Selling to Advised Buyers*

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Abstract

In many cases, buyers are not informed about their valuations and rely on experts, who are informed but biased for overbidding. We study auction design when selling to such "advised buyers". We show that a canonical dynamic auction, the English auction, has a natural equilibrium that outperforms standard static auctions in expected revenues and allocative efficiency. The ability to communicate as the auction proceeds allows for more informative communication and gives advisors the ability to persuade buyers into overbidding. The same outcome is the unique equilibrium of the English auction when bidders can commit to contracts with their advisors.

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1 Introduction

In many economic environments, agents that make purchase decisions have limited information about their valuations of the asset for sale. As a consequence, they rely on the advice of informed experts, who however often have misaligned preferences. For example, when a firm is competing for a target in a takeover contest, its board of directors has authority over submitting bids, while its managers are likely to be more informed about the valuation of the target. The managers, however, could be prone to overbidding because of career concerns and empire-building preferences. Other examples of advisors that have private information about bidders' valuations and advise them on bidding are research teams in telecommunication companies in spectrum auctions and realtors in real estate transactions.

The goal of this paper is to study how the seller should design the sale mechanism when the potential buyers are advised by informed but biased advisors. We study a canonical setting in which the seller has an asset to auction among a number of potential buyers with independent private values. We depart from it in one aspect: Each potential buyer is a pair of a bidder (female) and her advisor (male), where the bidder controls bidding decisions (e.g., the firm's board) but has no information about her valuation, while the advisor (e.g., the firm's manager) knows the valuation but has a conflict of interest. We focus on the advisors' bias toward overbidding: given value v to the bidder, the advisor's maximum willingness to pay is $v + b$ with $b > 0$. This specification captures empire-building motives of managers or career concerns of consulting companies.

Prior to the bidder submitting an offer, the advisor communicates with the bidder via a game of cheap talk. If the sale process consists of a single round of bidding, there is only one round of communication. In contrast, if it consists of multiple rounds, the advisor communicates with the bidder in each round. In this environment, communication and the design of the sale process interact. On one hand, communication from advisors affects bids and therefore revenues of each auction format. On the other hand, the auction format affects how advisors communicate information to bidders.

We first study static auctions. As one could expect from the classic game of cheap talk (Crawford and Sobel (1982)), communication takes a partition form: All types of the advisor are partitioned into intervals and types in each interval induce the same bid. Imposing the NITS ("no incentive to separate") condition from Chen et al. (2008), which in our setup boils down to the lowest type of the advisor getting a non-negative payoff, selects equilibria in which communication is relatively efficient. We prove a version of the revenue equivalence theorem for static auctions. Focusing on a large class of standard auctions with continuous payments introduced in Che and Gale (2006), which includes first-price, second-price, and all-pay auctions, we show that all static auctions in this class bring the same expected revenue and feature the same communication between bidders and advisors.

This conclusion changes drastically if the asset is sold via dynamic mechanisms. Consider the English (ascending-price) auction, in which the price continuously increases until only one bidder remains. From the position of a bidder and her advisor, bidding is a stopping time problem: At what price level to drop out. At each price level, the advisor advises his bidder about whether to quit the auction now or not. We show that the English auction has equilibria with the following structure. The advisor recommends to stay in the auction until the price reaches the advisor's maximum willingness to pay. In turn, the bidder follows the advisor's recommendation until the price reaches a high enough threshold, at which she drops out irrespectively of what the advisor says then. Thus, the advisor's types perfectly separate at the bottom of the distribution and pool at the top. Moreover, when the value is in the range of separation, the bidder overbids: She exits the auction at a higher price than she would had she known her value at the start of the game. Because the behavior in these equilibria is as if each bidder delegates bidding decisions to her advisor subject to a cap, we refer to them as the "capped delegation" equilibria.

The intuition why these equilibria exist in the English auction but not in static auctions lies in the irreversibility of the running price in the auction: While a bidder can always bid until a price level higher than the current price, she cannot exit at a price lower than the current price. Informally, she can improve her offer but cannot renege on past offers. If the advisor is biased for overbidding, he recommends the bidder to continue bidding and sends the recommendation to quit only when the price reaches the advisor's indifference point, i.e., when the price exceeds the buyer's value by the amount of the bias. When the bidder gets such a recommendation, she infers that her valuation is below the running price and, hence, quits the auction immediately. When the bidder gets the recommendation to stay in the auction, she trades off the continuation value of learning the advisor's private information against the cost of possibly overpaying. The solution is to act on the advisor's recommendation unless the running price reaches a high enough threshold. Thus, the English auction allows the advisor to persuade the bidder whose valuation v is not too high into overbidding - bidding until the price reaches $v + b$, rather than v, which is what the bidder would have done had she known her valuation at the start of the auction.

The main result of the paper is that under natural conditions on the distribution of types, the "capped delegation" equilibrium of the English auction outperforms in expected revenues any equilibrium of the static auctions satisfying the NITS condition. The key to the comparison is to view the seller's auction design problem as selling to advisors directly, where communication between advisors and bidders puts restrictions on what the selling mechanism can be. As in Myerson (1981), the expected revenues equal the expected virtual valuation of the winning advisor minus the expected payoff of the advisors with the lowest value. We show that the English auction has both a higher efficiency and a lower payoff of the lowest type of the advisor than static auctions. The English auction is more efficient both because types of the advisor below the cut-off fully separate and because the length of the pooling interval is below the length of the top interval of types in a static auction. In addition, in the English auction, the lowest type of the advisor never wins, so his payoff is zero. At the same time, the NITS condition implies that his payoff in the second-price auction cannot be negative.[1]

We further show that under weak distributional assumptions, imposing the NITS condition on equilibria in static auctions is not required for the revenue comparison if the auction is sufficiently competitive. In this case, the "capped delegation" equilibrium of the English auction yields higher expected revenues than any equilibrium of the second-price and, by revenue equivalence, any other static auction. Intuitively, as the number of bidders increases, the gain in expected revenues from the finer separation of high types in the English auction eventually outweighs the possible loss from extracting lower rents from low types.

