p_{i^\star}$. For any agent $i \neq i^\star$ that does not receive the item, $p_i = 0$.

We now consider the Nash Equilibrium (NE) of the All-Access mechanism. We assume that all the agents bid truthfully ($b_{i,1} = v_{i,1}$) in the auction because bidding truthfully is a weakly dominant strategy for any agent i, independent of his and other agents' decisions on obtaining information. Therefore, to characterize the Nash Equilibrium, we focus on the decision of the agents on information acquisition. We define $\tilde{e}_i(v_{i,0}) = e_i(v_{i,0}, b_{i,0} = 0, t_i = 0)$ as the investment strategy of agent i with initial valuation $v_{i,0}$ in the All-Access mechanism.

The next theorem shows that the All-Access mechanism always admits a pure strategy Nash equilibrium (NE).

THEOREM 2 **(Equilibria of the All-Access Mechanism)**. *The All-Access mechanism induces a game of incomplete information among agents where strategy of the agents are defined by $\tilde{e} = <\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n>$. For this game, there exists a pure strategy Nash equilibrium such that no agent i can gain by changing his investment strategy $\tilde{e}_i(\cdot)$ if the investment strategies of the other agents remain unchanged.*

All the proofs of this section are provided in Appendix B.

To gain insight into the All-Access mechanism, in the rest of this section, we consider the following example.

EXAMPLE 1. Assume that there are two agents that participate in the All-Access mechanism with no reserve. The cost of obtaining second signals for both agents is the same $c_1 = c_2$. The initial valuation of agents is drawn from a uniform distribution over $[0,1]$, i.e., Uniform$(0,1)$, and the second signals are drawn from Uniform$(-1,1)$.

Although the setting in Example 1 is seemingly simple, it highlights challenges in characterizing equilibrium of the All-Access mechanism. To characterize equilibrium of a game of incomplete information, the single crossing conditions are often used (Athey 2001). However, for the setting in Example 1, we show that the single crossing conditions do not hold; see Proposition 1 in the online appendix. In the light of this observation, in the next theorem, we present the equilibria of the All-Access mechanism for a wide range of the cost.

THEOREM 3. *Consider the All-Access mechanism with no reserve price and the setting in Example 1. Then,*

- *when cost $c \leq \frac{7}{96}$, there exists an equilibrium in which both agents always obtain the additional information, i.e., $\tilde{e}_i(v_{i,0}) = 1$ for $i = 1,2$ and $v_{i,0} \in [0,1]$, and*
- *when cost $c \geq \frac{7}{48}$, there exists an equilibrium in which none of the agents obtain the additional information, i.e., $\tilde{e}_i(v_{i,0}) = 0$, for $i = 1,2$ and $v_{i,0} \in [0,1]$.*

Theorem 3 characterizes the equilibrium of the All-Access mechanism when the cost is less than $\frac{7}{96} \approx 0.072$ and greater than $\frac{7}{48} \approx 0.145$. In order to compare the All-Access and the efficient mechanisms, we numerically (using an iterative procedure) find the equilibrium of the All-Access mechanism when the cost is within $(\frac{7}{96}, \frac{7}{48})$; see Online Appendix, Section C.1, for details.

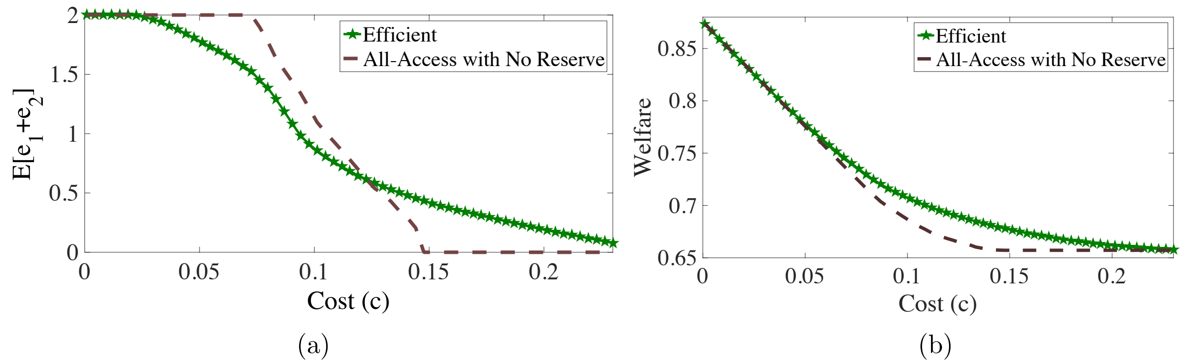


Figure 1 The average number of agents with the updated valuations and the average social welfare versus the cost in the efficient and All-Access (with no reserve) mechanisms with $n = 2$, $F = \text{Uniform}(0,1)$, and $G_i = \text{Uniform}(-1,1)$ for $i = 1,2$. Here, $c_1 = c_2 = c$.

Our numerical studies show that there exist equilibria in which agents follow *interval-based* investment decisions. An equilibrium with interval-based investment decisions can be defined with four parameters κ_1, κ_2, K_1, and K_2 where $\kappa_1 \le K_1 \in [0,1]$ and $\kappa_2 \le K_2 \in [0,1]$. In the equilibrium, each agent i obtains information only when his initial valuation $v_{i,0}$ lies in $[\kappa_i, K_i]$. We note that the range $[\kappa_1, K_1]$ is not necessarily equal to $[\kappa_2, K_2]$. That is, there exist equilibria in which the investment decisions of the agents are not symmetric.[5]

Under an interval investment strategy, agents do not obtain information when their initial valuations are too low or too high. The intuition is that an agent with high initial valuation does not have incentive to invest in information acquisition as he already has a high chance of winning the item without incurring the cost of information. On the other hand, an agent with low initial valuation is not willing to acquire costly information because he has a slim chance of winning the item.

Figures 1a and 1b compare the All-Access mechanism with the efficient mechanism. Observe that the equilibria of the All-Access mechanism may not be efficient as agents in the All-Access mechanism tend to over-invest in information acquisition when the cost is low, and they tend to under-invest when the cost is high. Yet another source of inefficiency comes from that fact that agents follow interval-based investment strategies and as a result, they may not acquire information when their initial valuations are high. This is in contrast with the efficient mechanism that motivates agents with higher initial valuations to acquire information; see Section 6.

[5] We note that when the cost is less than $\frac{7}{96}$, the All-Access mechanism has an interval-based equilibrium with $\kappa_1 = \kappa_2 = 0$ and $K_1 = K_2 = 1$. Similarly, when the cost is greater than $\frac{7}{48}$, the All-Access mechanism has an interval-based equilibrium with $\kappa_1 = K_1$ and $\kappa_2 = K_2$.

5. Maximizing Revenue

In the previous sections, we presented a welfare-maximizing (efficient) mechanism. Our goal here is to design a revenue-optimal mechanism. We start with a heuristic called "Sequential Second-Price (SSP) Mechanism." In this mechanism, the allocation and payments are determined via a second-price auction that makes the mechanism appealing from a practical perspective. Furthermore, this class of mechanisms is motivated in part by the structure of the efficient mechanism.

Despite the desirable properties of the SSP mechanism, it is not able to achieve the maximum revenue. Therefore, in Section 5.2, we present a revenue-optimal mechanism. The allocation rule of this mechanism favors agents with higher initial valuations and extracts more revenue from those agents in the first round. Both these mechanisms control agents' access to the additional information. In Section 5.3, we investigate the impacts of such a control by re-visiting the All-Access mechanism.

5.1. Sequential Second-Price Mechanisms

The second-price auctions and their variations are prevalent in online advertising and are used by Google and other major platforms. In this section, we present a class of mechanisms, called *Sequential Second-Price* (SSP), which extends the second-price auction to our setting with dynamic information acquisition.

The SSP mechanisms are similar to the efficient mechanism. The main difference is that the SSP mechanism sets a lower bound (reserve price) r on the final bid of the agents. That is, it allocates the item to the agent with the highest bid as long as his bid is greater than or equal to r. In fact, the mechanism selects agents in set $\mathcal{S}_r$, where

$$\mathcal{S}_r(b_0) \in \underset{S \subseteq \{1,\cdots,n\}}{\arg\max} \{\Omega_r(b_0, S)\}$$

Here,

$$\Omega_r(b_0, S) \;=\; \underset{S \subseteq \{1,\cdots,n\}}{\arg\max} \left\{ \mathrm{E}_S \left[\max\left\{ \max_{i \in S}\{b_{i,0} + \delta_i\}, \max_{i \notin S}\{b_{i,0}\}, r \right\} \right] - \sum_{i \in S} c_i \right\}.$$

Each selected agent i pays $t_i(b_0)$ in the first round, where $t_i(b_0)$ is given in Eq. (2).

The parameter r in the SSP mechanism can be optimized to maximize the revenue of the seller. In the online appendix, Section E.2, we compare the revenue of the SSP mechanism with the optimal reserve and the revenue-optimal mechanism. In our examples, the SSP mechanism yields more than 84% of the optimal revenue.

In practice, the SSP mechanism can be implemented via private auctions (Google AdX Documentation 2015) and pre-negotiated contracts that grant advertisers access to additional information in advance and advertisers bid for the impressions over time. For instance, consider a set of advertisers that are willing to display their ads on a specific website over a period of time. Their initial valuations, which depend on the contents of the website and advertisers' products and services, remain constant over time. However, advertisers' final valuation may vary over time because it also depends on the demographic or behavioral attributes of the user(s). In this case, there is no need for advertisers to report their initial valuations for every impression; they only need to report their updated valuations.

5.2. Revenue-Optimal Mechanism

For revenue maximization, without loss of generality, using the revelation principle (cf. Myerson (1986)), we focus on IC and IR mechanisms.

Definition 1 (Optimality). An incentive compatible and individually rational mechanism is *optimal* if it maximizes the revenue, equal to $\mathrm{E}\left[\sum_{i=1}^{n}(t_i + p_i)\right]$, among all incentive compatible and individually rational mechanisms.[6]

Let $\alpha(v_{i,0}) = -\frac{(1-F(v_{i,0}))}{f(v_{i,0})}$ be the negative of the inverse hazard rate associated with distribution F. We make the following assumption about $\alpha(\cdot)$.

Assumption 1 (**Monotone Hazard Rate**). Distribution F, with p.d.f. f, has a monotone hazard rate; that is, $\alpha(\cdot)$ is non-decreasing in $v_{i,0}$. Furthermore, assume that $\alpha(\cdot)$ is differentiable and $\sup_{v_{i,0}\in[\underline{v},\bar{v}]}\{\alpha'(v_{i,0})\} < \infty$.

The above assumption is standard in the revenue-optimal mechanism design and ensures that the virtual valuations of the agents are increasing in their initial valuations (Myerson 1981). We now present a revenue-optimal mechanism.

[6] The optimality is defined among all two-stage mechanisms. As discussed in the previous section, a mechanism with an adaptive selection rule may obtain higher revenue.

$\mathcal{M}^{\text{OPT}}$ **Mechanism:** The selection, allocation, and payment rules are defined as follows:

- **Selection:** Select the following set of agents:

$$\mathcal{S}_{\text{OPT}}(b_0) \in \underset{S\subseteq\{1,2,\ldots,n\}}{\arg\max} \left\{ \mathrm{E}_S \left[\max\left\{ \max_{i\in S}\{b_{i,0}+\alpha(b_{i,0})+\delta_i\}, \max_{i\notin S}\{b_{i,0}+\alpha(b_{i,0})\}, 0 \right\}\right] - \sum_{i\in S} c_i \right\}, \tag{4}$$

and t_i is defined the same as before; see Eq. (2).

- **Allocation and Payments:** Allocate the item to the agent with the highest non-negative weighted bid. More precisely, consider an agent $i^\star \in \operatorname{argmax}_i\{b_{i,1}+\alpha(b_{i,0})\}$. If $b_{i^\star,1}+\alpha(b_{i^\star,0}) \geq 0$, then the item is allocated to agent $i^\star$ at price $p_{i^\star} = \max\{\max_{i\neq i^\star}\{b_{i,1}+\alpha(b_{i,0})\}, r\} - \alpha(b_{i^\star,0})$ where $r:\mathbb{R}^n \to \mathbb{R}$ is a function of initial bids and is defined below.

With slight abuse of notations, let $\ell \in \arg\max_{j\in\mathcal{S}_{\text{OPT}}(b_0)}\{b_{j,0}+\alpha(b_{j,0})\}$ be an unselected agent with the highest weighted bid. Then, if $b_{\ell,0}+\alpha(b_{\ell,0})<0$ or all agents are selected, $r=0$. Otherwise, r solves the following equation

$$\int_{\max\{r,0\}}^{b_{\ell,0}+\alpha(b_{\ell,0})} \Pr\left[z \geq \max_{j\in\mathcal{S}_{\text{OPT}}(b_0)}\left\{b_{j,0}+\delta_j+\alpha(b_{j,0})\right\}\right]dz = \int_{\underline{v}}^{b_{\ell,0}} \mathrm{E}\left[q_\ell \middle| v_{\ell,0}=z, v_{-\ell,0}=b_{-\ell,0}\right]dz \tag{5}$$

Lemma 3 and Corollary 2 in the online appendix show that there exists an $r\in[0, b_{\ell,0}+\alpha(b_{\ell,0})]$ that solves the above equation. Note that similar to the efficient mechanism, r does not change the allocation or payment for selected agents.

Mechanism $\mathcal{M}^{\text{OPT}}$ is built upon the ideas of *virtual value* formulation of Myerson (1981) for static revenue-maximizing mechanism design. It allocates the item to the agent that has the highest (final) virtual valuation $v_{i,1}+\alpha(v_{i,0})$. The mechanism maximizes the virtual value of the winner minus the cost of information acquisition.

The following theorem establishes the optimality of $\mathcal{M}^{\text{OPT}}$.

THEOREM 4 **(Revenue-Optimal Mechanism)**. *Suppose Assumption 1 holds. Mechanism $\mathcal{M}^{\text{OPT}}$ described above is incentive compatible, individually rational, and optimal.*

In the online appendix, Section A, we show that mechanism $\mathcal{M}^{\text{OPT}}$ is IC and IR. To complete the proof of Theorem 4, in the following we show that mechanism $\mathcal{M}^{\text{OPT}}$ is revenue-optimal. First, via the following lemma, we show that the selection rule aims to optimize the revenue by maximizing the expected "virtual revenue."

Lemma 1 (**Revenue of $\mathcal{M}^{\text{Opt}}$**). *If all the agents follow the truthful strategy, then the expected revenue of $\mathcal{M}^{\text{Opt}}$ is equal to*

$$\mathrm{E}\left[\max_{S\subseteq\{1,2,\ldots,n\}}\left\{\mathrm{E}_S\left[\max\left\{\max_{i\in S}\{v_{i,0}+\alpha(v_{i,0})+\delta_i\},\max_{i\notin S}\{v_{i,0}+\alpha(v_{i,0})\},0\right\}\right]-\sum_{i\in S}c_i\right\}\right],\quad (6)$$

where the inner expectation is with respect to the second signals and the outer expectation is with respect to the initial valuations.

The proof is given in Appendix D.1. We now provide an upper bound on the revenue of any IC mechanisms that matches the revenue of mechanism $\mathcal{M}^{\text{Opt}}$.

Lemma 2 (**Upper Bound**). *The expected revenue of the seller is at most equal to Eq. (6).*

The upper bound is established using a closely related problem with fewer constraints called the relaxed problem, cf. Ëso and Szentes (2007), Kakade et al. (2013), and Pavan et al. (2014). In the relaxed problem, the mechanism, on the behalf of an agent, can decide to obtain information, and then both the agent and the mechanism learn his second signal. Because any mechanism that is IC in the original setting would also be IC in the relaxed setting, the revenue of the optimal relaxed mechanism provides an upper bound for the revenue of the revenue-optimal mechanism in the original setting; see Appendix D.1 for details.

The revenue-optimal mechanism is a generalization of the handicap mechanism of Ëso and Szentes (2007) and matches the mechanism when information acquisition is costless (i.e., $c_i = 0$) where the seller allows all agents access to information. We show that when the information is costly, the seller grants access to additional information only to a subset of agents. The intuition is that the seller indirectly bears the cost of information because the costs will affect the agents' (ex-ante) willingness to pay.

An important distinction between our mechanism and the handicap mechanism is the selection rule that grants access to additional information to a right set of agents. A technical challenge that we need to address is that the selection rule depends on the initial bids which, as we discuss in Section 6 is "non-monotone" in the initial valuations of the agents; also see Lemma 8 for more details.

5.3. What If All Agents Are Allowed to Access to the Additional Information?

Here, we re-visit the All-Access mechanism to study the impacts of controlling access to the additional information on the revenue-optimal mechanism. Specifically, we will introduce a reserve price r to the All-Access mechanism to ensure that the seller has a degree of freedom to extract more revenue from the agents. In this mechanism, privately informed agents decide on updating their valuations and then participate in a second price auction with reserve price r. One can easily extend Theorem 2 to show that the All-Access mechanism with a reserve price always admits a pure strategy Nash equilibrium.

In the following, we will re-examine Example 1 when agents participate in the All-Access mechanism with reserve price r.

Example 2. Assume that there are two agents that participate in the All-Access mechanism with reserve price $r \geq 0$. The cost of obtaining second signals for both agents is the same, $\delta_i \sim \text{Uniform}(-1,1)$, and $v_{i,0} \sim \text{Uniform}(0,1)$ for $i = 1,2$.

The following theorem, which generalizes Theorem 3, sheds light on the equilibria of the All-Access mechanism with reserve price r for a wide range of the cost.

Theorem 5. *Consider the All-Access mechanism with reserve price $r \in [0,1]$ and the setting in Example 2. Then,*

- *when cost $c \leq \min\{\frac{4r^3-3r^2-6r+5}{48}, \frac{8r^3+6r^2+7}{96}\}$, there exists an equilibrium in which both agents always obtain information, i.e., $\tilde{e}_i(v_{i,0}) = 1$, $i = 1,2$, for $v_{i,0} \in [0,1]$, and*
- *when $c \geq \frac{3r^4+8r^3+6r^2+7}{48}$ and $r \leq \sqrt{2}-1$, or $c \geq \frac{3r-r^3+1}{12}$ and $r \in (\sqrt{2}-1, 1]$, there exists an equilibrium in which none of the agents obtain information, i.e., $\tilde{e}_i(v_{i,0}) = 0$, $i = 1,2$, for $v_{i,0} \in [0,1]$.*

The proof is given in Section C.3 of the online appendix. Figure 2a depicts the results of Theorem 5.

Next, in Figure 2b, we compare the All-Access mechanism (with revenue-maximizing reserve price) with mechanism $\mathcal{M}^{\text{Opt}}$ in terms of their collected revenue for different values of the cost. As expected, the All-Access mechanism fails to obtain the maximum revenue. Surprisingly, the revenue of the All-Access mechanism is not monotone in the cost. For instance, the revenue of the All-Access mechanism when the cost c is 0.18 is less than when the cost is 0.2. This follows from agent's investment strategies. For $c = 0.2$, none of the agents acquire information while for $c = 0.18$, agents update their valuations when

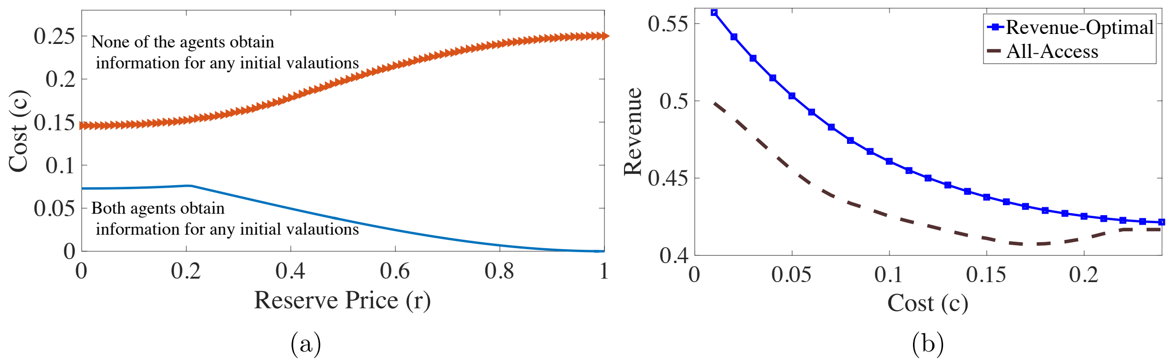


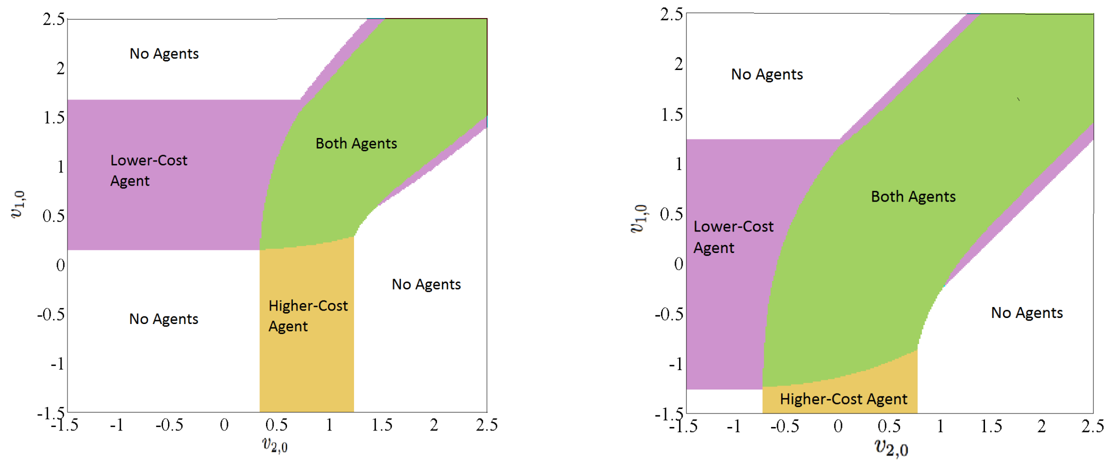
Figure 2 Figure 2a depicts the results in Theorem 5. Figure 2b shows the revenue of the revenue-optimal and All-Access (with revenue-maximizing reserve price) mechanisms versus the cost with $n = 2$, $F =$ Uniform$(0, 1)$, and $G_i =$ Uniform$(-1, 1)$ for $i = 1, 2$. Here, $c_1 = c_2 = c$.

their initial valuations are close to 0.5. For the latter case, it is likely that the updated valuations of the agents fall below the reserve price considering the fact that the revenue-maximizing reserve price is almost 0.5 and second signals are drawn from Uniform$(-1, 1)$. This will reduce the seller's revenue. In fact, the seller prefers that none of the agents obtain information.

6. Who Will Be Selected?

Here, we discuss the selection rule of our mechanisms in more detail. We first present an example that shows that the selection rule may not be monotone in initial valuations. We then show that under certain symmetry assumptions, the selection rule favors agents with higher initial valuations. Finally, we demonstrate how the selection rule reacts to increase in the cost and variance of the second signals.

Figures 3a and 3b depict the selected agents in the revenue-optimal and efficient mechanisms, respectively, for all realizations of $v_{1,0}$ and $v_{2,0}$ in the range of $[-1.5, 2.5]$ with $n = 2$, $F = N(0.5, 0.5)$, and $G_i = N(0, 0.5)$ for $i = 1, 2$. The cost of information for the first and second agents is, respectively, 0.01 and 0.05, that is, $c_1 = 0.01$ and $c_2 = 0.05$. The x-axis is the initial valuation of the second agent, and the y-axis is the initial valuation of the first agent. The areas in the figures are divided into several regions. In the white and green regions, the number of selected agents is zero and two, respectively. In the purple regions, only agent 1 whose cost of information is lower is selected while in the yellow regions, only agent 2 is selected.



(a) Optimal Mechanism

(b) Efficient Mechanism

Figure 3 Selected agents for different realizations of $v_{1,0}$ and $v_{2,0}$ with $n=2$, $c_1=0.01$, $c_2=0.05$, $F=N(0.5,0.5)$, and $G_i=N(0,0.5)$ for $i=1,2$.

Non-monotonicity of Selection Rule: Observe that the number of selected agents does not always increase as we move along one of the axes. Furthermore, when the initial valuation of a selected agent increases and the initial valuation of the other agent remains the same, he will not necessarily be selected. For instance, the efficient mechanism selects both agents when $v_{1,0}=v_{2,0}=1$ but does not select any agents when $v_{1,0}=2.4$ and $v_{2,0}=1$. The reason that the selection rule is not monotone is that all agents, including those who did not update their valuations, participate in the second round and have a chance to win the item; see Section 7.1. The selection rule will remain non-monotone even if the cost of information is the same across agents.

Also observe that the selection rule of both mechanisms favors agents with higher initial valuations and lower costs, as agents with high initial valuations are more likely to win the item in the second round. We can formalize this intuition under certain symmetric and independence assumptions.

THEOREM 6. *Suppose for each agent i, the second signal δ_i is drawn, independently of other agents' signals, from distribution $G_i=\mathcal{G}$. In addition, assume that distribution $\mathcal{G}$ is symmetric and $c_i=c$ for $i=1,2,\ldots,n$. If an agent is selected in the revenue-optimal or efficient mechanisms, then all agents with higher initial bids will also be selected.*

A distribution $\mathcal{G}$ is symmetric if $\mathcal{G}(-y)=1-\mathcal{G}(y)$. For instance, normal and uniform distributions satisfy the assumption. The proof is presented in Appendix B. The intuition is that the seller's objective is an increasing function of the maximum valuation of the agents. When agents are symmetric in terms of the cost and distribution of second signals, the seller would rather select agents with higher initial valuations. These are the agents who have a greater chance of receiving the item in the second round. We note that when the distribution of the valuations are asymmetric, it is likely that the mechanism selects an agent with lower initial valuation but higher variance of the second signal.

Theorem 6 provides a simple way to find the selected agents. One can sort the agents according to initial valuations (bids) in descending order and evaluate the value of the selection rule's objective function for each of the $n+1$ subsets $\emptyset$, $\{1\}$, $\{1,2\}$, $\cdots$, and $\{1,2,\cdots,n\}$ and then select the subset that maximizes the objective. See Guha et al. (2006) and Goel et al. (2010) for optimization problems similar to our selection rule in more general settings.

6.1. Impacts of the Cost and Variance of Second Signals on Selection Rules

To get more insight about the selection rule, we numerically study how the expected number of selected agents changes as the cost and variance of second signals increase. Figure 4a illustrates the impact of the variance of second signals, σ^2, on the average number of selected agents with $n=2$, $F=N(0.5,0.5)$, $G_i=N(0,\sigma^2)$, and $c_i=0.05$ for $i=1,2$[7] When the second signals are more "uncertain," the average number of selected agents in the revenue-optimal (OPT) and efficient (EFF) mechanisms increases. The intuition is that for larger variance, the seller anticipates seeing larger second signals and selects more agents.

Figure 4b depicts the impacts of the cost of information, $c_i=c$, on the average number of selected agents when $n=2$, $F=N(0.5,0.5)$, and $G_i=N(0,0.5)$. Both revenue-optimal and efficient mechanisms react to an increase in the cost of information by restricting the number of agents that can obtain information. Interestingly, to obtain higher revenue, the revenue-optimal mechanism selects fewer agents. This implies that the revenue-optimal mechanism distorts the revelation of information to extract more revenue from the agents.

[7] See Section E.3 of the online appendix regarding the impact of the number of agents.

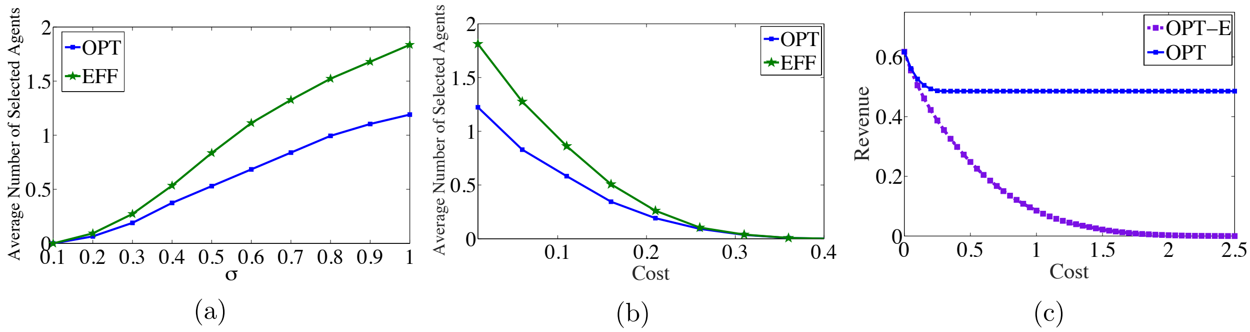


Figure 4 Figure 4a- Average number of selected agents in the revenue-optimal and efficient mechanisms versus the standard deviation of second signals, σ, with $G_i = N(0,\sigma^2)$ and $c_i = 0.05$ for $i = 1,2$. Figure 4b- Average number of selected agents in the revenue-optimal and efficient mechanisms versus the cost of information, c, with $G_i = N(0,0.5)$. Figure 4c- Average revenue of the OPT and OPT-E mechanisms versus the cost with $G_i = N(0,0.5)$, and $c_i = c$ for $i = 1,2$. In all the figures, $n = 2$ and $F = N(0.5,0.5)$.

7. Extensions

In this section, we discuss some of the extensions of our mechanisms. In particular, we extend our mechanism to a setting where agents are"extremely risk-averse" in a sense that they do not engage in the auction without obtaining additional information. Moreover, we generalize our mechanisms to a setting with multiple units for sale.

7.1. Information Acquisition as an Entry Cost

In our model, the seller may allocate the item to an agent who has not obtained additional information. However, in some applications, such as the sale of high-valued assets, buyers might face a significant risk if they purchase the item without gathering enough information. For these applications, one can interpret the cost of obtaining information as an entry cost. That is, buyers must invest in information to be considered in the allocation round.

Our proposed mechanisms can be extended to this setting by excluding unselected agents from the allocation stage. More precisely, an efficient mechanism in this setting selects the following set of agents

$$\mathcal{S}_{\text{EFF-E}}(b_0) = \underset{S\subseteq\{1,2,\ldots,n\}}{\arg\max} \left\{ \mathrm{E}_S \left[\max\left\{ \max_{j\in S}\{b_{j,0} + \delta_j\}, 0 \right\} \right] - \sum_{j\in S} c_j \right\},$$

and similarly, a revenue-optimal mechanism selects the following set of agents

$$\mathcal{S}_{\text{OPT-E}}(b_0) = \underset{S\subseteq\{1,2,\ldots,n\}}{\arg\max} \left\{ \mathrm{E}_S \left[\max\left\{ \max_{j\in S}\{b_{j,0} + \alpha(b_{j,0}) + \delta_j\}, 0 \right\} \right] - \sum_{j\in S} c_j \right\}.$$

Note that when $S=\emptyset$, the mechanisms do not allocate the item. We refer to the corresponding efficient and revenue-optimal mechanisms as EFF-E and OPT-E, respectively.

We note that mechanism OPT-E is a generalization of a mechanism proposed by Lu and Ye (2014), who study the problem of designing a revenue-maximizing mechanism with an entry cost when the cost of information is the same across agents. They show that the selection rule in this setting is monotone, that is, the number of selected agents decreases as the initial valuation of an agent increases. Using monotonicity in the selection rule, Lu and Ye (2014) show that their proposed mechanism is incentive compatible. Note that, as we saw in Section 6, the selection rule will not be monotone if the item can be allocated to an unselected agent.[8]

To get more insight, we numerically compare mechanisms $\mathcal{M}^{\text{OPT}}$ and OPT-E. We assume that the cost of information is the same across agents, the number of agents $n=2$, $F=N(0.5,0.5)$, and $G_i=N(0,0.5)$ for $i=1,2$. Figure 4c compares the revenue of the mechanisms. For any value of the cost, mechanism $\mathcal{M}^{\text{OPT}}$ yields more revenue than OPT-E. The revenue of the mechanism OPT-E approaches zero as acquiring information becomes more costly. However, mechanism $\mathcal{M}^{\text{OPT}}$ is more robust to the cost of information because, when the cost is high, it can allocate the item without allowing any agents to update their valuations.

7.2. Multi-unit

In this section, we discuss how our mechanisms can be extended to a setting where $m\geq 1$ units of the item are sold to $n>m$ agents. Specifically, we assume that each agent i needs at most one unit of the item, and his initial valuation for the item is $v_{i,0}\in[\underline{v},\overline{v}]$. Similar to our original setting, $v_{i,0}$ is agent i's private information and is drawn (independently) from distribution F.