To highlight the role of commitment, we next consider an "auction with contracts." Specifically, we assume that each bidder can commit to a contract that specifies the exit price in the English (or, equivalently, second-price) auction conditional on the advisor's report of the type. Under a mild distributional restriction, this auction with contracts has a unique undominated equilibrium, and it coincides with the capped delegation equilibrium of the English auction in the model without commitment. This result has two implications. First, it provides a foundation for our focus on capped delegation equilibria in the English auction with cheap talk.[2] If each bidder cannot commit to a contract but has the ability to select among the equilibria of the communication game with her advisor, this result suggests that the capped delegation equilibrium will arise as an outcome. Second, it reveals that the inability to lower the bid below the current running price in the English auction serves as an implicit commitment device for the

[1]Interestingly, for certain unbounded distributions of values, the threshold after which the bidder quits irrespective of the advisor's recommendation is infinite (i.e., there is full separation). This implies that the English auction is efficient. In this case, the English auction with an appropriately chosen reserve price extracts the highest expected revenues in the class of selling mechanisms that deliver a non-negative expected payoff to any type of the advisor.

[2]An alternative foundation, which is based on a dynamic extension of the NITS condition, is provided in Online Appendix B.

bidder to follow her advisor's recommendations, which is not feasible in static auctions. Once explicit commitment power is given to the bidders, the English and the second-price auctions become equivalent, as in the standard setting when buyers know their valuations.

Our paper is related to two strands of the literature: auction design and communication of non-verifiable information (cheap talk). Our contribution to the auction theory literature is to study the design of auctions when bidders are advised by informed experts. A fundamental result in auction theory is the celebrated revenue equivalence theorem (Myerson (1981); Riley and Samuelson (1981)), generalized to arbitrary type distributions by Che and Gale (2006). In our setting, it holds for static mechanisms, but breaks down for dynamic mechanisms.[3] Our paper is related to studies of information acquisition by bidders and information design by the seller. In particular, Compte and Jehiel (2007) show that multiple-round auctions bring higher expected revenues than static counterparts because of more flexible information acquisition.[4] While this result is similar to ours, it follows from a very different argument, which relies crucially on the asymmetry of bidders in information endowments and their knowledge of the number of remaining bidders in the auction. McAdams (2015) shows that multiple-round version of the second-price auction dominates the sealed-bid format when entry is costly. Bergemann and Pesendorfer (2007), Eso and Szentes (2007), Chakraborty and Harbaugh (2010), and Bergemann and Wambach (2015) study design of information by the auctioneer. Our difference from this literature is in how bidders get information: from biased experts as opposed to the seller. Burkett (2015) studies a principal-agent relationship in auctions where the principal optimally constrains an agent with a budget and shows revenue equivalence of first- and second-price auctions. Burkett (2016) shows that the optimality of constraining a bidder using a simple budget extends to a large class of selling mechanisms. Differently from us, he focuses exclusively on the setup with commitment and assumes that the agent's bias vanishes as the value converges to the lowest value.

Second, our paper is related to the literature on cheap talk. In addition to the classic paper by Crawford and Sobel (1982), two papers that relate the most to our paper are Chen et al. (2008) and Grenadier et al. (2016). First, because some of our main results about the comparison of expected revenues rely on the NITS condition, our paper builds on Chen et al. (2008), who introduce it.[5] Second, our paper builds on Grenadier et al. (2016) who study

[3]Existing reasons for the failure of revenue equivalence include affiliation of values (Milgrom and Weber (1982)), bidder asymmetries (Maskin and Riley (2000)), and budget constraints (Che and Gale (1998, 2006); Pai and Vohra (2014)), among others.

[4]Other papers on information acquisition by bidders in auctions include Persico (2000), Bergemann and Välimäki (2002), Bergemann et al. (2009), Crémer et al. (2009), and Shi (2012).

[5]It is also related to Kartik (2009) and Chen (2011) who study perturbed versions of the classic cheap talk

a cheap talk game in the context of an option exercise problem and show that, when the sender is biased for delaying exercise, it leads to different equilibria than the static counterpart: separation up to a cut-off.[6] Our main contribution is that rather than taking the game as given, we compare auction designs from the perspective of maximizing expected revenues. A number of papers study cheap talk models with less related dynamic aspects of communication.[7]

Finally, several papers study other effects of cheap talk in auctions. Matthews and Postlewaite (1989) study pre-play communication in a two-person double auction. Ye (2007) and Quint and Hendricks (forthcoming) study two-stage auctions, where the actual bidding is preceded by the indicative stage, which takes form of cheap talk between bidders and the seller. Kim and Kircher (2015) study how auctioneers with private reservation values compete for potential bidders by announcing cheap-talk messages. Several papers also study the role of cheap talk in non-auction trading environments.[8]

The structure of the paper is as follows. Section 2 introduces the model. Section 3 illustrates our main results in a simple example. Section 4 compares the auction formats under cheap-talk communication and presents our main results. Section 5 studies bidding with contracts. Section 6 concludes. The Appendix contains the proofs. Online Appendices A and B contain technical details of the proofs and additional results.

2 Model

Consider the standard setting of symmetric bidders with independent private values. There is a single indivisible asset for sale. Its value to the seller is normalized to zero. There are N potential buyers (bidders). The valuation of bidder i, v_i, is an i.i.d. draw from distribution with c.d.f. F and p.d.f. f. The distribution F has full support on $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} < \overline{v} \leq \infty$ and satisfies $\int_{\underline{v}}^{\overline{v}} v dF(v) < \infty$.

The novelty of our setup is that each bidder i does not know her valuation v_i, but consults advisor i who does. Advisor i knows v_i, but has no information about v_j, $j \neq i$ except for their distribution F. While advisor i knows v_i, he is biased. Specifically, the payoffs from the auction

game with lying costs and behavioral players, respectively, since both variations can be used to motivate the NITS condition.

[6]See also Guo (2016) for a related result in the optimal delegation (rather than cheap-talk) problem.

[7]See Sobel (1985); Morris (2001); Golosov et al. (2014); Ottaviani and Sørensen (2006a,b); Krishna and Morgan (2004); Aumann and Hart (2003).

[8]E.g., Koessler and Skreta (2016); Inderst and Ottaviani (2013); Levit (2017).

are:

$$\text{Bidder } i\text{'s payoff}: \ I_i v_i - p_i; \qquad \text{Advisor } i\text{'s payoff}: \ I_i (v_i + b) - p_i, \tag{1}$$

where I_i is the indicator variable that bidder i obtains the asset, p_i is the payment of bidder i to the seller, and b is the advisor's bias. Bias b is commonly known.

Motivated by applications described in the introduction, we assume that advisors have a bias for overbidding, i.e., $b > 0$. For example, consider a publicly traded firm bidding for a target. The board of the firm has formal authority over the bidding process, maximizes firm value, but does not know valuation v_i. Suppose that a risk-neutral CEO of the firm knows v_i, but is biased. Specifically, if the CEO owns fraction α of the stock of the company and gets a private benefit of B from acquiring the target and managing a larger company, his payoff is $\alpha(v_i - p) + B$. Normalizing this payoff by α and denoting $b = \frac{B}{\alpha}$, we obtain formulation (1).

Our objective is to analyze how communication between biased advisors and bidders affects expected revenues and efficiency of different selling mechanisms. We model communication as a game of cheap talk. If the auction format is static (i.e., it consists of a single round of bidding), the timing of the game is as follows:

1. Advisor i sends a private message $\tilde{m}_i \in M$ to bidder i where M is some infinite set of messages.

2. Having observed message $\tilde{m}_i$, bidder i chooses what bid $\beta_i \in \mathbb{R}_+$ to submit.

3. Given all submitted bids $\beta_1, ..., \beta_N$, the asset is allocated and payments are made according to the rule of the auction.

In contrast, if the auction format is dynamic, the advisor sends a message to the bidder before each round of bidding.

Static Auctions If the communication has a partition form as in Crawford and Sobel (1982) (which it will in equilibrium as we show below), then after receiving a message, the bidder updates her expected value to one of a finite number of values. It is well-known that when the distribution of bidder's values is discrete, the revenue equivalence need not hold. Thus, within static auctions, we consider a rich class of auctions for which the revenue equivalence theorem holds for arbitrary distributions of values in the standard setting where bidders are informed about their valuations (Che and Gale (2006)):

Definition 1 (Che and Gale, 2006)**.** *An auction is a* standard auction with continuous payments *if it satisfies the following conditions:*

1. *the highest bid wins and ties are broken randomly;*

2. *the payment depends only on the bidder's own and the highest competing bid, i.e., bidder i pays $\tau_w(\beta_i, \beta_{m(i)})$, if she wins, and $\tau_l(\beta_i, \beta_{m(i)})$, if she loses, where $\beta_{m(i)} = \max_{j \neq i} \beta_j$;*

3. *$\tau_w(0,0) = \tau_l(0,\cdot) = 0$ and $\tau_k(\cdot, \beta_{m(i)})$ is continuous for $k = w, l$, in the relevant domain.*

This is a rich class of auctions that includes common formats, such as first-price, second-price, and all-pay auctions. For example, in the first-price auction, $\tau_w\left(\beta_i, \beta_{m(i)}\right) = \beta_i$ and $\tau_l(\cdot) = 0$, while in the second-price auction, $\tau_w\left(\beta_i, \beta_{m(i)}\right) = \beta_{m(i)}$ and $\tau_l(\cdot) = 0$. For conciseness, we refer to standard auctions with continuous payments in which there is only one round of bidding (and, hence, communication) as simply *static auctions.*

We consider Perfect Bayesian Equilibria (PBE) of static auctions. Since all bidders are symmetric, we focus on symmetric PBEs in which all advisors play the same communication strategy $m : [\underline{v}, \overline{v}] \rightarrow M$ and all bidders play the same bidding strategy, which maps messages in M to distributions over bids. We refer to an equilibrium as *babbling* if regardless of the message received, each bidder plays the same strategy.

There is a multiplicity of equilibria in cheap talk games. To select among them, we impose the "no incentive to separate" (NITS) condition of Chen et al. (2008). According to the NITS condition, the equilibrium payoff to the "weakest" type of the advisor, $\underline{v}$, cannot be below his payoff if he credibly revealed himself (and had the bidder best-respond to that information). Intuitively, every type of the advisor wants to convince the bidder to bid more than the bidder would bid if she knew her value. Thus, it is natural to assume that the recommendation to bid the lowest possible amount would be perceived as credible by the bidder. Chen et al. (2008) show that NITS can be justified by perturbations of the cheap-talk game with non-strategic players or costs of lying. Further, as we shall see, the NITS condition in our model boils down to the requirement that advisor type $\underline{v}$ gets non-negative expected utility from the auction. This is akin to the participation constraint, which is automatically implied if the advisor can quit and obtain the payoff of zero after learning v. This provides another justification for our use of NITS in static auctions.

English Auction We focus on the English auction among dynamic mechanisms. The seller continuously increases price p, which we refer to as the *running price,* starting from zero. Each bidder decides whether to continue participating or to quit the auction. A bidder who quit the auction cannot re-enter. Once only one bidder remains, she wins and pays the running price.

The advisor sends a message to the bidder before each round of bidding. We index rounds by corresponding running prices p. We assume that bidders and advisors only observe the running price p, but not the actions of other bidders.[9] The history h of the bidder i includes the current running price p and messages $(m_t)_{t<p}$ sent by advisor i up to round p. Denote the set of all histories by $H = \{(p, (m_t)_{t<p})\}$.

A strategy of advisor i is a measurable mapping $m : [\underline{v}, \overline{v}] \times H \to M$ from the advisor's private information about the valuation v and a history h into a message $m(v, h)$ sent to bidder i after that history. In the English auction, the only actions are to stay or to quit labelled 0 and 1, respectively. A strategy of bidder i is a measurable mapping $a : H \times M \mapsto \{0, 1\}$ from a history h and a current message $\tilde{m}$ into the action $a(h, \tilde{m})$ chosen by the bidder. A bidder's posterior belief process is a measurable mapping $\tilde{\mu} : H \times M \to \Delta([\underline{v}, \overline{v}])$ from a history h and current message $\tilde{m}$ into the posterior distribution over $[\underline{v}, \overline{v}]$, $\tilde{\mu}(h, \tilde{m})$.

We focus on symmetric Perfect Bayesian equilibria in pure Markov strategies (PBEM) where the state consists of the auction round p and a bidder's posterior belief about her valuation v. Communication strategy $m(v, p, \mu)$ gives the message sent in round p when bidder's posterior is μ and the advisor's type is v. Belief mapping $\tilde{\mu}(p, \mu, \tilde{m})$ gives the bidder's posterior in round p after observing message $\tilde{m}$ given that the posterior in the beginning of round p is μ. Bidding strategy $a(p, \tilde{\mu})$ gives the bidder's decision in round p to quit/stop the auction ($a = 1$) or continue ($a = 0$), when her beliefs are $\tilde{\mu}$ ($\tilde{\mu}$ is an updated version of μ after observing the advisor's last message).