We start with an efficient mechanism. The mechanism allocates the item to m agents with the highest (non-negative) bids. In case the number of agents with a positive bid is less than m, the mechanism allocates the item to agents with positive bids. The efficient mechanism selects the following set of agents

$$\max_{S\subseteq\{1,2,\dots,n\}}\left\{\mathrm{E}_S\left[\max_{A\subseteq\{1,2,\dots,n\},|A|\leq m}\sum_{i\in A}b_{i,1}\right]-\sum_{i\in S}c_i\right\}, \tag{7}$$

[8] We also relax an assumption in Lu and Ye (2014) where the mechanism can verify whether or not a selected agent has invested in obtaining information.

where A is the set of agents that the item will be allocated to, $|A|$ is the cardinality of set A, and the condition $|A| \leq m$ implies that we cannot sell more than m units. In addition, $b_{i,1} = b_{i,0} + \delta_i$ if $i \in S$, and $b_{i,0}$ otherwise. It is easy to see that the selection rule of the efficient mechanism selects a welfare-maximizing set of agents provided that it is incentive compatible.

The initial payment of the selected agents is the same as before and is given in Eq. (2). In addition, only agents that receive the item will pay in the second round. Specifically, any selected agent that wins the item pays $\max\{b^{(m+1)}, 0\}$, where with slight abuse of notations, $b^{(m+1)}$ is the $(m+1)^{\text{th}}$ highest final bid. Furthermore, any unselected agent j that receives the item pays $\max\{b^{(m+1)}, r_j\}$, where r_j solves the following equation

$$\int_{\max\{r_j,0\}}^{b_{j,0}} \Pr\left[z \geq b_{-j}^{(m)}\right] dz = \int_{\underline{v}}^{b_{j,0}} \mathrm{E}\left[q_j \middle| v_{j,0} = z, v_{-j,0} = b_{-j,0}\right] dz \ , \tag{8}$$

where $b_{-j}^{(m)}$ is the m^{th} highest final bid among all agents except for agent j. The term inside the integral of the l.h.s., $\Pr\left[z \geq b_{-j}^{(m)}\right]$, is the cumulative distribution function of random variable $b_{-j}^{(m)}$ at point z. Using a similar argument in Lemma 3 in the online appendix, one can show that there exists $r_j \in [0, b_{j,0}]$ that solves the above equation.

Next, we will present a revenue-optimal mechanism in this setting. Similar to mechanism $\mathcal{M}^{\text{OPT}}$, the mechanism allocates the item to m agents with the highest (non-negative) weighted bid, where the weighted bid of an agent i is $b_{i,1} + \alpha(b_{i,0})$. The mechanism allows the following set of agents

$$\max_{S \subseteq \{1,2,\ldots,n\}} \left\{ \mathrm{E}_S \left[\max_{A \subseteq \{1,2,\ldots,n\}, |A| \leq m} \sum_{i \in A} \left(b_{i,1} + \alpha(b_{i,0})\right) \right] - \sum_{i \in S} c_i \right\} \tag{9}$$

to acquire information. The initial payment is given in Eq. (2). In the second round, any selected agent that receives the item has to pay the $(m+1)^{\text{th}}$ highest weighted bid if the $(m+1)^{\text{th}}$ highest weighted bid is positive and is zero otherwise. In addition, any unselected agent j that wins the item has to pay a maximum of $(m+1)^{\text{th}}$ highest weighted bid and r_j. Here, r_j solves Eq. (8), where $b_{-j}^{(m)}$ should be replaced by the m^{th} highest final weighted bid among all agents except for agent j. For any agents that do not receive the item, their payment in the second round is zero.

8. Conclusion

Information structure plays a crucial role in the outcome of auctions. This role becomes even more important when information acquisition is costly. We observe that in such environments, agents may over or under-invest in information. We also presented efficient and revenue-optimal mechanisms that shows how auctioneer should control the access to information via a selection rule and prices.

In the previous section, we discussed some of the extensions of our mechanism. An important direction for future research is to extend the results to settings with adaptive selection rules where information is disclosed sequentially over time, and the mechanism makes "selection decisions" based on updated reported valuations of the agents (McAfee and McMillan 1988). Another important direction is studying environments where the agents are risk-averse.

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Auctions with Dynamic Costly Information Acquisition: Online Appendix

Appendix A: Sequential Weighted Second-Price Mechanism & Proofs of Theorems 1 and 4

In this section, we present a parameterized class of mechanisms called Sequential Weighted Second-Price (SWSP), which is denoted by $\mathcal{M}(\rho,\beta)$. Weight function $\beta:\mathbb{R}\to\mathbb{R}$ connects the bids in the first and the second rounds by manipulating the final allocation and payments in the favor of agents with higher initial valuations. Parameter $\rho\in\mathbb{R}$ specifies a lower bound on weighted bids the seller is willing to accept for the item.

As it will be more clear later, this class of mechanism includes the efficient and optimal mechanisms. We will show that the SWSP mechanisms are incentive compatible and individually rational. As a corollary of this result, we will conclude that the efficient and optimal mechanisms are IC and IR; that is, Theorems 1 and 4 hold.

We make the following assumption on function β.

Assumption 2. Weight function β is *non-decreasing* and differentiable with bounded derivatives, that is, $\sup_z\{\beta'(z)\}<\infty$, $z\in[\underline{v},\overline{v}]$.

Note that as it will be more clear later, the non-decreasing function β alters the social welfare and the seller's revenue by distorting the allocation via favoring agents with higher valuations. The weight function β and parameter ρ can be used to adjust the social welfare and the revenue of the mechanism. For instance, as we show later, for the efficient mechanism, function $\beta(\cdot)$ and ρ are equal to 0.

Let us start with the description of the mechanism.

Sequential Weighted Second-Price Mechanism $\mathcal{M}(\rho,\beta)$: The selection, allocation, and payment rules are defined as follows:

- Selection: Select the following set of agents

$$\mathcal{S}_{\rho,\beta}(b_0)\in \underset{S\subseteq\{1,\cdots,n\}}{\arg\max}\left\{\Omega_{\rho,\beta}(b_0,S)\right\}\ , \tag{10}$$

where $\Omega_{\rho,\beta}(b_0,S)$, the *weighted surplus*, is defined as follows

$$\Omega_{\rho,\beta}(b_0,S)=\mathrm{E}_S\Big[\max\Big\{\max_{i\in S}\{b_{i,0}+\beta(b_{i,0})+\delta_i\},\max_{i\notin S}\{b_{i,0}+\beta(b_{i,0})\},\rho\Big\}\Big]-\sum_{i\in S}c_i. \tag{11}$$

The expectation is with respect to the second signals of the selected agents; in case of ties, we will choose one of the sets at random. Each selected agent i pays $t_i(b_0)$ to the seller; see Eq. (2).

• Allocation and Payments: Agents participate in a "weighted second-price" auction with a reserve price ρ, where the mechanism allocates the item to the agent with the highest weighted bid as along as it exceeds ρ. More precisely, consider an agent $i^\star \in \text{argmax}_i\{b_{i,1} + \beta(b_{i,0})\}$. If $b_{i^\star,1} + \beta(b_{i^\star,0}) \geq \rho$, then the item is allocated to agent $i^\star$. If agent $i^\star$ was a selected agent, then he pays $p_{i^\star} = \max\{\max_{i\neq i^\star}\{b_{i,1} + \beta(b_{i,0})\}, \rho\} - \beta(b_{i^\star,0})$. If agent $i^\star$ is not a selected agent, then he pays $p_{i^\star} = \max\{\max_{i\neq i^\star}\{b_{i,1} + \beta(b_{i,0})\}, r\} - \beta(b_{i^\star,0})$, where $r : \mathbb{R}^n \to \mathbb{R}$ will be defined below.

Note that by letting $\rho = 0$ and $\beta(\cdot) = 0$, we can implement mechanism $\mathcal{M}^{\text{EFF}}$. Furthermore, by setting $\rho = 0$ and $\beta(\cdot) = \alpha(\cdot)$, we have mechanism $\mathcal{M}^{\text{OPT}}$.

Let ℓ be an unselected agent with the highest weighted bid, i.e., $\ell \in \arg\max_{j\notin\mathcal{S}_{\rho,\beta}(b_0)}\{b_{j,0} + \beta(b_{j,0})\}$. Then, if $b_{\ell,0} + \beta(b_{\ell,0}) < \rho$ or all agents are selected, $r = \rho$. Otherwise, r solves the following equation

$$\int_{\max\{r,\rho\}}^{b_{\ell,0}+\beta(b_{\ell,0})} \Pr\left[z \geq \max_{j\in\mathcal{S}_{\rho,\beta}(b_0)}\left\{b_{j,0} + \delta_j + \beta(b_{j,0})\right\}\right] dz = \int_{\underline{v}}^{b_{\ell,0}} \mathrm{E}\left[q_\ell \middle| v_{\ell,0} = z, v_{-\ell,0} = b_{-\ell,0}\right] dz\ . \quad (12)$$

The next lemma shows that there exists an $r \in [\rho, b_{\ell,0} + \beta(b_{\ell,0})]$ that solves the above equation.

Lemma 3. *Consider agent $\ell \in \arg\max_{j\notin\mathcal{S}_{\rho,\beta}(b_0)}\{b_{j,0} + \beta(b_{j,0})\}$ in mechanism $\mathcal{M}(\rho,\beta)$. If $b_{\ell,0} + \beta(b_{\ell,0}) \geq \rho$, then there exists $r \in [\rho, b_{\ell,0} + \beta(b_{\ell,0})]$ that satisfies Eq. (12).*

The following results are immediate corollaries of Lemma 3.

Corollary 1. *Let $\ell \in \arg\max_{j\notin\mathcal{S}_{\text{EFF}}(b_0)}\{b_{j,0}\}$ be an unselected agent ℓ with the highest initial bid in mechanism $\mathcal{M}^{\text{EFF}}$. Then, if $b_{\ell,0} \geq 0$, then there exists $r \in [0, b_{\ell,0}]$ that satisfies Eq. (3)*

Corollary 2. *Let $\ell \in \arg\max_{j\notin\mathcal{S}_{\text{OPT}}(b_0)}\{b_{j,0} + \alpha(b_{j,0})\}$ be an unselected agent ℓ with the highest weighted bid in mechanism $\mathcal{M}^{\text{OPT}}$. Then, if $b_{\ell,0} + \alpha(b_{\ell,0}) \geq 0$, then there exists $r \in [0, b_{\ell,0} + \alpha(b_{\ell,0})]$ that satisfies Eq. (5).*

We now present the main result of this section, which shows that the proposed mechanism is IC and IR.

Theorem 7 **(Incentive Compatibility)**. *Suppose $\rho \geq 0$ and function β satisfies Assumption 2. Then, the Sequential Weighted Second-Price mechanism $\mathcal{M}(\rho,\beta)$ is incentive compatible and individually rational.*

The proof of the theorem is given in Appendix A.1.

If mechanism $\mathcal{M}(\rho,\beta)$ is incentive compatible, it maximizes the *weighted surplus* defined below as

$$\Omega_{\rho,\beta}(v_0) = \arg\max_{S\subseteq\{1,\ldots,n\}}\{\Omega_{\rho,\beta}(v_0, S)\}$$

The assumption that the derivatives of the function β are bounded will ensure us that the weighted surplus, $\Omega_{\rho,\beta}(v_0)$, is absolutely continuous in the initial valuations of the agents. Note that $\Omega_{\rho=0,\beta=0}(v_0)$ and $\Omega_{\rho=0,\beta=\alpha}(v_0)$ are, respectively, the maximum social welfare and virtual revenue. Therefore, Theorem 7 implies mechanisms $\mathcal{M}^{\text{EFF}}$ and $\mathcal{M}^{\text{OPT}}$ are efficient and optimal, respectively.

A.1. Proof of Theorem 7

In this section, we prove Theorem 7. We start with incentive compatibility and show that no agents would prefer to deviate from the truthful strategy, as long as all other agents are truthful. We prove this by going over the strategy of an agent in a backward manner. First, using Lemma 4, we show that agents bid truthfully in the second round. Then, we prove that a selected agent obtains the additional information (Lemma 5). Finally, in Lemma 8 we show that agents will be better off by being truthful in the first round. We present the proof of Lemma 8 in Section A.2 since it our key technical lemma. The proofs of other lemmas are relegated to Section D.

The key challenging part is to show that agents bid truthfully in the first round. The reason is that the effects of initial bids are twofold. First, they determine the set of selected agents. Second, they influence the final allocation of the item.

The following lemma shows that agents who can bid in the second round will be truthful even if they were untruthful in the first round. Precisely, we will show that

$$v_{i,1} = \arg\max_{b_{i,1}} \left\{ q_i v_{i,1} - p_i - t_i - c_i e_i \right\}. \tag{13}$$

Note that unselected agents do not bid in the second round; that is, their initial bids are considered as their final bids.

Lemma 4 **(Truthfulness in the Second Round)**. *Under mechanism $\mathcal{M}(\rho,\beta)$, for any agent that is allowed to update his bid in the second round of bidding, truthfulness is a weakly dominant strategy, even if the agent has not been truthful in the first round.*

From a technical perspective, one of the aspects that differentiates our work from the previous work on dynamic mechanism design, in particular Ëso and Szentes (2007), is that the deviation strategies of the agents, in addition to misreporting his valuations, include the decision on obtaining information. In the following lemma, we show that a selected agent i will acquire the additional information when he bids truthfully in the first round, and all other agents follow the truthful strategy, i.e.,

$$1 = \arg\max_{e_i \in \{0,1\}} \left\{ q_i v_{i,1} - p_i - t_i - e_i c_i \middle| b_0 = v_0,\ b_{-i,1} = v_{-i,1},\ e_j = 1 \text{ if } s_j = 1,\ j \neq i \right\}, \tag{14}$$

The conditions in the above equation imply that agent i bids truthfully in the first round, and all other agents follow the truthful strategy; that is, they bid truthfully in both rounds and if they get selected, they obtain information.

LEMMA 5 **(Obtaining Additional Information)**. *Consider a selected agent i who bids truthfully in the first round. Assuming all other agents are truthful, agent i would incur cost c_i to obtain signal δ_i.*

Lemma 5 implies that a selected agent i "who bids truthfully in the first round" will obtain information. However, if agent i bids untruthfully in the first round is selected, he will not necessarily obtain information. We will show in Lemma 8 that agent i will not gain from bidding untruthfully in the first round regardless of his decision to obtain information.

The proof of Lemma 5is provided in Section D.3. To obtain the result, we show that the incentive of the selected agent i gets aligned with the selection rule. Thus, the selected agent prefers to incur cost c_i and acquire information; that is, $e_i = 1$.

The final step is to show that an agent i will bid truthfully in the first round. Let $U_i(x_i, \hat{x}_i)$ be the utility of agent i with initial valuation x_i when he bids $\hat{x}_i$ in the first round and follows the "optimal strategy" afterwards (assuming other agents are truthful). More precisely,

$$U_i(x_i, \hat{x}_i) = \max_{b_{i,1}, e_i} \left\{ \mathrm{E}\left[q_i\big((\hat{x}_i, v_{-i,0}), (b_{i,1}, v_{-i,1})\big) v_{i,1} - e_i c_i - s_i t_i\big((\hat{x}_i, v_{-i,0})\big) - p_i\big((\hat{x}_i, v_{-i,0}), (b_{i,1}, v_{-i,0})\big) \right] \right\}, \tag{15}$$

where the expectation is taken assuming that all agents except for agent i are truthful. Then, considering the fact that initial valuation $v_0 = x$, for any $j \neq i$, we have $v_{j,1} = x_j + \delta_j$ if agent j is selected and x_j otherwise. Note that after initial bidding, agent i optimizes over $(e_i, b_{i,1})$ to obtain his best (utility-maximizing) strategy.

We start with characterizing $U_i(x_i, x_i)$.

LEMMA 6. *If the vector of initial valuations is given by x, and all agents except for agent i are truthful, then the expected utility of agent i who bid truthfully in the first round, denoted by $U_i(x_i, x_i)$, is equal to*

$$U_i(x_i, x_i) = \int_{\underline{v}}^{x_i} \mathrm{E}\left[q_i \middle| v_{i,0} = z,\ v_{-i,0} = x_{-i} \right] dz, \tag{16}$$

In addition, $U_i(x_i, x_i)$ is non-decreasing in x_i.

In the proof of Lemma 6, we use lemmas 4 and 5 where we show that if agent i bids truthfully in the first round, he will prefer to follow the truthful strategy afterwards.

Lemma 6 implies that the utility of truthful agent i is fully determined by his allocation probability for different initial valuations. Furthermore, the higher his probability of allocation is, the more utility he earns. In fact, Eq. (16) is analogous to the utility of an agent in standard static incentive compatible mechanisms (see Myerson (1981)).