From now on, we refer to the equilibria we restrict attention to as simply *equilibria*.

3 Example: Two Bidders with Uniformly Distributed Valuations

We start the analysis with a simple example that illustrates the results of the paper. There are two bidders ($N = 2$), each valuation is an i.i.d. draw from the uniform distribution over $[0, 10]$, and the advisors' bias is $b = 1$.

First, consider the second-price auction. Because of the bias, the advisor cannot credibly communicate the valuation to the bidder, and the equilibrium must have an interval partition

[9] This assumption simplifies the analysis. However, equilibria that we consider are also equilibria in the model in which the number of remaining rivals is observed by bidders and advisors.

structure. Consider the conditions that characterize an equilibrium with K intervals, $[\omega_0, \omega_1]$, ...,$[\omega_{K-1}, \omega_K]$, with $\omega_0 = 0$ and $\omega_K = 10$. Given the advisor's message that conveys that the valuation is in the k^{th} interval, the best response of the bidder is to bid the updated expected valuation, $(\omega_{k-1} + \omega_k)/2$. This bid is the winning bid with probability one, if the valuation of the rival bidder is below ω_{k-1}, with probability 50%, if it is between ω_{k-1} and ω_k, and with probability zero, if it is above ω_k. By inducing the bidder to bid $(\omega_k + \omega_{k+1})/2$ instead of $(\omega_{k-1} + \omega_k)/2$, the advisor increases the probability of winning against types $[\omega_{k-1}, \omega_k]$ from 50% to one and against types $[\omega_k, \omega_{k+1}]$ from zero to 50%. Hence, for the cut-off type of the advisor ω_k, the additional payoff from a higher probability of winning against types $[\omega_{k-1}, \omega_k]$ must equal the cost from overpaying when the bidder wins against types $[\omega_k, \omega_{k+1}]$:

$$\frac{\omega_k - \omega_{k-1}}{10}\left(\omega_k + b - \frac{\omega_{k-1} + \omega_k}{2}\right) = \frac{\omega_{k+1} - \omega_k}{10}\left(\frac{\omega_k + \omega_{k+1}}{2} - \omega_k - b\right),\ k = 1, ..., K-1.$$

We will refer to the equilibrium with the highest number of intervals as the most informative. In this example, this equilibrium has three intervals, $\left[0, 1\frac{1}{3}\right], \left[1\frac{1}{3}, 4\frac{2}{3}\right]$, and $\left[4\frac{2}{3}, 10\right]$. The corresponding bids are $\frac{2}{3}$, 3, and $7\frac{1}{3}$ (see Figure 1). Since the lowest bid is below $b = 1$, this equilibrium satisfies the NITS condition: The weakest type of the advisor ($v = 0$) is better off inducing bid $\frac{2}{3}$ than communicating that $v = 0$. There exist two other equilibria: one with two intervals ($[0, 4]$ and $[4, 10]$) and the uninformative equilibrium. Since the lowest bid (2 in the former case; 5 in the latter) exceeds $b = 1$, these equilibria violate the NITS condition.

Next, in the English auction a bidder faces a stopping time problem: At each price p, she decides whether to quit the auction or stay for a little longer. Consider the following "delegation-like" equilibrium. Suppose that an advisor with type v plays the threshold strategy of recommending to stay in the auction, if $p < v+1$, and to quit once p hits $v+1$ (see Figure 1). Given this, what is the optimal strategy of the bidder? If she gets the recommendation to quit when the running price is $p \in [1, 11]$, she infers that her valuation is $v = p - 1$. Since p exceeds this valuation, the bidder finds it optimal to quit the auction immediately. If she has received recommendations to continue bidding, she trades off the value of waiting for more information against the possibility of overpaying for the asset. As the running price p increases, the support of bidder's beliefs, $[p-1, 10]$, shrinks. Therefore, the best response of the bidder is to stay in the auction, as long as $p \leq \hat{p}$, given by $0 = \mathbb{E}\left[v | v \geq \hat{p} - 1\right] - \hat{p}$. Hence, $\hat{p} = 9$. Intuitively, $\hat{p} = 9$ is exactly the price at which the bidder is indifferent between winning the auction and getting the valuation of 9 on average (when the auction reaches this price, the bidder's posterior is that $v \in [8, 10]$) and quitting it.

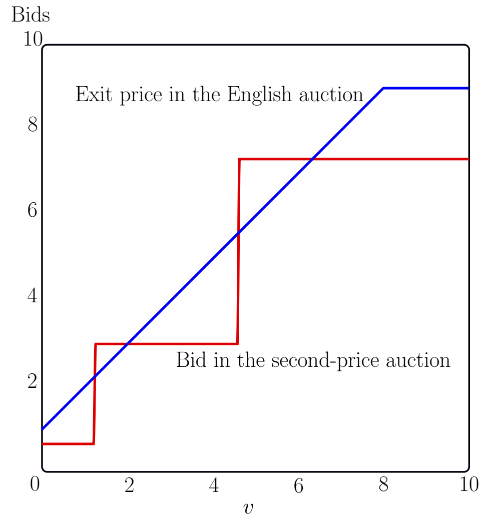

Figure 1: Equilibrium bids in second-price and ascending-price auctions

The "delegation-like" equilibrium in the English auction is very different from the equilibrium of the second-price auction. What does this imply for the comparison of revenues and efficiency? It is clear from Figure 1 that the English auction is more efficient: Not only there is a separation of types up to $v = 8$, but the pooling interval $[8, 10]$ is contained in the pooling interval in the top interval in the second-price auction $\left[4\frac{2}{3}, 10\right]$. The English auction also generates higher expected revenues than the second-price auction: $4\frac{23}{75}$ versus $3\frac{88}{135}$. The comparison of revenues is not obvious at first glance, because one distribution of bids does not dominate the other. Nevertheless, higher expected revenues in the English auction is a rather general result.

4 Bidding under Cheap-Talk Communication

This section solves for equilibria. Subsection 4.1 shows that the revenue equivalence theorem extends to the setting when bidders rely on biased advisors, if the auction is static. Subsection 4.2 shows that the English auction has "delegation-like" equilibria that dominate equilibria in the second-price auction in terms of both expected revenues and efficiency.