We now consider the utility of agent i when he bids untruthfully in the first round and follows his optimal strategy thereafter, i.e., $U_i(x_i, \hat{x}_i)$ defined in Eq. (15). By Lemma 4, if agent i with initial bid $\hat{x}_i \neq x_i$ gets selected, he bids truthfully in the second round, i.e., $b_{i,1} = v_{i,1}$. But, untruthful agent i will not necessarily obtain information if the mechanism selects him. In the following, we denote his best investing strategy by $e_i(v_{i,0} = x_i, b_{i,0} = \hat{x}_i, t_i)$.

The next lemma establishes an upper bound on $U_i(x_i, \hat{x}_i)$.

LEMMA 7. *Suppose the vector of initial valuations is given by x, and all agents except agent i are truthful. We have $U_i(x_i, \hat{x}_i) \leq \max\left\{U_i(\hat{x}_i, \hat{x}_i) + \int_{\hat{x}_i}^{x_i} \Pr\left[z + e_i\delta_i + \beta(\hat{x}_i) \geq \omega_{-i}\right] dz, 0\right\}$, where ω_{-i} is the maximum weighted bids of all agents but agent i when he misreports $\hat{x}_i$ in the first round and other agents are truthful, i.e.,*

$$\omega_{-i} = \max\left\{\max_{j \in \mathcal{S}_{\rho,\beta}(\hat{x}_i, x_{-i}), j \neq i}\{x_j + \beta(x_j) + \delta_j\}, \max_{j \notin \mathcal{S}_{\rho,\beta}(\hat{x}_i, x_{-i}), j \neq i}\{x_j + \beta(x_j)\}, \rho\right\} \tag{17}$$

and $e_i = e_i(v_{i,0} = x_i, b_{i,0} = \hat{x}_i, t_i)$.

The term inside the integral, i.e., $\Pr\left[z + e_i\delta_i + \beta(\hat{x}_i) \geq \omega_{-i}\right]$, is the probability that agent i with final weighted bid $z + e_i\delta_i + \beta(\hat{x}_i)$ wins the item when agents in set $\mathcal{S}_{\rho,\beta}(\hat{x}_i, x_{-i})\backslash\{i\}$ obtain information and agent i follows an investing strategy associated with $e_i = e_i(v_{i,0} = x_i, b_{i,0} = \hat{x}_i, t_i)$.

Next, we show that $U_i(x_i, x_i) \geq U_i(x_i, \hat{x}_i)$; that is an agent i prefers to bid truthfully in the first round. In Lemma 7, we find an upper bound for $U_i(x_i, \hat{x}_i)$. Then, when $U_i(x_i, \hat{x}_i) = 0$, immediately we have $U_i(x_i, x_i) \geq U_i(x_i, \hat{x}_i) = 0$. Now we show that even the upper bound of $U_i(x_i, \hat{x}_i)$, i.e., $U_i(\hat{x}_i, \hat{x}_i) + \int_{\hat{x}_i}^{x_i} \Pr\left[z + e_i\delta_i + \beta(\hat{x}_i) \geq \omega_{-i}\right] dz$, is smaller than $U_i(x_i, x_i)$, where $e_i = e_i(v_{i,0} = x_i, b_{i,0} = \hat{x}_i, t_i)$.

We start with defining a suboptimal selection rule $\hat{\mathcal{S}}_{y_1,y_2}(z, x_{-i})$ for any nonzero measure interval $[y_1, y_2]$ such that $y_1 \leq y_2$ and $[y_1, y_2] \subseteq [\min\{x_i, \hat{x}_i\}, \max\{x_i, \hat{x}_i\}]$, where

$$\hat{\mathcal{S}}_{y_1,y_2}(z, x_{-i}) = \begin{cases} \mathcal{S}_{\rho,\beta}(\hat{x}_i, x_{-i})\backslash\{i\} & \text{if } e_i = 0 \text{ and } z \in [y_1, y_2]; \\ \mathcal{S}_{\rho,\beta}(\hat{x}_i, x_{-i}) & \text{if } e_i = 1 \text{ and } z \in [y_1, y_2]; \\ \mathcal{S}_{\rho,\beta}(z, x_{-i}) & \text{otherwise,} \end{cases}$$

where $e_i = e_i(v_{i,0} = x_i, b_{i,0} = \hat{x}_i, t_i)$. Note that the suboptimal selection rule $\hat{\mathcal{S}}_{y_1,y_2}$ follows the optimal selection rule everywhere except interval $[y_1, y_2]$. In the interval $[y_1, y_2]$, the set of selected agents is the set of agents that update their valuations when all agents except for agent i follow the truthful strategy, and agent i with initial valuation x_i misreports $\hat{x}_i$ in the first round and follows his best strategy afterward with regard to obtaining the additional information. The suboptimal selection rule captures the untruthful behavior of agent i in the first round while other agents are truthful.

In the next lemma, by characterizing the difference between the weighted surplus under the selection rule of the SWSP mechanism, $\mathcal{S}_{\rho,\beta}(x)$, and suboptimal selection rule $\hat{\mathcal{S}}_{y_1,y_2}(x)$, we will show that agent i prefers to bid truthfully in the first round; that is, $U_i(x_i, x_i) \geq U_i(x_i, \hat{x}_i)$.

LEMMA 8. *For any interval $[y_1, y_2]$, consider the suboptimal selection rule $\hat{\mathcal{S}}_{y_1,y_2}$ described above. Then, for any $x \in [\underline{v}, \bar{v}]^n$, we have $\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x)) - \Omega_{\rho,\beta}(x, \mathcal{S}_{\rho,\beta}(x)) \le 0$, and as a result, agent i prefers to bid truthfully in the first round; that is, $U_i(x_i, x_i) \ge U_i(x_i, \hat{x}_i)$.*

Note that by Lemma 8, an agent i who misreports his initial valuation will be worse off regardless of his decision to acquire information.

A.2. Proof of Lemma 8

Throughout the proof, to simplify our notations, we denote $\mathcal{S}_{\rho,\beta}$ by $\mathcal{S}$. Furthermore, without loss of generality, we assume that $\hat{x}_i < x_i$. A similar argument can be applied when $\hat{x}_i > x_i$.

We first show that the weighted surplus under the selection rule of the SWSP mechanism, $\mathcal{S}(x)$, i.e., $\Omega_{\rho,\beta}(x, \mathcal{S}(x))$, is less than the weighted surplus under selection rule $\hat{\mathcal{S}}_{y_1,y_2}(x)$, i.e., $\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x))$. Then, by characterizing the difference between $\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x))$ and $\Omega_{\rho,\beta}(x, \mathcal{S}(x))$ as a function of allocation probabilities, and using the fact that $\beta(\cdot)$ is an increasing function, we show that $U_i(x_i, x_i) \ge U_i(x_i, \hat{x}_i)$.

By definition, $\Omega_{\rho,\beta}(x, \mathcal{S}(x)) = \Omega_{\rho,\beta}(x)$. Then, since the selection rule $\mathcal{S}(x)$ maximizes the weighted surplus, we have $\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x)) - \Omega_{\rho,\beta}(x) \le 0$. Next, in Lemma 9, we characterize $\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x)) - \Omega_{\rho,\beta}(x)$. The proof which uses the Envelope Theorem (cf. Milgrom and Segal (2002)) is provided in Section D.4.

LEMMA 9. *For any interval $[y_1, y_2]$, consider the suboptimal selection rule $\hat{\mathcal{S}}_{y_1,y_2}$. Then, we have*

$$\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x)) - \Omega_{\rho,\beta}(x)$$
$$= \int_{y_1}^{y_2} \left(1 + \beta'(z)\right) \left(\mathrm{E}\left[q_i \middle| v_{i,0} = z,\ v_{-i,0} = x_{-i},\ \hat{\mathcal{S}}_{y_1,y_2}(z, x_{-i}) \right] - \mathrm{E}\left[q_i \middle| v_{i,0} = z,\ v_{-i,0} = x_{-i},\ \mathcal{S}(z, x_{-i}) \right] \right) dz,$$

where $\mathrm{E}\left[q_i \middle| v_{i,0} = z,\ v_{-i,0} = x_{-i},\ S \right]$ is the probability that truthful agent i with initial valuation z receives the item when other agents bid truthfully and agents in set S update their valuations.

In the following, using Lemma 9 and the fact that $\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x)) \le \Omega_{\rho,\beta}(x)$, we will show that agent i prefers to bid truthfully in the first round. Precisely, we will show that

$$\int_{z=\hat{x}_i}^{x_i} \mathrm{E}\left[q_i \middle| v_{i,0} = z,\ v_{-i,0} = x_{-i} \right] dz \ge \int_{z=\hat{x}_i}^{x_i} \Pr\left[z + e_i\delta_i + \beta(\hat{x}_i) \ge \omega_{-i} \right] dz \ , \tag{18}$$

where by Lemma 6, the l.h.s. is $U_i(x_i, x_i) - U_i(\hat{x}_i, \hat{x}_i)$. In addition, by Lemma 7, the r.h.s. is an upper bound of $U_i(x_i, \hat{x}_i) - U_i(\hat{x}_i, \hat{x}_i)$. Thus the above equation implies that agent i does not have any incentive to bid untruthfully in the first round.

By definition, $\mathrm{E}\left[q_i \middle| v_{i,0} = z,\ v_{-i,0} = x_{-i},\ \mathcal{S}(z, x_{-i}) \right] = \mathrm{E}\left[q_i \middle| v_{i,0} = z,\ v_{-i,0} = x_{-i} \right]$. Then, by Lemma 9 and the fact that $\Omega_{\rho,\beta}(x, \hat{\mathcal{S}}_{y_1,y_2}(x)) \le \Omega_{\rho,\beta}(x)$, we have

$$\int_{y_1}^{y_2} \left(1 + \beta'(z)\right) \left(\mathrm{E}\left[q_i | v_{i,0} = z,\ v_{-i,0} = x_{-i} \right] - \mathrm{E}\left[q_i | v_{i,0} = z,\ v_{-i,0} = x_{-i},\ \hat{\mathcal{S}}_{y_1,y_2}(z, x_{-i}) \right] \right) dz \ge 0 \tag{19}$$

Note that for any $y_1 \leq z \leq y_2$, we have

$$\begin{aligned} \mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\ ,\hat{\mathcal{S}}_{y_1,y_2}(z,x_{-i})\right] &= \Pr\left[z+e_i\delta_i+\beta(z)\geq \omega_{-i}\right] \\ &\geq \Pr\left[z+e_i\delta_i+\beta(\hat{x}_i)\geq \omega_{-i}\right], \end{aligned} \tag{20}$$

Here, again, $e_i = e_i(v_{i,0} = x_i, b_{i,0} = \hat{x}_i, t_i)$, and the equality follows from the construction of the suboptimal selection rule and the definition of ω_{-i} in Eq. (17). In addition, the inequality holds because the weight function $\beta(\cdot)$ is non-decreasing. Applying Eq. (20) in Eq. (19), we obtain

$$\int_{y_1}^{y_2} \left(1+\beta'(z)\right)\left(\mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\right] - \Pr\left[z+e_i\delta_i+\beta(\hat{x}_i)\geq \omega_{-i}\right]\right) dz \geq 0\ .$$

Then, Eq. (18) follows from the fact that the above equation holds for any nonzero measure interval $[y_1, y_2] \subseteq [\hat{x}_i, x_i]$, and the weight function $\beta(\cdot)$ is non-decreasing. Therefore, agent i prefers to bid truthfully in the first round.

Appendix B: Proof of Theorem 6

We show this result for any mechanism which selection rule maximizes the weighted surplus $\Omega_{\rho,\beta}(b_0, S)$, where $\Omega_{\rho,\beta}(b_0, S)$ is defined in Eq. (13), $\rho \geq 0$, and $\beta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function from an initial bid to a weight. Note that the efficient and optimal mechanisms select a set of agents that maximize $\Omega_{\rho=0,\beta=0}(b_0, S)$ and $\Omega_{\rho=0,\beta=\alpha}(b_0, S)$, respectively.

Consider two unselected agents i, j such that $b_{i,0} > b_{j,0}$. Assume that agents in set S are already selected. We will show that when the cost of information is the same for all agents, $\Omega_{\rho,\beta}(b_0, S \cup \{i\})$ is greater than or equal to $\Omega_{\rho,\beta}(b_0, S \cup \{j\})$; that is, the seller prefers to add agent i to set S rather than agent j.

By definition,

$$\Omega_{\rho,\beta}(b_0, S\cup\{i\}) = \mathrm{E}\left[\max\left\{\max_{k\neq i,j}\{b_{k,1}+\beta(b_{k,0})\},\ b_{i,0}+\beta(b_{i,0})+\delta_i,\ b_{j,0}+\beta(b_{j,0}),\ \rho\right\}\right] - c\times(|S|+1)\ ,$$

where the expectation is with respect to second signals and $b_{k,1} = b_{k,0} + \delta_k$ if $k \in S$ and is $b_{k,0}$ otherwise. For any realizations of second signals, let $Y_{-i} = \max\big\{\max_{k\neq i,j}\{b_{k,1} + \beta(b_{k,0})\}, b_{j,0} + \beta(b_{j,0}), \rho\big\} - b_{i,0} - \beta(b_{i,0})$. Then, $\Omega_{\rho,\beta}(b_0, S \cup \{i\})$ is given by

$$\mathrm{E}\Big[\big(b_{i,0}+\beta(b_{i,0})+\delta_i\big)\ \mathbb{1}\{\delta_i \geq Y_{-i}\}\ +\ \big(Y_{-i}+b_{i,0}+\beta(b_{i,0})\big)\ \mathbb{1}\{\delta_i < Y_{-i}\}\Big] - c\times(|S|+1)\ .$$

After some manipulations, it can be rewritten as

$$b_{i,0}+\beta(b_{i,0})+Y_{-i}\ \mathcal{G}(Y_{-i}) + \int_{Y_{-i}}^{\bar{\delta}} z\ d\mathcal{G}(z) - c\times(|S|+1)\ ,$$

where $\mathcal{G}$ is the distribution of δ_i. Likewise,

$$\Omega_{\rho,\beta}(b_0, S\cup\{j\}) = b_{j,0} + \beta(b_{j,0}) + Y_{-j}\ \mathcal{G}(Y_{-j}) + \int_{Y_{-j}}^{\bar{\delta}} z\ d\mathcal{G}(z) - c\times(|S|+1),$$

where $Y_{-j} = \max\big\{\max_{k\neq i,j}\{b_{k,1}+\beta(b_{k,0})\}, b_{i,0}+\beta(b_{i,0}),\rho\big\} - b_{j,0} - \beta(b_{j,0})$. Using integration by part, one can easily show that $\Omega_{\rho,\beta}(b_0, S\cup\{i\}) - \Omega_{\rho,\beta}(b_0, S\cup\{j\}) = b_{i,0}+\beta(b_{i,0}) - b_{j,0} - \beta(b_{j,0}) - \int_{Y_{-i}}^{Y_{-j}}\mathcal{G}(z)dz$. To show the result we need to consider the following two cases.

- $Y_{-i} - Y_{-j} = b_{j,0}+\beta(b_{j,0}) - b_{i,0} - \beta(b_{i,0})$: In this case, $\max\big\{\max_{k\neq i,j}\{b_{k,1}+\beta(b_{k,0})\},\rho\big\}$ is greater than $b_{j,0}+\beta(b_{j,0})$ and $b_{i,0}+\beta(b_{i,0})$. That is, $Y_{-i} = \max\big\{\max_{k\neq i,j}\{b_{k,1}+\beta(b_{k,0})\},\rho\big\} - b_{i,0} - \beta(b_{i,0})$ and $Y_{-j} = \max\big\{\max_{k\neq i,j}\{b_{k,1}+\beta(b_{k,0})\},\rho\big\} - b_{j,0} - \beta(b_{j,0})$. Thus,

$$\Omega_{\rho,\beta}(b_0, S\cup\{i\}) - \Omega_{\rho,\beta}(b_0, S\cup\{j\}) \geq b_{i,0}+\beta(b_{i,0}) - b_{j,0} - \beta(b_{j,0}) - (b_{i,0}+\beta(b_{i,0}) - b_{j,0} - \beta(b_{j,0})) = 0,$$

where the inequality follows from the fact that for any z, $\mathcal{G}(z)\leq 1$.

- $Y_{-i} - Y_{-j} \neq b_{j,0}+\beta(b_{j,0}) - b_{i,0} - \beta(b_{i,0})$: In this case, $\max\big\{\max_{k\neq i,j}\{b_{k,1}+\beta(b_{k,0})\},\rho\big\}$ is less than $b_{i,0}+\beta(b_{i,0})$. That is, $Y_{-j} = b_{i,0}+\beta(b_{i,0}) - b_{j,0} - \beta(b_{j,0})$ and $Y_{-i} \geq b_{j,0}+\beta(b_{j,0}) - b_{i,0} - \beta(b_{i,0})$. Then,

$$\Omega_{\rho,\beta}(b_0, S\cup\{i\}) - \Omega_{\rho,\beta}(b_0, S\cup\{j\}) \geq b_{i,0}+\beta(b_{i,0}) - b_{j,0} - \beta(b_{j,0}) - \int_{b_{j,0}+\beta(b_{j,0})-b_{i,0}-\beta(b_{i,0})}^{b_{i,0}+\beta(b_{i,0})-b_{j,0}-\beta(b_{j,0})}\mathcal{G}(z)dz.$$

Note that the upper level of the integral equals negative the lower level of the integral. Thus, by the fact the $\mathcal{G}(-z) = 1-\mathcal{G}(z)$, we have $\int_{b_{j,0}+\beta(b_{j,0})-b_{i,0}-\beta(b_{i,0})}^{b_{i,0}+\beta(b_{i,0})-b_{j,0}-\beta(b_{j,0})}\mathcal{G}(z)dz = b_{i,0}+\beta(b_{i,0}) - b_{j,0} - \beta(b_{j,0})$; that is, $\Omega_{\rho,\beta}(b_0, S\cup\{i\}) - \Omega_{\rho,\beta}(b_0, S\cup\{j\})$ is at least zero, which is the desired result.

Appendix C: Appendix to Sections 4.1 and 5.3

In this section, we first present the proof of Theorem 2. Then, we show that the single crossing conditions might not hold in the All-Access mechanism. We then provide the proof of Theorems 3 and 5.

C.1. Proof of Theorem 2

To show the result, we consider the following procedure: At each round, one agent presumes that other agents are playing stationary strategies. That is, he assumes that the maximum bid of other agents, $B = \max\{\max_{j\neq i}\{v_{j,0}+\delta_j\tilde{e}_j(v_{j,0})\},0\}$, is drawn from a stationary distribution. Then, he best responds to the strategies of other agents. We show this procedure terminates in an equilibrium. To this aim, we establish that when an agent updates his strategy in any round, he increases a bounded potential function $\bar{\Omega}:\{(\tilde{e}_1,\tilde{e}_2,\ldots,\tilde{e}_n)\}\to\mathbb{R}$, defined below,

$$\bar{\Omega}(\tilde{e}_1,\tilde{e}_2,\ldots,\tilde{e}_n) = \mathrm{E}\left[\max\left\{\max_{j=1,2,\ldots,n}\{v_{j,0}+\delta_j\tilde{e}_j(v_{j,0})\},0\right\} - \sum_{j=1}^{n} c_j\tilde{e}_j(v_{j,0})\right],$$

where $\tilde{e}_j = \tilde{e}_j(\cdot),\ j = 1,2,\ldots,n$, is the investment strategy of agent j, and the expectation is with respect to $v_{j,0}$ and δ_j for $j = 1,2,\ldots,n$. Note that the potential function is the average social welfare of the auctioneer and agents when agents follow investment strategies $< \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n >$. Then, considering the fact that $\bar{\Omega}(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n) \le \mathrm{E}\left[\max_{S\subseteq\{1,2,\ldots\}}\{\Omega(v_0, S)\}\right] < \infty$, the potential function is bounded, and as a result, the process of updating strategies will eventually result in an equilibrium.

For any initial valuation $v_{i,0}$, agent i selects $\tilde{e}_i(v_{i,0}) \in \{0,1\}$ that maximizes his utility. Let $\tilde{e}_{-i}$ be investment strategies of all agents except of agent i. Given $\tilde{e}_{-i}$, the utility of agent i with initial valuation $v_{i,0}$ when he does not obtain information $(\tilde{e}_i(v_{i,0}) = 0)$ can be written as

$$\begin{aligned} u_i\big(v_{i,0},\ \tilde{e}_i(v_{i,0}) = 0,\ \tilde{e}_{-i}\big) &= \mathrm{E}_{v_{-i,0},\tilde{e}_{-i}}\left[\max\{v_{i,0} - B\ ,0\}\right] \\ &= \mathrm{E}_{v_{-i,0},\tilde{e}_{-i}}\left[\max\left\{v_{i,0} - \max\left\{\max_{j\neq i}\{v_{j,0} + \delta_j\tilde{e}_j(v_{j,0})\},0\right\}\ ,0\right\}\right], \end{aligned} \tag{21}$$

where $\mathrm{E}_{v_{-i,0},\tilde{e}_{-i}}$ denotes the expectation with respect to the initial valuations and second signals of all agents except for agent i while taking into account their investment strategies, i.e., $\tilde{e}_{-i}$. Recall that the agents bid truthfully in the second price auction. Similarly, when agent i obtains information, his expected utility is given by

$$u_i\big(v_{i,0},\ \tilde{e}_i(v_{i,0}) = 1,\ \tilde{e}_{-i}\big) = \mathrm{E}_{v_{-i,0},\tilde{e}_{-i}}\left[\mathrm{E}\left[\max\left\{(v_{i,0} + \delta_i) - \max\left\{\max_{j\neq i}\{v_{j,0} + \tilde{e}_j(v_{j,0})\delta_j\},0\right\},0\right\}\right]\right] - c_i\ , \tag{22}$$

where the inner expectation is with respect to δ_i. In the rest of the proof, we denote all the expectations by E. Then, for any initial valuation $v_{i,0}$, we have

$$\begin{aligned} \tilde{e}_i(v_{i,0}) &= \arg\max_{e\in\{0,1\}}\{u_i\big(v_{i,0}, e,\ \tilde{e}_{-i}\big)\} = \arg\max_{e\in\{0,1\}}\{\mathrm{E}\left[\max\{v_{i,0} + \delta_i e - B, 0\}\right] - c_i e\} \\ &= \arg\max_{e\in\{0,1\}}\{\mathrm{E}\left[\max\{(v_{i,0} + \delta_i e), B\} - B - c_i e\right]\} \\ &= \arg\max_{e\in\{0,1\}}\left\{\mathrm{E}\left[\max\{(v_{i,0} + \delta_i e), B\} - c_i e - \sum_{j\neq i} c_j\tilde{e}_j(v_{j,0})\right]\right\} \\ &= \arg\max_{e\in\{0,1\}}\left\{\mathrm{E}\left[\max\left\{(v_{i,0} + \delta_i e), \max\{\max_{j\neq i}\{v_{j,0} + \delta_j\tilde{e}_j(v_{j,0})\},0\}\right\} - c_i e - \sum_{j\neq i} c_j\tilde{e}_j(v_{j,0})\right]\right\}\ , \end{aligned}$$

where the last equation follows from the definition of B. It is easy to observe that at any round, when an agent i updates his strategy, the potential function is increased. Then, by the fact that the potential function is bounded, the process of updating strategies will eventually terminate in an equilibrium point.

C.2. The All-Access Mechanism and Single Crossing Conditions

The following proposition shows that in the setting in Example 1, the All-Access mechanism does not admit any equilibrium with increasing investment decisions.

PROPOSITION 1. *Consider the All-Access mechanism with no reserve price and the setting in Example 1. Then, there exists no equilibrium such that both agents follow increasing investment decisions. That is, if an agent $i = 1, 2$ acquires information only when his initial valuation is greater than $\mu_i \in (0, 1)$, i.e., $\tilde{e}_i(v_{i,0}) = 1$ for $v_{i,0} \geq \mu_i$, and $\tilde{e}_i(v_{i,0}) = 0$ for $v_{i,0} < \mu_i$, then the other agent $j \neq i$ will not choose the following increasing investment decision: $\tilde{e}_j(v_{j,0}) = 1$ for $v_{j,0} \geq \mu_j$, and $\tilde{e}_j(v_{j,0}) = 0$ for $v_{j,0} < \mu_j$, where $\mu_j \in (0, 1)$.*

C.2.1. Proof of Proposition 1 To show the result, we will assume that an agent $i = 1, 2$ follows an increasing investment decision. That is, he only obtains the additional information when his initial valuation is greater than $\mu_i \in (0, 1)$. Then, we will establish that agent $j \neq i$ will not follow an increasing investment decision.

Throughout the proof, for simplicity, we denote μ_i by μ. We define $W(v_{j,0})$ as the difference between the utility of agent j when he obtains information and the utility of agent j when he does not obtain information given that agent $i \neq j$ only obtains information when his initial valuation is greater than μ, i.e.,

$$W(v_{j,0}) = u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 1,\ \tilde{e}_i) - u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 0,\ \tilde{e}_i)\ ,$$

where $u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 1,\ \tilde{e}_i)$ and $u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 0,\ \tilde{e}_i)$ are defined in equations (22) and (21). Here, $\tilde{e}_i(v) = 1$ for any $v \in [\mu, 1]$, and $\tilde{e}_i(v) = 0$ for $v \in [0, \mu)$. We will show that $W(\cdot)$ is a unimodular function and obtains its unique maximum at $\hat{v} \in (0, \mu)$. Precisely, we will show that $\frac{\partial W(v_{j,0})}{\partial v_{j,0}} \geq 0$ for $v_{j,0} \in [0, \hat{v}]$ and $\frac{\partial W(v_{j,0})}{\partial v_{j,0}} \leq 0$ for $v_{j,0} \in [\hat{v}, 1]$. This implies that for any values of the cost, agent j will not follow an increasing investment decision. To see why note that agent j with initial valuation $v_{j,0}$ updates his valuation if $W(v_{j,0}) \geq 0$. Then, considering the fact that $W(\cdot)$ is unimodular, $\{v_{j,0} : W(v_{j,0}) \geq 0\}$ cannot be in the form of $[\mu_j, 1]$ where $\mu_j \in (0, 1)$.

Let $H(\cdot)$ be the distribution of the maximum bid that agent j competes against, i.e., $B = \max\{(v_{i,0} + \delta_i \tilde{e}_i(v_{i,0})), 0\}$ where $\tilde{e}_i(v_{i,0}) = 1$ for $v_{i,0} \in [\mu, 1]$, and is zero otherwise. Then

$$W(v_{j,0}) \ = \ \mathrm{E}\left[\int_{x=0}^{\max\{v_{j,0}+\delta_j,0\}} (v_{j,0} + \delta_j - x) dH(x)\right] - \int_{x=0}^{v_{j,0}} (v_{j,0} - x) dH(x) - c,$$

By Leibniz's integral rule, $\frac{\partial W(v_{j,0})}{\partial v_{j,0}}$ can be written as

$$\frac{\partial W_r(v_{j,0})}{\partial v_{j,0}} \ = \ \mathrm{E}[H(\max\{v_{j,0} + \delta_j, 0\})] - H(v_{j,0}) \ = \ \mathrm{E}[H(v_{j,0} + \delta_j)] - H(v_{j,0})$$

The second equality holds because $B \geq 0$ and as a result $H(x) = 0$ for any $x < 0$. The next Lemma characterizes the distribution $H(\cdot)$.

LEMMA 10. *Suppose that $v_{i,0} \sim u(0,1)$, $\delta_i \sim u(-1,1)$, $\tilde{e}_i(v_{i,0}) = 1$ for $v_{i,0} \in [\mu, 1]$, and $\tilde{e}_i(v_{i,0}) = 0$ for $v_{i,0} \in [0, \mu)$. Then, the distribution of $B = \max\{(v_{i,0} + \tilde{e}_i(v_{i,0})\delta_i),\ 0\}$, denoted by H, is given by*

$$H(x) = \begin{cases} 0 & x < 0. \\ \frac{3-\mu}{2}x + \frac{(\mu-1)^2}{4} & 0 \leq x < \mu. \\ \frac{1-\mu}{2}x + \mu + \frac{(\mu-1)^2}{4} & \mu \leq x < \mu+1 \\ -\frac{x^2}{4} + x & \mu+1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

The proof is straightforward. Thus, it is omitted.

To simplify our notations, in the rest of the proof, we denote δ_j and $v_{j,0}$ with δ and v, respectively. In the following, using Lemma 10, we will show that there exists $\hat{v} \in [0, \mu]$ such that for $v \in [0, \hat{v})$ $W(v)$ is increasing in v, and it is non-increasing otherwise. By Lemma 10, for any $v \in [0, \mu)$, we have

$$\begin{aligned} \frac{\partial W(v)}{\partial v} &= \mathrm{E}[H(v+\delta)] - H(v) = \frac{1}{2}\int_{-v}^{\mu-v}\left(\frac{3-\mu}{2}(v+\delta) + \frac{(\mu-1)^2}{4}\right)d\delta \\ &+ \frac{1}{2}\int_{\mu-v}^{1}\left(\frac{1-\mu}{2}(v+\delta) + \mu + \frac{(\mu-1)^2}{4}\right)d\delta - \left(\frac{3-\mu}{2}v + \frac{(\mu-1)^2}{4}\right) \\ &= \frac{v^2(1-\mu) + (\mu^2+4\mu-9)v - 3\mu^2 + 5\mu}{8} \end{aligned}$$

Since $\frac{\partial W(v)}{\partial v}$ is quadratic in v, it is easy to verify that $\frac{\partial W(v)}{\partial v}$ is decreasing in $v \in [0, \mu]$. Then, by the fact that $\frac{\partial W(v)}{\partial v}\Big|_{v=0} = \frac{-3\mu^2+5\mu}{8} > 0$ and $\frac{\partial W(v)}{\partial v}\Big|_{v=\mu} = \frac{\mu(\mu-2)}{4} < 0$, we can conclude that there exists $\hat{v} \in (0, \mu)$ such that $\frac{\partial W(v)}{\partial v} \geq 0$ for $v \in [0, \hat{v}]$, and $\frac{\partial W(v)}{\partial v} < 0$ for $v \in (\hat{v}, \mu)$.

The last step of the proof is to show that $W(v)$ is decreasing in v when $v > \mu$. Lemma 10 implies that for any $v > \mu$, we have

$$\begin{aligned} \frac{\partial W(v)}{\partial v} &= \mathrm{E}[H(v+\delta)] - H(v) = \frac{1}{2}\int_{-v}^{\mu-v}\left(\frac{3-\mu}{2}(v+\delta) + \frac{(\mu-1)^2}{4}\right)d\delta \\ &+ \frac{1}{2}\int_{\mu-v}^{1+\mu-v}\left(\frac{1-\mu}{2}(v+\delta) + \mu + \frac{(\mu-1)^2}{4}\right)d\delta \\ &+ \frac{1}{2}\int_{1+\mu-v}^{1}\left(-\frac{(v+\delta)^2}{4} + (v+\delta)\right)d\delta - \left(\frac{1-\mu}{2}v + \mu + \frac{(\mu-1)^2}{4}\right) \\ &= \frac{\mu^3 + 12\mu v - 9\mu^2 - 9\mu}{24} + \frac{(3v^2 - 3v - v^3)}{24}\ . \end{aligned}$$

By the fact that $3v^2 - 3v - v^3$ is decreasing in $v \in (\mu, 1]$ and $v > \mu$, we have

$$\frac{\partial W(v)}{\partial v} \leq \frac{\mu^3 + 12\mu v - 9\mu^2 - 9\mu}{24} + \frac{(3\mu^2 - 3\mu - \mu^3)}{24} = \frac{12\mu v - 6\mu^2 - 12\mu}{24} \leq \frac{-6\mu^2}{24} < 0,$$

where the second inequality follows from the fact that $v \leq 1$.

C.3. Proof of Theorems 3 and 5

Theorem 3 can be seen as a corollary of Theorem 5. Therefore, in the following, we will verify Theorem 5.

The proof is naturally divided into two parts. In the first part, we show that when the cost is small, there exists an equilibrium in which both agents obtain the additional information, i.e., $\tilde{e}_i(v_{i,0}) = 1$ for $i = 1, 2$ and $v_{i,0} \in [0,1]$. In the second part, we show that when the cost is large, there exists an equilibrium in which none of the agents obtain the additional information, i.e., $\tilde{e}_i(v_{i,0}) = 0$ for $i = 1, 2$ and $v_{i,0} \in [0,1]$.

Part 1: We will show that when an agent $i = 1, 2$ obtains the additional information for any $v_{i,0} \in [0,1]$, then agent $j \neq i$ also has incentive to obtain information for any $v_{j,0} \in [0,1]$ as long as cost $c \leq \min\{\frac{4r^3-3r^2-6r+5}{48}, \frac{8r^3+6r^2+7}{96}\}$.