4.1 Static Auctions

After a bidder gets message $\tilde{m}$ from her advisor, she updates her belief about her value and decides on the bid. By risk-neutrality, the bidder cares only about her posterior expected value, which we refer to as her *type* $\theta \equiv \mathbb{E}[v|\tilde{m}] \in [\underline{v}, \overline{v}]$. Let F_θ denote the distribution of a bidder's types, induced by equilibrium at the communication stage (by symmetry, F_θ is the same for all bidders). The next proposition establishes revenue equivalence for static auctions, and shows that communication takes a partition form similar to standard cheap-talk games.

Proposition 1. *If there is a single round of communication, then*

1. *for any equilibrium in a static auction there exists an equilibrium of the second-price auction that generates the same allocation, expected revenues, and equilibrium distribution of bidders' expected values, F_θ, after the communication stage;*

2. *in any equilibrium, the communication takes an interval partition form $(\omega_k)_{k=0}^K$, in which $\omega_0 = \underline{v}$, $\omega_K = \overline{v}$, and types $v \in [\omega_{k-1}, \omega_k)$, $k = 1, \ldots, K$ induce the same bid.*

Our main question is whether the choice of the auction format affects its expected revenues and efficiency. Part 1 of Proposition 1 tells us that it does not if one restricts attention to static auctions. Intuitively, the advisor's decision of what message to send depends only on how it affects the probability of winning and expected payment. From Che and Gale (2006), we know that both are the same for any fixed distribution F_θ. Therefore, the advisor's problem of choosing what message to send is also the same.

Part 2 of Proposition 1 states that in static auctions the conflict of interest results in coarse information transmission from advisors to bidders. After the communication, each bidder updates her expected value to one of finite values $\mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$, $k = 1, \ldots, K$, and bids it. Hence, ties arise with positive probability, and the asset is sometimes allocated inefficiently. The interval partition structure of communication is similar to Crawford and Sobel (1982), though it does not follow from it directly because the payoffs of each bidder and advisor depend on bids of rival bidders that are outcomes of the communication game. Instead, we show that the advisor's payoff generally satisfy the appropriate single-crossing condition in Kartik et al. (2017), which implies the interval partition structure.[10] Note that an equilibrium with a higher number of intervals need not imply higher expected revenues to the seller.

4.2 Comparison of Auction Formats

Unlike static auctions, the English auction also admits an equilibrium of a very different form:

Definition 2. *The equilibrium of the English auction is a* capped delegation equilibrium *if it is outcome-equivalent to the equilibrium in which for some $v^* \in (\underline{v}, \overline{v}]$ on the equilibrium path:*

[10] Chen et al. (2008) show that in the standard cheap talk game, there always exist equilibria satisfying the NITS condition, which is also true in our model (Proposition 5 in Online Appendix A).

- *advisor type v recommends "stay" in the auction when the running price is below his most preferred exit price $v + b$, and recommends "quit" when it reaches $v + b$;*

- *the bidder quits the auction if either the running price increases to $v^* + b$ or she receives message "quit" from the advisor, whichever happens earlier.*

If $v^* = \overline{v}$, we call the equilibrium the full delegation equilibrium.

In a capped delegation equilibrium, the advisor's types below v^* fully separate over the course of the auction, provided that it reaches price $v^* + b$. In contrast, types above v^* pool, because the bidder exits the auction at price $v^* + b$ irrespectively of the advisor's recommendation then. We call this equilibrium capped delegation, because if the advisor submitted the bids himself, he would stay in the auction until price $v + b$. Thus, even though the bidder has formal authority over the bidding decisions, she effectively delegates them to the advisor with the restriction that he cannot stay in the auction beyond price $v^* + b$ ("cap"). In Online Appendix B, we show that if an equilibrium of the English auction satisfies a dynamic version of the NITS condition, then it must be a capped delegation equilibrium.

For a large class of distributions, the capped delegation equilibrium is unique. Let $MRL(s) \equiv \mathbb{E}[v|v \geq s] - s$ be the *mean residual lifetime* function, which is well-studied in industrial engineering and economics (Bagnoli and Bergstrom (2005)). Then, a sufficient condition for uniqueness is one of the following two assumptions:[11]

Assumption A. $MRL\,(s) > b$ for all $s \in [\underline{v}, \overline{v}]$ (and so $\overline{v} = \infty$).

Assumption B. $MRL\,(s)$ is strictly decreasing in s.

Assumption A is satisfied for distributions with weakly increasing $MRL\,(s)$ if the bias b is not too high. For example, it holds for exponential, Pareto, and truncated from below log-normal distributions. Decreasing $MRL\,(s)$ is a particularly natural property. In industrial engineering, where $MRL\,(s)$ captures the expected time before a machine of age s breaks down, decreasing $MRL\,(s)$ means that the machine gets less durable as it ages. In our context, it means that winning at a higher price is worse news for the bidder than winning at a lower price. It holds for uniform, normal, logistic, extreme value, and many other distributions.

We now turn to our main comparison results. Let $\varphi(v) \equiv v + b - \frac{1-F(v)}{f(v)}$ denote the virtual valuation of the advisor with type v.

[11] Proposition 6 in Online Appendix A shows that with two bidders our results generalize to cases in which neither assumption A nor B is satisfied.

Theorem 1. *Suppose that Assumption A holds. Then, there exists the full delegation equilibrium in the English auction. It is fully efficient: the winner of the auction is always the bidder with the highest valuation. If, in addition, $\varphi(\cdot)$ is strictly increasing, then it brings higher expected revenues than any NITS equilibrium in the second-price auction. Finally, no other capped delegation equilibrium exists.*

Unlike in static auctions, full separation of advisor types is possible in the English auction. This immediately leads to full efficiency. The difference arises because of communication during the course of the auction. When the advisor recommends the bidder to quit the auction at the current price p, the bidder learns that her valuation is $p - b < p$ and thus exits immediately. As she gets recommendations to stay in the auction, she updates her belief that the valuation is not too low. Her decision whether to continue bidding trades off the benefit of waiting for more information against the cost of possibly overpaying. If the bidder wins when the strongest rival's bid is $s + b$, she pays $s + b$ and gets on average $\mathbb{E}[v|v \geq s]$. Under Assumption A, $\mathbb{E}[v|v \geq s]$ is always above $s + b$. Thus, following the advisor's recommendation is always optimal for the bidder. Given that such a reaction of the bidder implements the advisor's unconstrained optimal bidding strategy of bidding up to $v + b$, the advisor does not want to deviate from this communication strategy. This full delegation equilibrium is not possible in static auctions because of the commitment problem: The bidder would not follow the advisor's recommendation. In contrast, in the English auction the advisor can make the bidder bid above her (unknown) valuation by delaying the recommendation to quit the auction.

The revenue comparison result is surprising: It is a priori not clear why the English auction should bring higher expected revenue. To maintain the indifference of cut-off types ω_k in the second-price auction, it is necessary that bids in the second-price and the English auction are not clearly ordered. Relatedly, optimal mechanisms usually impose inefficiencies to limit information rents of bidders. The key idea is to view the seller's problem as the problem of selling directly to informed advisors, where communication between advisors and bidders puts a restriction on the set of outcomes that can be implemented. By the envelope formula in Myerson (1981), we can write the seller's expected revenues as the expected virtual valuation of the winning advisor less the payoff of the lowest type:

$$\mathbb{E}\left[\sum_{i=1}^{N} \varphi\left(v_i\right) p\left(v_i, \boldsymbol{v}_{-i}\right)\right] - N \cdot U_A\left(\underline{v}\right), \tag{2}$$

where $p\left(v_i, \boldsymbol{v}_{-i}\right)$ is the probability that a bidder with valuation v_i wins the auction if the vector of types of rival bidders is $\boldsymbol{v}_{-i}$ and $U_A(v)$ is the expected payoff of advisor type v. In (2), the auction format determines $p(\cdot)$ and $U_A(\underline{v})$. Higher efficiency of the English auction together with increasing virtual valuation implies the first term in (2) is higher in the English auction than in the second-price auction. The NITS condition guarantees that the expected payoff of the lowest type is non-negative in the second-price auction, while it is zero in the English auction. Together, these two effects imply that the English auction generates higher expected revenues.

The English auction can be easily modified to allow for a reservation price. Then, under Assumption A, the English auction with an appropriate reserve price becomes optimal in a very large class of selling mechanisms. Specifically, consider any selling mechanism in which in each round each bidder i privately communicates with advisor i via cheap talk, and which in equilibrium delivers a non-negative expected payoff to any type of the advisor: $U_A(v) \geq 0$ for all $v \in [\underline{v}, \overline{v}]$. The expected revenues from this mechanism can be written as (2). Thus, the seller's problem of selling to bidders relying on the advice of informed advisors is a constrained problem of selling to advisors directly, so the optimal mechanism in the former problem cannot generate higher expected revenues than the optimal mechanism in the latter problem. We know from Myerson (1981) that if the seller sells directly to informed advisors, the English auction with a reserve price $r = \varphi^{-1}(0) + b$ achieves the highest expected revenues among all mechanisms satisfying $U_A(v) \geq 0$ for all $v \in [\underline{v}, \overline{v}]$. However, under Assumption A, the English auction in which the seller sells to bidders relying on advisors is identical to selling to advisors directly. Thus, we get the following theorem:

Theorem 2. *Suppose that Assumption A holds and $\varphi(\cdot)$ is strictly increasing. Then, the English auction with a reserve price $r = \varphi^{-1}(0) + b$ is optimal among all selling mechanisms that in equilibrium generate a non-negative expected utility to any type of the advisor: $U_A(v) \geq 0$ for all $v \in [\underline{v}, \overline{v}]$.*

The next theorem generalizes the uniform example in Section 3 and compares auction formats under Assumption B. We say that an equilibrium in one auction is *more efficient* than an equilibrium in another auction if the former results in a higher expected valuation of the winning bidder.

Theorem 3. *Suppose that Assumption B holds. If $b \in \left(\lim_{v \to \overline{v}} MRL(v), MRL(\underline{v})\right)$, then there is a unique capped delegation equilibrium in the English auction, and the cut-off type is $v^* = MRL^{-1}(b) < \overline{v}$. This equilibrium in the English auction is more efficient than any equilibrium of the second-price auction. If, in addition, $\varphi(\cdot)$ is strictly increasing, it brings higher expected revenue than any NITS equilibrium of the second-price auction. Furthermore, it brings higher expected revenue than any equilibrium of static auctions if N is sufficiently high.*

Under Assumption B, in the English auction the value of the option to wait for advisor's recommendation is strictly positive at any price below v^*+b. However, when the price p exceeds v^*+b, the bidder learns that her valuation is in a narrow enough interval $[p-b,\bar{v}]$ so that the risk of overpaying outweighs the value of additional information. Thus, there is necessarily pooling at the top above $v^*$and the full delegation is not possible. This makes the efficiency comparison more nuanced, because both English and second-price auctions misallocate the asset with positive probability. Nevertheless, we show that the pooling region is always smaller in the English auction. To see this result, consider the advisor's indifference condition that determines intervals in the second-price auction. For advisor with type ω_{K-1} to be indifferent, the highest bid must exceed the maximum willingness to pay of the advisor type ω_{K-1}: $\mathbb{E}\left[v|v\geq\omega_{K-1}\right]>\omega_{K-1}+b$, or, equivalently, $MRL\left(\omega_{K-1}\right)>b$. Hence, in the English auction the bidder's option value of waiting is positive at price $\omega_{K-1}+b$. Consequently, types just above ω_{K-1} would recommend the bidder to stay in the English auction at this price, and the bidder would follow the recommendation, implying a smaller pooling region and higher efficiency. Once we obtain the efficiency ranking, the revenue comparison follows by the same argument as under assumption A.

The last result in Theorem 3 highlights the role of NITS condition in static auctions. If an equilibrium in the second-price auction violates the NITS condition, $U_A(\underline{v})$ in (2) is negative, so that the the second-price auction could generate higher expected revenues despite its lower efficiency. Indeed, in the example of Section 3, the babbling equilibrium in the second-price auction generates $\mathbb{E}\left[v\right]=5$ in revenues, which exceeds the expected revenues of the capped delegation equilibrium of the English auction ($4\frac{23}{75}$). The NITS condition deems equilibria with negative payoff of advisor unreasonable, implying that a NITS equilibrium in the second-price auction cannot be too inefficient. This in turn implies that the information rents of advisors cannot be too low and leads to the ranking of expected revenues.