With slightly abuse of notation, we define $W_r(v_{j,0})$ as the difference of the utility of agent $j = 1, 2$ with initial valuation $v_{j,0}$ when he obtains information, and his utility when he does not obtain information given that the reserve price is r, and agent $i \neq j$ obtains the additional information for any initial valuation. That is,

$$W_r(v_{j,0}) = u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 1,\ \tilde{e}_i) - u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 0,\ \tilde{e}_i)\ ,$$

where $u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 1,\ \tilde{e}_i)$ and $u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 0,\ \tilde{e}_i)$ are defined in equations (22) and (21). Here, $\tilde{e}_i(v) = 1$ for any $v \in [0,1]$. We will show that $W_r(v_{j,0}) \geq 0$ for any $v_{j,0} \in [0,1]$; that is, for any initial valuation, agent j prefers to update his valuation. To this end, we will verify that $W_r(\cdot)$ achieves its minimum at either $v_{j,0} = 0$ or $v_{j,0} = 1$. Then by showing that $W_r(1), W_r(0) \geq 0$, we can conclude that $W_r(v_{j,0}) \geq 0$ for any $v_{j,0} \in [0,1]$.

Similar to the proof of Theorem 1, we write $W_r(\cdot)$ as a function of the distribution of the competing bid B, where $B = \max\{(v_{i,0} + \tilde{e}_i(v_{i,0})\delta_i), r\} = \max\{v_{i,0} + \delta_i, r\}$[9]. This follows because a second price auction is a truthful mechanism, and agent $i \neq j$ obtains the additional information for any value of $v_{i,0}$. We denote the distribution of B by H_r. Then

$$W_r(v_{j,0}) = \mathrm{E}\left[\int_{x=r}^{\max\{v_{j,0}+\delta_j, r\}} (v_{j,0} + \delta_j - x)dH_r(x)\right] - \int_{x=r}^{v_{j,0}} (v_{j,0} - x)dH_r(x) - c,$$

where the expectation is with respect to δ_j. We will show that the derivative of $W_r(v_{j,0})$ with respect to $v_{j,0}$ is positive for small values of $v_{j,0}$ and is non-positive for large values of $v_{j,0}$. This implies that $W_r(v_{j,0}) \geq \min\{W_r(0), W_r(1)\}$ for any $v_{j,0} \in [0,1]$. By Leibniz's integral rule, $\frac{\partial W_r(v_{j,0})}{\partial v_{j,0}}$ can be written as $\frac{\partial W_r(v_{j,0})}{\partial v_{j,0}} = \mathrm{E}[H_r(v_{j,0} + \delta_j)] - H_r(v_{j,0})$. The next lemma characterizes the distribution H_r.

[9] We can assume that the seller is one of the opponents with submitted bid of r.

LEMMA 11. *If $v_{i,0} \sim u(0,1)$ and $\delta_i \sim u(-1,1)$, the distribution of $B = \max\{(v_{i,0}+\delta_i), r\}$, denoted by H_r, is given by*

$$H_r(x) = \begin{cases} 0 & x < r; \\ \frac{1}{4} + \frac{x}{2} & r \le x \le 1; \\ x - \frac{x^2}{4} & 1 < x \le 2; \\ 1 & x > 2 \end{cases}$$

The proof is straightforward. Thus, it is omitted.

In the following, using Lemma 11, we will show that $\arg\min_{v\in[0,1]}\{W_r(v)\}$ is either 0 or 1. To this aim, we will show that $W_r(v)$ is increasing in v given that $v < r$, and is decreasing in v otherwise.

Observe that for any $v \in [0,r)$, $\frac{\partial W_r(v)}{\partial v} = \mathrm{E}[H_r(v+\delta)] \ge 0$. Furthermore, by Lemma 11, for any $v \in (r,1]$, we have

$$\begin{aligned}\frac{\partial W_r(v)}{\partial v} &= \mathrm{E}[H_r(v+\delta)] - H_r(v) = \frac{1}{2}\int_{r-v}^{1-v}\left(\frac{1}{4}+\frac{v+x}{2}\right)dx + \frac{1}{2}\int_{1-v}^{1}\left((v+x) - \frac{(v+x)^2}{4}\right)dx - \left(\frac{1}{4}+\frac{v}{2}\right) \\ &= \frac{3v^2 - v^3 - 3v}{24} + \frac{-3r^2-3r}{24} \le \frac{3r^2 - r^3 - 3r}{24} + \frac{-3r^2-3r}{24} = \frac{-r^3-6r}{24} \le 0\end{aligned}$$

The first inequality holds because $3v^2 - v^3 - 3v$ is decreasing in v and $v > r$. We established that $W_r(\cdot)$ gets minimized either at $v = 0$ or $v = 1$. Then, in the last step, we will verify $W_r(0), W_r(1) \ge 0$. By definition,

$$\begin{aligned} W_r(0) &= \mathrm{E}\left[(\delta - B)^+\right] - c = \Pr[B = r] \times \int_r^1 \frac{1}{2}(\delta - r)d\delta + \frac{1}{4}\int_r^1\int_r^\delta (\delta - x)dxd\delta - c \\ &= \frac{4r^3 - 3r^2 - 6r + 5}{48} - c \end{aligned}$$

Similarly,

$$\begin{aligned} W_r(1) &= \mathrm{E}\left[(1+\delta-B)^+\right] - \mathrm{E}\left[(1-B)^+\right] - c \\ &= \Pr[B=r] \times \mathrm{E}\left[(1+\delta-r)^+\right] + \frac{1}{4}\int_{\delta=r-1}^{1}\int_{x=r}^{\min\{1+\delta,1\}}(1+\delta-x)dxd\delta \\ &\quad - \Pr[B=r]\times(1-r) - \frac{1}{2}\int_{x=r}^{1}(1-x)dx = \frac{8r^3+6r^2+7}{96} - c \end{aligned}$$

Therefore, when $c \le \min\{\frac{4r^3-3r^2-6r+5}{48}, \frac{8r^3+6r^2+7}{96}\}$, $W_r(v) \ge 0$ for any $v \in [0,1]$. As a result, the agent is willing to obtain information regardless of his initial valuation.

Part 2: In this part, we will show that when the reserve price $r \le \sqrt{2}-1$ and an agent $i = 1,2$ does not obtain the additional information regardless of his initial valuation, then agent $j \ne i$ also does not have any incentive to obtain information for any $v_{j,0} \in [0,1]$ as long as cost $c \ge \frac{3r^4+8r^3+6r^2+7}{48}$. Similarly, when $r > \sqrt{2}-1$ and the cost is greater than $\frac{3r-r^3+1}{12}$, there exists an equilibrium such that none of the agents obtain the additional information.

With slightly abuse of notation, we define $W_r(v_{j,0})$ as the difference between the utility of agent j when he obtains information and the utility of agent j when he does not obtain information given that agent $i \neq j$ does not acquire the additional information for any initial valuation, i.e., $W_r(v_{j,0}) = u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 1,\ \tilde{e}_i) - u_j(v_{j,0},\ \tilde{e}_j(v_{j,0}) = 0,\ \tilde{e}_i)$, where $\tilde{e}_i(v) = 0$ for any $v \in [0,1]$. We will show that $W_r(v_{j,0}) \leq 0$ for any $v_{j,0} \in [0,1]$ when i- $c \geq \frac{3r^4+8r^3+6r^2+7}{48}$ and $r \leq \sqrt{2}-1$, or ii- $c \geq \frac{3r-r^3+1}{12}$ and $r > \sqrt{2}-1$. To this aim, we will verify that $\max_{v\in[0,1]}\{W_r(v)\} = \frac{3r^4+8r^3+6r^2+7}{48} - c \leq 0$ when $r \leq \sqrt{2}-1$, and $\max_{v\in[0,1]}\{W_r(v)\} = \frac{3r-r^3+1}{12} - c \leq 0$ when $r > \sqrt{2}-1$. This implies that the agent does not have any incentive to update his valuation for any initial valuation.

Specifically, we will show that $W_r(v)$ is increasing in v when $v \in [0, \max\{r, \frac{1-r^2}{2}\}]$, and is decreasing otherwise. That is, $\arg\max_{v\in[0,1]}\{W_r(v)\} = \frac{1-r^2}{2}$ if $r \leq \sqrt{2}-1$, and $\arg\max_{v\in[0,1]}\{W_r(v)\} = r$ if $r > \sqrt{2}-1$. We show these statements by characterizing $\frac{\partial W_r(v)}{\partial v}$.

By part 1, $\frac{\partial W_r(v)}{\partial v} = \mathrm{E}[H_r(v+\delta)] - H_r(v)$, where $H_r(\cdot)$ is the distribution of the bid that agent j competes against. Then, considering the fact that $v_{i,0} \sim u(0,1)$, and the other agent does not obtain information, i.e., $B = \max\{v_{i,0}, r\}$, we have

$$H_r(x) = \begin{cases} 0 & x < r; \\ x & x \in [r,1]; \\ 1 & x > 1. \end{cases}$$

This implies that for any $v \in [0,1]$

$$\mathrm{E}[H_r(v+\delta)] = \Pr[\delta + v \geq 1] + \frac{1}{2}\int_{r-v}^{1-v} (v+x)dx = \frac{v}{2} + \frac{1-r^2}{4}.$$

Thus, for any $v < r$, we have $\frac{\partial W_r(v)}{\partial v} = \mathrm{E}[H_r(v+\delta)] = \frac{v}{2} + \frac{1-r^2}{4} \geq 0$ and for any $v > r$,

$$\frac{\partial W_r(v)}{\partial v} = \mathrm{E}[H_r(v+\delta)] - H(v) = \frac{-v}{2} + \frac{1-r^2}{4}\ .$$

We point that for any $v \geq r$, we have $\frac{\partial W_r(v)}{\partial v} \leq 0$ as long as $r > \sqrt{2}-1$. To see why note that

$$\frac{\partial W_r(v)}{\partial v} = \frac{-v}{2} + \frac{1-r^2}{4} \leq \frac{-r}{2} + \frac{1-r^2}{4} \leq 0\ ,$$

where the last inequality holds because $r > \sqrt{2}-1$. On the other hand, when $r \leq \sqrt{2}-1$, $\frac{dW_r(v)}{dv} \geq 0$ for $v \in [r, \frac{1-r^2}{2}]$ and $\frac{\partial W_r(v)}{\partial v} \leq 0$ for $v \geq \frac{1-r^2}{2}$. This implies that we have $\arg\max_{v\in[0,1]}\{W_r(v)\} = r$ when $r > \sqrt{2}-1$, and $\arg\max_{v\in[0,1]}\{W_r(v)\} = \frac{1-r^2}{2}$ when $r \leq \sqrt{2}-1$. The proof will be completed by showing that i- $W_r(\frac{1-r^2}{2}) \leq 0$ when $c \geq \frac{3r^4+8r^3+6r^2+7}{48}$ and $r \leq \sqrt{2}-1$, and ii- $W_r(r) \leq 0$ when $c \geq \frac{3r-r^3+1}{12}$ and $r > \sqrt{2}-1$.

For $r \leq \sqrt{2}-1$, we have

$$\max_{v\in[0,1]}\{W_r(v)\} \ = \ W_r(\frac{1-r^2}{2}) \ = \ \mathrm{E}\left[\left(\frac{1-r^2}{2} + \delta - B\right)^+\right] - \mathrm{E}\left[\left(\frac{1-r^2}{2} - B\right)^+\right] - c$$

$$= \Pr[B=r] \times \mathrm{E}\left[\left(\frac{1-r^2}{2}+\delta-r\right)^+\right] + \frac{1}{2}\int_{\delta=r-\frac{1-r^2}{2}}^{1}\int_{x=r}^{\min\{\frac{1-r^2}{2}+\delta,1\}}\left(\frac{1-r^2}{2}+\delta-x\right)dxd\delta$$

$$-\Pr[B=r]\times\left(\frac{1-r^2}{2}-r\right) - \int_{x=r}^{\frac{1-r^2}{2}}\left(\frac{1-r^2}{2}-x\right)dx \;=\; \frac{3r^4+8r^3+6r^2+7}{48}-c \le 0\ ,$$

where the inequality holds because $c \ge \frac{3r^4+8r^3+6r^2+7}{48}$. Similarly, when $r > \sqrt{2}-1$, we have

$$\max_{v\in[0,1]}\{W_r(v)\} \;=\; W_r(r) \;=\; \mathrm{E}\left[(r+\delta-B)^+\right] - \mathrm{E}\left[(r-B)^+\right] - c$$

$$= \Pr[B=r]\times \mathrm{E}\left[(\delta)^+\right] + \frac{1}{2}\int_{\delta=0}^{1}\int_{x=r}^{\max\{r+\delta,1\}}(r+\delta-x)dxd\delta - c \;=\; \frac{3r-r^3+1}{12}-c \;\le\; 0\ ,$$

where the inequality follows from the fact that $c \ge \frac{3r-r^3+1}{12}$.

Appendix D: Technical Proofs

D.1. Proof from Section 5

Proof of Lemma 1 Let x be the vector of the initial valuations. Given that mechanism $\mathcal{M}^{\text{OPT}}$ is incentive compatible, the revenue of the seller is given by

$$\mathrm{E}\left[\sum_{i=1}^{n} t_i + p_i\right] = \mathrm{E}\left[\sum_{i=1}^{n} v_{i,1}q_i - c_i \times s_i - U_i(x_i,x_i)\Big| v_0 = b_0 = x\right], \tag{23}$$

where $v_{i,1} = x_i + \delta_i$ if $i \in \mathcal{S}_{\text{OPT}}(x)$ and x_i otherwise, and the expectations are with respect to initial valuations and second signals. Note that the sum of the first and second terms is the social welfare of the agents and the seller. In the following, we compute the last term in the r.h.s. of the above equation, that is, $\mathrm{E}[U_i(x_i,x_i)]$, where the expectation is with respect to initial valuations. By Lemma 6, we have $\mathrm{E}[U_i(x_i,x_i)] = \int_{x_i=\underline{v}}^{\bar{v}}\int_{z=\underline{v}}^{x_i}\mathrm{E}\left[q_i\Big|v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]dz\ dF(x_i)$, where the expectation inside the integral is with respect to $v_{-i,0}$. Changing the order of integrals, we get

$$\int_{z=\underline{v}}^{\bar{v}}\int_{x_i=z}^{\bar{v}} dF(x_i)\mathrm{E}\left[q_i\Big|v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]dz = \int_{z=\underline{v}}^{\bar{v}}(1-F(z))\mathrm{E}\left[q_i\Big|v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]dz.$$

By multiplying and dividing the r.h.s. of the equation above by the probability density $f(z)$, we obtain $\mathrm{E}[U_i(x_i,x_i)] = \mathrm{E}\left[\frac{1-F(x_i)}{f(x_i)}q_i\Big|v_0=x\right] = \mathrm{E}\left[-\alpha(x_i)q_i\Big|v_0=x\right]$. Substituting $\mathrm{E}[U_i(x_i,x_i)]$ in Eq. (23), the expected revenue of the seller is given by $\mathrm{E}\left[\sum_{i=1}^{n}(v_{i,1}+\alpha(x_i))q_i - c_i\times s_i\Big|v_0=x\right]$. Finally, the result follows from applying the selection and allocation rules. □

Proof of Lemma 2 To find an upper bound, we consider a relaxed environment in which the seller observes the additional information of selected agents, and she can force agents to update their valuations. It is easy to see that the maximum achievable revenue in this environment is an upper bound on the revenue of the seller in the original environment.

By the revelation principle, we focus on direct incentive compatible mechanisms that consist of transfer scheme $\bar{t}_i : \mathbb{R}^n \to \mathbb{R}$, selection rule $\bar{s}_i : \mathbb{R}^n \to \{0,1\}$, and allocation rule $\bar{q}_i : \mathbb{R}^{2n} \to \mathbb{R}^+$, where

$\bar{s}_i$ is 1 when agent i is selected and is 0 otherwise. Note that the payment and selection rules are only functions of initial bids, and the allocation rule is a function of the initial bids and the second signals observed by the seller.