While the NITS condition seems to be a sensible restriction in static cheap talk games, the last statement of Theorem 3 shows that the revenues comparison result becomes selection-free if the auction is sufficiently competitive: the capped delegation equilibrium of the English auction generates higher expected revenues than *any* equilibrium of the second-price auction (and by Proposition 1, any other static auction in a very large class). Intuitively, as N increases, valuation of the second-highest bidder is more likely to be high. Thus, the seller eventually cares much more about the finer separation of types in the English auction than about the extraction of extra rents from the lower types in the second-price auction. In the example of Section 3, already for $N=3$ expected revenues of the English auction are higher than in any equilibrium of the second-price auction.[12]

Given that the English auction is attractive from both efficiency and revenues dimensions, it is interesting to explore how they depend on the magnitude of the advisors' bias. In particular, does the seller benefit from advisors being more biased for overpaying? The next proposition sheds light on this question:

Proposition 2. *Suppose that assumption B holds. Then, in the unique capped delegation equilibrium of the English auction:*

1. *The expected valuation of the winning bidder is strictly decreasing in b on $\left(\lim_{v\to\bar{v}} MRL(v), MRL(\underline{v})\right)$.*
2. *The expected revenues are strictly increasing in b in the neighborhood of $b = 0$ and strictly decreasing in b in the neighborhood of $b = MRL(\underline{v})$.*
3. *For any $b > 0$, if $\bar{v} < \infty$, and for any $b > \lim_{v\to\infty} MRL(v)$, if $\bar{v} = \infty$, there exists $N(b)$ such that for all $N > N(b)$, the expected revenues strictly increase with a marginal decrease in b.*

The first result of the proposition is that efficiency of the auction decreases with the advisors' bias. This is because a higher bias increases the size of the pooling region. More interestingly, the second result shows that the effect of a bias on revenues is non-monotone. A higher bias has two opposite effects. On one hand, it leads to a more aggressive bidding when the valuation is in the separating region, $v < v^*(b)$, since the advisor recommends to quit the auction at a higher price. On the other hand, a higher bias leads to a less aggressive bidding when the valuation is in the pooling region, $v > v^*(b)$, since the bidder stops listening to the advisor's recommendation earlier. The former effect dominates when the size of the pooling region is small, which is the case when the bias is low, while the latter effect dominates when it is high. In the example of Section 3, expected revenues are single-peaked in b, reaching the maximum at $b \approx 3.54$.

The last result of Proposition 2 implies that for any bias level, expected revenues decrease in the bias if the auction is sufficiently competitive. Intuitively, if the auction is sufficiently competitive, the valuations of the strongest two bidders are very likely to be in the pooling region, which implies that more aggressive bidding by high types is more important than more

[12] The capped delegation equilibrium of the English auction yields the expected revenues of ≈ 5.93. The highest expected revenues in the second-price auction are ≈ 5.17, which is attained in the most informative equilibrium with four partition intervals.

aggressive bidding by low types. Therefore, a lower bias increases expected revenues in sufficiently competitive auctions. Overall, our results suggest that the seller benefits from a higher bias if the bias is moderate and the auction is not too competitive.

5 Bidding with Contracts

We have shown that when bidders rely on cheap talk communication with their advisors, there is an equilibrium in the English auction, but not in the second-price and other static auctions, in which bidders behave as if they delegate bidding to advisors with caps on bids. This equilibrium results in more efficient allocations and higher expected revenues to the seller. This section considers a model in which bidders can commit to contracts with their advisors. It shows that commitment makes the English and the second-price auctions equivalent, and that their equilibrium features the same bidding behavior and outcomes as the "capped delegation" equilibrium of the English auction in the "cheap talk" model.

Formally, we consider the following *auction with contracts*. At the initial date, each bidder i simultaneously and privately commits to a contract that maps each report of her advisor of valuation $w_i \in [\underline{v}, \overline{v}]$ into the exit price in the English auction $\theta_i(w_i)$. After the contracts are committed to, each advisor sends a private report of his valuation to his bidder, and bidders bid in the auction abiding to their contracts. The optimal contract of bidder i maximizes her expected payoff subject to providing the advisor with incentives to report the valuation truthfully, $w_i = v_i$, taking as given contracts of other bidders, $\theta_j(w_j)$, $j \neq i$. An equilibrium in this game is a set of contracts, $\theta_j^*(w_j)$, $j = 1, ..., N$, which satisfy the property that each bidder j finds contract $\theta_j^*(w_j)$ optimal, given that she expects other bidders offer their equilibrium contracts.

As an intermediate step, consider the optimal contracting problem of a single bidder i, fixing bidding strategies of all rival bidders $j \neq i$. Suppose that bidder i expects the distribution (c.d.f.) of the highest rival bid to be $y(\cdot)$ and $y(\cdot)$ is strictly increasing in the range $\theta \in [\underline{v} + b, \overline{v}]$. The next proposition shows that under an additional condition on the distribution of valuations, the solution to bidder i's problem is to bid her advisor's maximum willingness to pay up to a certain cut-off. Furthermore, this cut-off does not depend on the distribution $y(\cdot)$ of the highest rival bid, and in fact, coincides with the cut-off in the capped delegation equilibrium of the cheap talk game of Section 4:

Proposition 3. *Consider the optimal contracting problem of bidder i for any distribution $y(\cdot)$ of the highest rival bid. Suppose that Assumption B holds and $F(v)+bf(v)$ is strictly increasing* *in $v \in [\underline{v}, \overline{v}]$. Then, contract $\theta_i(w_i) = b + \min\{w_i, v^*\}$, where $v^* \equiv MRL^{-1}(b)$, is optimal. If, in addition, $y(\theta)$ is strictly increasing in the range $\theta \in [\underline{v} + b, \overline{v}]$, then this contract is the unique optimal contract.*[13]

Thus, if the bidder can commit to any way she responds to recommendations from her advisor, she strictly prefers a capped delegation contract, provided that the probability of winning is strictly increasing in the bid in the relevant range. If the distribution of the highest rival bid $y(\cdot)$ has flat regions in the range $[\underline{v} + b, \overline{v}]$, then the contract from Proposition 3 is also optimal, but not necessarily uniquely optimal.[14] Proposition 3 is similar to Proposition 2 in Burkett (2016), but our proposition has a result about uniqueness, which is important for our claim about the equilibrium uniqueness in the next proposition. Our proof is different from Burkett (2016), because he assumes that the advisor's bias goes to zero as $v \to \underline{v}$, while in our setup the bias is constant. In the proof, we follow Melumad and Shibano (1991) to derive the general shape of incentive compatible contracts, and then show that any contract that does not take capped delegation form with cap $b + v^*$ can be profitably modified by the bidder.