To compute the upper bound on the revenue of any incentive compatible mechanism in the relaxed environment, we first need to characterize the utility of each agent i. Assume that agent i with initial valuation x_i reports $\hat{x}_i$, and other agents report truthfully. Then, his utility is given by

$$U_i(x_i,\hat{x}_i) = \mathrm{E}\Big[\bar{q}_i \times (x_i + \delta_i \bar{s}_i) - \bar{t}_i - c_i\bar{s}_i \Big| b_{i,0} = \hat{x}_i,\ v_{i,0} = x_i,\ v_{-i,0} = b_{-i,0} = x_{-i}\Big],$$

where the expectation is with respect to the second signals. Incentive compatibility implies that

$$U_i(x_i,x_i) - U_i(\hat{x}_i,\hat{x}_i) \le U_i(x_i,x_i) - U_i(\hat{x}_i,x_i) = (x_i - \hat{x}_i)\mathrm{E}\Big[\bar{q}_i \Big| b_{i,0} = x_i,\ b_{-i,0} = x_{-i}\Big],$$

and similarly, $U_i(x_i,x_i) - U_i(\hat{x}_i,\hat{x}_i) \ge (x_i - \hat{x}_i)\mathrm{E}\Big[\bar{q}_i \Big| b_{i,0} = \hat{x}_i,\ b_{-i,0} = x_{-i}\Big]$. Without loss of generality, we assume that $x_i > \hat{x}_i$. Then, using the above equations,

$$\mathrm{E}\Big[\bar{q}_i \Big| b_{i,0} = \hat{x}_i,\ b_{-i,0} = x_{-i}\Big] \le \frac{U_i(x_i,x_i) - U_i(\hat{x}_i,\hat{x}_i)}{x_i - \hat{x}_i} \le E\Big[\bar{q}_i \Big| b_{i,0} = x_i,\ b_{-i,0} = x_{-i}\Big].$$

Finally by taking the limit as $\hat{x}_i \to x_i^-$, we get $U_i(x_i,x_i) = U_i(\underline{v},\underline{v}) + \int_{\underline{v}}^{x_i} \mathrm{E}\left[\bar{q}_i \middle| v_{i,0} = z, b_{-i,0} = x_{-i}\right] dz$.

We are now ready to compute the upper bound of the revenue. By using the same arguments as in Lemma 1, it can be shown that for any selection rule $\bar{s}_i$ and allocation rule $\bar{q}_i$ revenue of the seller when agents are truthful is given by

$$\mathrm{E}\Big[\sum_{i:\bar{s}_i(x)=1} \big(x_i + \alpha(x_i) + \delta_i\big)\bar{q}_i + \sum_{i:\bar{s}_i(x)=0} \big(x_i + \alpha(x_i)\big)\bar{q}_i - \sum_{i=1}^{n} c_i\bar{s}_i - \sum_{i=1}^{n} U_i(\underline{v},\underline{v}) \big| b_0 = x\Big], \tag{24}$$

where the expectation is with respect to the first and second signals. Because the mechanism should be individually rational, we set $U_i(\underline{v},\underline{v}) = 0$ for all i. Then, to maximize the revenue, the item should be allocated to the agent with the highest non-negative virtual valuation, that is,

$$\bar{q}_i = 1 \quad \text{if} \quad i \in \arg\max_j \left\{ \big(x_j + \alpha(x_j) + \delta_j \bar{s}_j(x)\big) \right\}^{10}$$

and 0 otherwise. Therefore, the expected revenue of the seller can be written as

$$\mathrm{E}\left[\max\left\{ \max_{i:\bar{s}_i(x)=1} \{x_i + \alpha(x_i) + \delta_i\}, \max_{i:\bar{s}_i(x)=0} \{x_i + \alpha(x_i)\}, 0 \right\} - \sum_{i=1}^{n} c_i\bar{s}_i \middle| b_0 = x \right].$$

So, if agents in set $\mathcal{S}_{\text{relaxed}}$, defined below, obtain the additional information, the revenue gets maximized:

$$\mathcal{S}_{\text{relaxed}}(x) = \arg\max_{S\subset\{1,\cdots,n\}} \left\{ \mathrm{E}\left[\max\Big\{\max_{i\in S}\{x_i + \delta_i + \alpha(x_i)\}, \max_{i\notin S}\{x_i + \alpha(x_i)\}, 0\Big\}\right] - \sum_{i\in S} c_i \right\}.$$

Finally, the result follows by plugging the selection rule $\mathcal{S}_{\text{relaxed}}(x)$ and allocation rule $\bar{q}_i$ in Eq. (24). □

[10] In case of ties, we choose one of them randomly.

D.2. Proofs from Section A

Proof of Lemma 3 The basic idea is to establish an upper bound on the r.h.s. of Eq. (12). We will show the upper bound is larger than the l.h.s. of Eq. (12) at $r=\rho$. This will imply that Eq. (12) is satisfied at some $r\geq\rho$.

We first show that the r.h.s. of Eq. (12), i.e., $\int_{\underline{v}}^{b_{\ell,0}} \mathrm{E}\left[q_\ell \middle| v_{\ell,0}=z, v_{-\ell,0}=b_{-\ell,0}\right] dz$, is less than or equal to $\int_{\underline{v}}^{b_{\ell,0}} \Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz$, where $\omega_{-\ell}=\max\left\{\max_{j\in\mathcal{S}_{\rho,\beta}(b_0)}\left\{b_{j,0}+\delta_j+\beta(b_{j,0})\right\},\rho\right\}$. Because agent ℓ has the largest weighted bid among unselected agents, $\omega_{-\ell}=\max\left\{\max_{j\in\mathcal{S}_{\rho,\beta}(b_0)}\left\{b_{j,0}+\delta_j+\beta(b_{j,0})\right\},\max_{j\notin\mathcal{S}_{\rho,\beta}(b_0),j\neq\ell}\left\{b_{j,0}+\beta(b_{j,0})\right\},\rho\right\}$. Then, by Lemma 7,

$$\begin{aligned}
U_\ell(\underline{v},b_{\ell,0}) \;&\leq\; U_\ell(b_{\ell,0},b_{\ell,0})-\int_{\underline{v}}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz \;\leq\; U_\ell(\underline{v},\underline{v})\\
&=\; U_\ell(b_{\ell,0},b_{\ell,0})-\int_{\underline{v}}^{b_{\ell,0}}\mathrm{E}\left[q_\ell\middle|v_{\ell,0}=z,\; v_{-\ell,0}=b_{-\ell,0}\right]dz,
\end{aligned}$$

where the second inequality follows from Lemma 8, and the equality follows from Lemma 6. By the above equation, we can conclude that $\int_{\underline{v}}^{b_{\ell,0}} \mathrm{E}\left[q_\ell\middle|v_{\ell,0}=z,\; v_{-\ell,0}=b_{-\ell,0}\right]dz$ is less than $\int_{\underline{v}}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz$.

Next, we will show that the l.h.s. of Eq. (12) at $r=\rho$ is greater than the upper bound, i.e., $\int_{\underline{v}}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz$. Then, considering the fact that the upper bound is not a function of r, the l.h.s. is a non-increasing function of r, and is zero at $r=b_{\ell,0}+\beta(b_{\ell,0})$, we conclude that there exists an $r\in[\rho,b_{\ell,0}+\beta(b_{\ell,0})]$ that satisfies Eq. (12).

By changing variable, the l.h.s. at $r=\rho$ can be written as

$$\int_{\rho-\beta(b_{\ell,0})}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\max\left\{\max_{j\in\mathcal{S}_{\rho,\beta}(b_0)}\left\{b_{j,0}+\delta_j+\beta(b_{j,0})\right\},\rho\right\}\right]dz=\int_{\rho-\beta(b_{\ell,0})}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz,$$

Then, because $b_{\ell,0}+\beta(b_{\ell,0})\geq\rho$, we have

$$\begin{aligned}
\int_{\rho-\beta(b_{\ell,0})}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz &\geq \int_{\max\{\rho-\beta(b_{\ell,0}),\underline{v}\}}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz\\
&=\int_{\underline{v}}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz-\int_{\underline{v}}^{\max\{\rho-\beta(b_{\ell,0}),\underline{v}\}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz\\
&=\int_{\underline{v}}^{b_{\ell,0}}\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]dz,
\end{aligned}$$

where the last equality holds because $\omega_{-\ell}\geq\rho$ and for any $z\leq\rho-\beta(b_{\ell,0})$, $\Pr\left[z+\beta(b_{\ell,0})\geq\omega_{-\ell}\right]=0$. The last equation shows that the l.h.s. of Eq. (12) at $r=\rho$ is greater than the r.h.s. of Eq. (12).

□

D.3. Proofs from Section A.1

Proof of Lemma 4 Consider a selected agent i with initial bid $b_{i,0}$. If agent i wins the item, his payment in the second round, p_i, would be equal to $\max\{\max_{j\neq i}\{b_{j,1}+\beta(b_{j,0})\},\rho\}-\beta(b_{i,0})$. Note that p_i is independent of $b_{i,1}$. Therefore, an agent cannot change his price for the item. However, the agent can change the probability of the allocation. It is easy to see that underbidding may only result in losing the item. On the other hand, over bidding may yield a negative utility. Note that overbidding can make a difference only when

$$v_{i,1}+\beta(b_{i,0}) \;\leq\; \max\{\max_{j\neq i}\{b_{j,1}+\beta(b_{j,0})\},\rho\} \;\leq\; b_{i,1}+\beta(b_{i,0}).$$

In this case, the utility of agent i would be non-positive:

$$v_{i,1}-p_i \;=\; v_{i,1}+\beta(b_{i,0})-\max\{\max_{j\neq i}\{b_{j,1}+\beta(b_{j,0})\},\rho\} \;\leq\; v_{i,1}+\beta(b_{i,0})-(v_{i,1}+\beta(b_{i,0})) \;=\; 0.$$

Therefore, a weekly dominate strategy of agent i is to be truthful. □

Proof of Lemma 5 Consider agent $i\in\mathcal{S}_{\rho,\beta}(x)$ who bids truthfully in the first round, where x is the initial valuations of agents. In the proof, to simplify the notations, we denote $\mathcal{S}_{\rho,\beta}(x)$ by $\mathcal{S}(x)$. We will show that agent i will learn his second signal, i.e., $e_i=1$, given that other agents are truthful. To this aim, we will prove that for agent i, the marginal value of changing his decision to obtain information is identical to the change in the weighted surplus $\Omega_{\rho,\beta}$. More precisely, the difference between the utility of agent i when he obtains information and that when he does not is equal to $\Omega_{\rho,\beta}(x,\mathcal{S}(x))-\Omega_{\rho,\beta}(x,\mathcal{S}(x)\backslash\{i\})$. Then, since the SWSP mechanism maximizes the weighted surplus, i.e., $\Omega_{\rho,\beta}(x,\mathcal{S}(x))\geq\Omega_{\rho,\beta}(x,\mathcal{S}(x)\backslash\{i\})$, we conclude that agent i prefers to update his valuation.

We first characterize the utility of agent i when he does not obtain information. Let Y be the random variable corresponding to the maximum weighted bid of all agents except for agent i, i.e., $Y=\max\Big\{\max_{j\in\mathcal{S}(x),j\neq i}\big\{x_j+\delta_j+\beta(x_j)\big\},\max_{j\notin\mathcal{S}(x)}\big\{x_j+\beta(x_j)\big\},\rho\Big\}$. Then, by Lemma 4, when agent i does not update his valuation, his utility is given by

$$\begin{aligned}\mathrm{E}\left[\max\left\{x_i+\beta(x_i)-Y,0\right\}\right]-t_i &= \mathrm{E}\big[\max\{Y,x_i+\beta(x_i)\}-Y\big]-t_i,\\ &= \Omega_{\rho,\beta}(x,\mathcal{S}(x)\backslash\{i\})+\sum_{j\in\mathcal{S}(x)\backslash\{i\}}c_j-\mathrm{E}[Y]-t_i \qquad (25)\end{aligned}$$

Note that in computing utility of agent i, we use the fact that other agents are truthful. In addition, the second equality holds because

$$\begin{aligned}&\Omega_{\rho,\beta}(x,\mathcal{S}(x)\backslash\{i\})\\ &=\mathrm{E}\left[\max\Big\{\max_{j\in\mathcal{S}(x),j\neq i}\{x_j+\beta(x_j)+\delta_j\},\max_{j\notin\mathcal{S}(x)}\{x_j+\beta(x_j)\},x_i+\beta(x_i),\rho\Big\}\right]-\sum_{j\in\mathcal{S}(x)\backslash\{i\}}c_j\\ &=\mathrm{E}\left[\max\{Y,x_i+\beta(x_i)\}\right]-\sum_{j\in\mathcal{S}(x)\backslash\{i\}}c_j\ ,\end{aligned}$$

Similarly, when agent i obtains the additional information, his utility is equal to

$$\begin{aligned}\mathrm{E}\left[\max\left\{x_i+\delta_i+\beta(x_i)-Y,0\right\}\right]-t_i-c_i &= \mathrm{E}\left[\max\{Y,x_i+\delta_i+\beta(x_i)\}-Y\right]-t_i-c_i \\ &= \Omega_{\rho,\beta}(x,\mathcal{S}(x))+\sum_{j\in\mathcal{S}(x)\setminus\{i\}} c_j-\mathrm{E}[Y]-t_i\ . \end{aligned} \tag{26}$$

The first expression follows from Lemma 4 where we show that agent i bids truthfully in the second round. The second equality holds because

$$\begin{aligned}\Omega_{\rho,\beta}(x,\mathcal{S}(x)) &= \mathrm{E}\left[\max\Big\{\max_{j\in\mathcal{S}(x),j\neq i}\{x_j+\beta(x_j)+\delta_j\},\max_{j\notin\mathcal{S}(x)}\{x_j+\beta(x_j)\},x_i+\delta_i+\beta(x_i),\rho\Big\}\right]-\sum_{j\in\mathcal{S}(x)} c_j \\ &= \mathrm{E}\left[\max\{Y,x_i+\delta_i+\beta(x_i)\}\right]-\sum_{j\in\mathcal{S}(x)} c_j\ .\end{aligned}$$

In addition, note that t_i in Eq. (26) is the same as t_i in Eq. (25) since agent i's decision to obtain information, i.e., e_i, is not observable by the mechanism. By equations (25) and (26), the difference between the utility of agent i when he updates his valuation and his utility when he does not is $\Omega_{\rho,\beta}(x,\mathcal{S}(x))-\Omega_{\rho,\beta}(x,\mathcal{S}(x)\setminus\{i\})$. Then, considering the fact that the SWSP mechanism maximizes the weighted surplus, i.e., $\Omega_{\rho,\beta}(x,\mathcal{S}(x))\geq\Omega_{\rho,\beta}(x,\mathcal{S}(x)\setminus\{i\})$, we conclude that agent i prefers to learn his second signal. □

Proof of Lemma 6 We first show that for any $x_{-i}\in[\underline{v},\bar{v}]^{n-1}$ and any $i=1,2,\ldots,n$, $\mathrm{E}\left[q_i|v_{i,0}=x_i,\ v_{-i,0}=x_{-i}\right]$ is a non-decreasing function of x_i. Observe that the weighted surplus is the maximum of affine functions of $v_{i,0}+\beta(v_{i,0})$. Thus, it is a convex function of $v_{i,0}+\beta(v_{i,0})$. Furthermore, the weighted surplus is a continuos function of $v_{i,0}+\beta(v_{i,0})$, and its derivative with respect to $v_{i,0}+\beta(v_{i,0})$ at $v_{i,0}=x_i$, if exists, is equal to $\mathrm{E}\left[q_i|v_{i,0}=x_i,\ v_{-i,0}=x_{-i}\right]$.[11] This implies that $\mathrm{E}\left[q_i|v_{i,0}=x_i,\ v_{-i,0}=x_{-i}\right]$ is a non-decreasing function of $v_{i,0}+\beta(v_{i,0})$. Finally, considering the fact function β is non-decreasing, we can conclude that $\mathrm{E}\left[q_i|v_{i,0}=x_{i,0},\ v_{-i,0}=x_{-i}\right]$ is a non-decreasing function of $v_{i,0}$.

Next we show that the utility of an agent i that bids truthfully in the first round and follows the optimal strategy afterwards follows from Eq. (16) when all other agents are truthful. By Lemma 5, agent i that bids truthfully in the first round will obtain information if he gets selected. Furthermore, Lemma 4 implies that agent i will bid truthfully in the second round if he is allowed to update his bid in the second round. Therefore, agent i that bids truthfully in the first round stays truthful.

Now, we are ready to show the result. We consider the following cases. Throughout this proof, for simplicity, we drop the subscript of $\mathcal{S}_{\rho,\beta}(x)$, and denote it by $\mathcal{S}(x)$.

[11] To see that note $\mathrm{E}\left[q_i|v_{i,0}=x_i,\ v_{-i,0}=x_{-i}\right]$ is equal to $\Pr[v_{i,1}+\beta(x_i)\geq\max\{\max_{j\in\mathcal{S}(x)}\{x_j+\delta_j+\beta(x_{j,0})\},\max_{j\notin\mathcal{S}(x)}\{x_j+\beta(x_{j,0})\},\rho\}]$, where $v_{i,1}=x_i+\delta_i$ if $i\in S(x)$ and it is x_i otherwise.

i) $i \in \mathcal{S}(x)$: By lemmas 4 and 5, selected agent i learns his second signal and reports it truthfully in the second round. Thus, his utility is given by $\mathrm{E}\left[q_i(x_i+\delta_i) - p_i - t_i - c_i \middle| v_0 = x_0\right]$. The claim follows from plugging t_i from Eq. (2).

ii) $i \notin \mathcal{S}(x)$ and $x_i + \beta(x_i) < \max\left\{\max_{j\notin\mathcal{S}(x), j\neq i}\{x_j+\beta(x_j)\}, \rho\right\}$: In this case, the utility of agent i and his allocation probability is 0. By the fact that $\mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]$ is an increasing function of z and $\mathrm{E}\left[q_i \middle| v_{i,0}=x_i,\ v_{-i,0}=x_{-i}\right] = 0$, we can write the utility of agent i as $\int_{\underline{v}}^{x_i} \mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\right] dz = 0$.

iii) $i \notin \mathcal{S}(x)$ and $x_i + \beta(x_i) \geq \max\left\{\max_{j\notin\mathcal{S}(x), j\neq i}\{x_j+\beta(x_j)\}, \rho\right\}$: In this case, when unselected agent i wins the item, he has to pay maximum of r and the second highest weighted bid. Therefore, $U_i(x_i,x_i) = \mathrm{E}\left[q_i \times \left(x_i + \beta(x_i) - \max\left\{\max_{j\in\mathcal{S}(x)}\left\{x_j+\delta_j+\beta(x_j)\right\}, r\right\}\right)\right]$, where $U_i(x_i,x_i)$ is defined in Eq. (15).

Let $Y = \max_{j\in\mathcal{S}(x)}\{x_j+\delta_j+\beta(x_j)\}$, and let H be the distribution of Y. Then, $U_i(x_i,x_i)$ can be written as

$$\begin{aligned}
&\mathrm{E}\Big[\big(x_i+\beta(x_i)-Y\big)\times \mathbb{1}\{x_i+\beta(x_i)\geq Y\geq r\} + \big(x_i+\beta(x_i)-r\big)\times\mathbb{1}\Big\{x_i+\beta(x_i)\geq r\geq Y\Big\}\Big] \\
&= (x_i+\beta(x_i))\big(H(x_i+\beta(x_i))-H(r)\big) - \int_r^{x_i+\beta(x_i)} z dH(z) + (x_i+\beta(x_i)-r)H(r) \\
&= \big(x_i+\beta(x_i)\big)H(x_i+\beta(x_i)) - rH(r) - \int_r^{x_i+\beta(x_i)} z dH(z) \;=\; \int_r^{x_i+\beta(x_i)} H(z)dz,
\end{aligned}$$

where in the first equation, the expectation is with respect to the second signals. The last equality is followed from the integration by part. Therefore, using Eq. (12), we get

$$U_i(x_i,x_i) = \int_r^{x_i+\beta(x_i)} \Pr\left[z \geq \max_{j\in\mathcal{S}(x)}\left\{x_j+\delta_j+\beta(x_j)\right\}\right]dz = \int_{\underline{v}}^{x_i} \mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\right] dz.$$

□

Proof of Lemma 7 Throughout the proof, all the expectations are with respect to the second signals. Consider an untruthful agent i with initial valuation x_i who bids $\hat{x}_i$ in the first round. We establish an upper bound on his utility. We consider the following two cases, $s_i = 1$ and $s_i = 0$.

$s_i = 1$: When agent i is selected, $s_i = 1$, he can either obtain information or not. Given his investing decision $e_i = e_i(v_{i,0} = x_i, b_{i,0} = \hat{x}_i, t_i)$, by Lemma 4, his utility can be written as

$$\begin{aligned}
U_i(x_i,\hat{x}_i) &= \mathrm{E}\Big[(x_i+e_i\delta_i) - p_i - t_i - e_ic_i \;\Big| v_0 = x,\ b_{i,0}=\hat{x}_i,\ b_{-i,0}=x_{-i}\Big] \qquad (27)\\
&= \mathrm{E}\left[\max\left\{x_i+\beta(\hat{x}_i)+e_i\delta_i-\omega_{-i}, 0\right\} - t_i - e_ic_i\right] ,
\end{aligned}$$

where ω_{-i} is defined in Eq. (17), and the expectation is taken assuming the all agents except for agent i are truthful. Note that for abbreviation, we omit the condition in the second equation and in the rest of the proof. Then, by adding and subtracting $\mathrm{E}[\max\{\hat{x}_i+\beta(\hat{x}_i)+e_i\delta_i-\omega_{-i},0\}]$, the utility can be rewritten as

$$\begin{aligned}U_i(x_i,\hat{x}_i) &= \mathrm{E}\Big[\max\{\hat{x}_i+\beta(\hat{x}_i)+e_i\delta_i-\omega_{-i},0\}-t_i-e_ic_i\\&\quad-\Big(\max\{\hat{x}_i+\beta(\hat{x}_i)+e_i\delta_i-\omega_{-i},0\}-\max\{x_i+\beta(\hat{x}_i)+e_i\delta_i-\omega_{-i},0\}\Big)\Big]\\&=\mathrm{E}\Big[\max\{\hat{x}_i+\beta(\hat{x}_i)+e_i\delta_i-\omega_{-i},0\}-t_i-e_ic_i\Big]+\int_{\hat{x}_i}^{x_i}\Pr[z+\beta(\hat{x}_i)+e_i\delta_i\geq\omega_{-i}]\,dz\end{aligned}$$

When $e_i=1$ the first term in the last line is $U_i(\hat{x}_i,\hat{x}_i)$. Otherwise, it is the utility of selected agent i with initial $\hat{x}_i$ who bids truthfully, gets selected, but does not learn his second signal, which is by Lemma 5 is less than or equal to $U_i(\hat{x}_i,\hat{x}_i)$. Thus, the utility is at most $U_i(\hat{x}_i,\hat{x}_i)+\int_{\hat{x}_i}^{x_i}\Pr[z+\beta(\hat{x}_i)+e_i\delta_i\geq\omega_{-i}]\,dz$, which is the desired result.

$s_i=0$: Note that $s_i=0$ means agent i is not selected. Then, if $\hat{x}_i+\beta(\hat{x}_i)<\max\left\{\max_{j\notin\mathcal{S}_{\rho,\beta}(\hat{x}_i,x_{-i})}\{x_j+\beta(x_j)\},\rho\right\}$, his utility is zero. If not, the utility of agent i given that he stays in the game can be written as

$$\mathrm{E}\left[q_i\times\Big(x_i+\beta(\hat{x}_i)-\max\left\{\omega_{-i},r\right\}\Big)\right].$$

By adding and subtracting $\mathrm{E}[q_i\times\hat{x}_i]$, and by the fact that agent i receives the item if $\hat{x}_i+\beta(\hat{x}_i)$ is greater than ω_{-i}, we have

$$\mathrm{E}\left[\max\left\{(\hat{x}_i+\beta(\hat{x}_i)-\max\left\{\omega_{-i},r\right\},0\right\}+(x_i-\hat{x}_i)\times\mathbb{1}\left\{\hat{x}_i+\beta(\hat{x}_i)\geq\omega_{-i}\right\}\right].$$

The first term is $U_i(\hat{x}_i,\hat{x}_i)$. Since $U_i(\hat{x}_i,\hat{x}_i)\geq 0$ and agent i can exit the game if his utility gets negative, he can at most yield $\max\{U_i(\hat{x}_i,\hat{x}_i)+\int_{\hat{x}_i}^{x_i}\Pr\left\{\hat{x}_i+\beta(\hat{x}_i)\geq\omega_{-i}\right\}dz,0\}$, which is less than $\max\{U_i(\hat{x}_i,\hat{x}_i)+\int_{\hat{x}_i}^{x_i}\Pr\left\{z+\beta(\hat{x}_i)\geq\omega_{-i}\right\}dz,0\}$.

□

D.4. Proofs from Section A.2

Proof of Lemma 9 We establish the following two claims.

- **Claim 1:** For any set $S\subseteq\{1,2,\ldots,n\}$

$$\Omega_{\rho,\beta}((x_i,x_{-i}),S)-\Omega_{\rho,\beta}((\hat{x}_i,x_{-i}),S)=\int_{\hat{x}_i}^{x_i}(1+\beta'(z))\mathrm{E}\left[q_i\middle|v_{i,0}=z,\ v_{-i,0}=x_{-i},\ S\right]dz\ .$$

- **Claim 2:** Suppose Assumption 2 holds. Then weighted surplus is an absolutely continuous and convex function of $v_{i,0}+\beta(v_{i,0})$, and it is given by

$$\Omega_{\rho,\beta}(x)=\Omega_{\rho,\beta}((\hat{x}_i,x_{-i}))+\int_{\hat{x}_i}^{x_i}(1+\beta'(z))\,\mathrm{E}\left[q_i\middle|v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]dz\ .$$

The proof of claims follows from Theorems 1 and 2 in Milgrom and Segal (2002). Thus, we do not repeat it here. By Claims 1 and 2 and the fact that $\mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i},\ \hat{\mathcal{S}}_{y_1,y_2}(z,x_{-i})\right] = \mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]$ for $z<y_1$ and $z>y_2$, we have

$$\begin{aligned}\Omega_{\rho,\beta}(x,\hat{\mathcal{S}}_{y_1,y_2}(x)) = \Omega_{\rho,\beta}((\hat{x}_i,x_{-i}),\hat{\mathcal{S}}_{y_1,y_2}(\hat{x}_i,x_{-i})) &+ \int_{\hat{x}_i}^{y_1}\big(1+\beta'(z)\big)\mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]dz\\ &+\int_{y_1}^{y_2}\big(1+\beta'(z)\big)\mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i},\ \hat{\mathcal{S}}_{y_1,y_2}(z,x_{-i})\right]dz\\ &+\int_{y_2}^{x_i}\big(1+\beta'(z)\big)\mathrm{E}\left[q_i \middle| v_{i,0}=z,\ v_{-i,0}=x_{-i}\right]dz.\end{aligned}$$

Then, the result follows from Claim 2 and the fact that, by construction, $\Omega_{\rho,\beta}(\hat{x}_i,x_{-i}) = \Omega_{\rho,\beta}((\hat{x}_i,x_{-i}),\hat{\mathcal{S}}_{y_1,y_2}(\hat{x}_i,x_{-i}))$. □

Appendix E: Numerical Experiments

In Section E.1, we depict the initial payments. Section E.2 compares the SSP mechanism with the optimal mechanism in terms of the revenue of the seller. In Section E.3, we study impacts of the number of agents, n.

E.1. Payments in the First Round

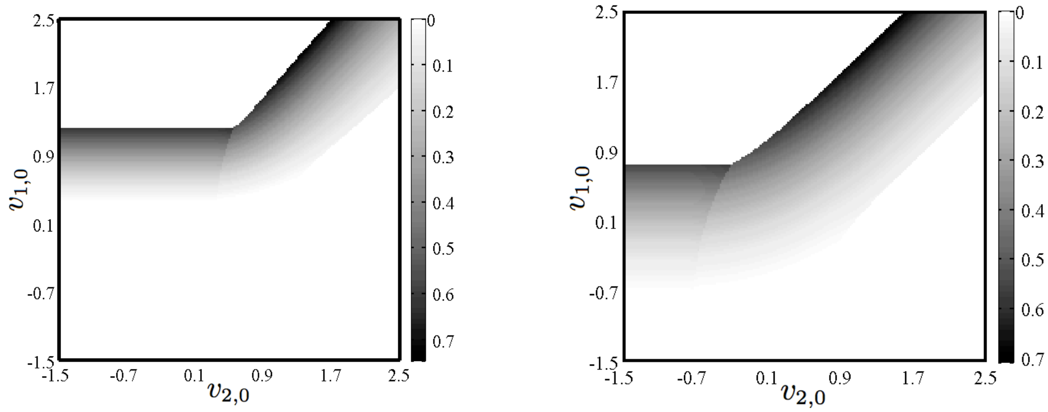
Recall that the initial payment t_i incentivizes agents to be truthful. In this section, we investigate how much the OPT and EFF mechanisms charge each agent i upfront for different realizations of initial valuations. As usual, $n=2$, $F=N(0.5,0.5)$, $G_i=N(0,0.5)$, and $c_i=0.05$ for $i=1,2$.

The initial payment for the first agent, t_1, in the OPT and EFF mechanisms for all realizations of $v_{1,0}$ and $v_{2,0}$ in the range of $[-1.5,2.5]$ is shown in Figures 5a and 5b, respectively. The x-axis is $v_{2,0}$, and the y-axis is $v_{1,0}$. Here, different shades of gray mean different initial payment as defined in the color bars next to the figures. By construction, the initial payment of the first agent is zero if he is not selected. Furthermore, when he is selected, t_1 is an increasing function of $v_{1,0}$.

E.2. The SSP Mechanism versus the optimal Mechanism

In this section, we seek to understand how the SSP mechanism performs in compare with mechanism $\mathcal{M}^{\textsc{Opt}}$. To this aim, we report the revenue of the SSP mechanism under four problem classes, corresponding with cost 0.02 and 0.05, and number of agents of 2 and 3. Here, $F=N(0.5,0.5)$ and $G_i=N(0,\sigma^2)$, where $\sigma^2=0.5,1,1.5$, and 2.

In Table 1, for each problem class, we present the revenue of the SSP mechanism with revenue-maximizing r as a percentage of the optimal revenue, averaged over 2000 instances in each problem class. We observe that the SSP mechanism performs better as the number of agents gets larger, σ^2 becomes smaller, and the additional information gets more costly. In addition, the SSP mechanism yields more than 84% of the optimal revenue.

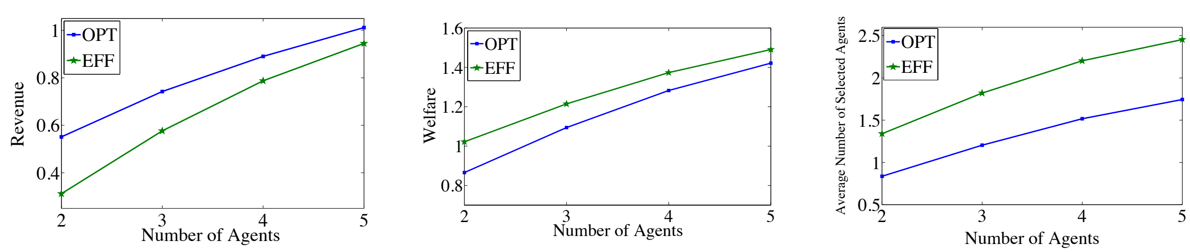


(a) Optimal mechanism (b) Efficient mechanism

Figure 5 The payment of agent 1 in the first round, t_1, for different realizations of $v_{1,0}$ and $v_{2,0}$ with $n = 2$, $c = 0.05$, $F = N(0.5, 0.5)$, and $G_i = N(0, 0.5)$.

Problem Class		$G_i = N(0,\sigma^2)$			
n	cost	$\sigma^2 = 0.5$	$\sigma^2 = 1$	$\sigma^2 = 1.5$	$\sigma^2 = 2$
2	0.02	94	90	87	84
	0.05	95	92	90	87
3	0.02	95	93	89	88
	0.05	96	94	92	90

Table 1 Revenue of the SSP mechanism (with revenue-maximizing r) as a percentage of the optimal revenue with $F = N(0.5, 0.5)$. Here, the standard errors of all numbers are less than 1%.



(a) (b) (c)

Figure 6 Revenue, social welfare, and average number of selected agents of the optimal and efficient mechanisms versus number of agents, n, with $c = 0.05$, $F = N(0.5, 0.5)$, and $G_i = N(0, 0.5)$.

E.3. More Agents

In this section, we investigate how the number of agents can affect the outcome of the OPT and EFF mechanisms. Again, $F = N(0.5, 0.5)$, $G_i = N(0, 0.5)$, and $c_i = 0.05$.

Figure 6 shows the average number of selected agents, revenue, and social welfare versus the number of agents, n. As the number of agents increases, the revenue and social welfare, and average number of selected agents in all considered mechanisms rise. However, even in the EFF mechanism, the average number of selected agents is sub-linear (concave) in n.