Next, consider the auction with contracts in which each bidder j simultaneously commits to some contract $\theta_j(\cdot)$. Proposition 3 implies that for each bidder i, the strategy of choosing contract $\theta_i(w_i) = b + \min\{\omega_i, v^*\}$ is weakly dominant in the following sense: It earns bidder i an expected payoff of at least as high as any other contract, regardless of the contracts that the other bidders commit to. It follows that the auction with contracts has a unique undominated equilibrium, i.e., an equilibrium in which no bidder chooses a weakly dominated contract, and it is given by all bidders committing to contract $\theta^*(\omega) = b + \min\{\omega, v^*\}$.[15]

Corollary 1. *Suppose that assumption B holds and $F(v) + bf(v)$ is strictly increasing in v. Then, the unique undominated equilibrium in the auction with contracts is a symmetric one in which all bidders offer contract $\theta^*(\omega) = b + \min\{\omega, v^*\}$.*

Thus, if bidders could commit to contracts, then the second-price auction would result in exactly the same bidding behavior and allocation as the English auction in the model with cheap-talk communication.

Therefore, this section achieves two objectives. First, it illustrates the role of the lack of commitment for our results about the comparison of auction formats – the irreversibility of

[13] As always, uniqueness means uniqueness within the class of direct revelation contracts.

[14] Intuitively, if it is a zero probabilily event that the highest rival bid is in an interval $[x - \varepsilon, x + \varepsilon]$ for some $\varepsilon > 0$ and $x \in (\underline{v} + b, v^* + b)$, then the contract from Proposition 3 yields the same payoff to bidder i as an otherwise identical contract that pools types close enough to type $x - b$.

[15] This is also a unique equilibrium in a perturbed version of the game, in which with probability $\varepsilon > 0$ there is an additional "behavioral" bidder, whose bid distribution has c.d.f. that is strictly increasing on $[\underline{v} + b, \overline{v}]$.

the price in the English auction gives commitment power to the bidder for free. Second, the result that delegated bidding up to a cap is the optimal contract from the bidder's point of view provides another justification for the capped delegation equilibrium in the English auction in the model with cheap-talk communication. If a bidder has ability to "influence" what equilibrium of the communication game with her advisor is played, Proposition 3 suggests that the bidder will have strong incentives to favor the capped delegation equilibrium.

6 Conclusion

The goal of the paper is to understand how to sell assets when potential buyers rely on the advice of biased experts. We analyze this problem in the canonical framework of symmetric independent private values. We show that when the communication takes form of cheap-talk, the revenue equivalence theorem holds in static auctions. However, the English auction is, quite generally, more efficient and also results in higher expected revenues than static auctions. This is because by communicating his information later in the game rather than in the beginning, advisors are able to persuade their bidders to stay in the auction longer. When all bidders can commit to contracts, the revenue equivalence of the second-price and English auctions is restored and the communication there is the same as in the English auction with cheap-talk.

Our analysis points to several directions for future research. First, the analysis of bidder asymmetries, in particular in the biases of their advisors, is relevant in applications and can be fruitful. Second, since our focus is on the comparison of static and dynamic formats, we do not solve for the optimal mechanism, except for the case of assumption A. Solving for the optimal mechanism in the general case is thus an avenue for future research. We conjecture that the optimality of English auction with an appropriate reserve price generalizes beyond assumption A. Finally, many applications in which bidders rely on biased advisors may have valuations with a common component: corporate takeovers and real estate transactions are two examples. Thus, it can be interesting to extend the model beyond the independent private values framework.

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Appendix

In the analysis, we will frequently refer to the distribution of valuation of the strongest opponent of a bidder. We denote by $\hat{v}$ the maximum of $N-1$ i.i.d. random variables distributed according to F and its c.d.f. by G: $G(\hat{v}) = F(\hat{v})^{N-1}$. We also use $F(a,b) = F(b) - F(a)$ to denote the probability that a random variable distributed according to F falls in the interval $[a,b]$. Similarly, $G(a,b) = G(b) - G(a)$.

Proof of Proposition 1. **Part 1:** Consider a standard static auction $\mathcal{A}$ with continuous payments and an equilibrium in it. Let $m_{\mathcal{A}} : [\underline{v}, \overline{v}] \mapsto M$ be the equilibrium communication strategy. Let $F_{\theta,\mathcal{A}}$ be the distribution of each bidder's types generated by $m_{\mathcal{A}}$, $\Theta_{\mathcal{A}}$ be the support of $F_{\theta,\mathcal{A}}$, and $\beta_{\mathcal{A}} : \Theta_{\mathcal{A}} \mapsto \Delta(\mathbb{R}_+)$ be the equilibrium bidding strategy. Let $x(\theta)$ and $t(\theta)$ be type θ's equilibrium expected probability of winning and expected payment, respectively.

We first use the results of Che and Gale (2006) to argue that if bidders' types are drawn i.i.d. from $F_{\theta,\mathcal{A}}$, the equilibrium $\beta_{\mathcal{S}}$ in the second-price auction $\mathcal{S}$ implies the same expected probabilities of winning and payments $x(\theta)$ and $t(\theta)$. Since this result follows directly from Che and Gale (2006), we simply outline the argument. Lemma 2 in Che and Gale (2006) shows that a symmetric equilibrium of a standard auction with continuous payments admits an efficient allocation, i.e., for any realization of bidders' types (which in our case are drawn i.i.d. from $F_{\theta,\mathcal{A}}$), a bidder with the highest type wins the auction. This implies that function $x(\theta)$ is the same across such auctions. Proposition 1 in Che and Gale (2006) shows that for standard auctions with continuous payments their conditions (A1) and (A2) hold. Condition (A1) implies that their inequality (3) holds as equality. This in conjunction with the envelope condition for the bidder's payoff (their equation (4)) and condition (A2) implies that function $t(\theta)$ is the same across standard auctions with continuous payments.

We next show that the communication strategy $m_{\mathcal{A}}$ is also an equilibrium communication strategy in the second-price auction. Consider any type v contemplating to send message $m' \neq m_{\mathcal{A}}(v)$. First, consider $m' = m_{\mathcal{A}}(v')$ for some other type $v' \neq v$. Then, message m' generates some bidder's type $\theta' \in \Theta_{\mathcal{A}}$. Since type v is better off sending message $m_{\mathcal{A}}(v)$ than message $m_{\mathcal{A}}(v')$ in auction $\mathcal{A}$, it must be that $(v+b)\,x(\theta) - t(\theta) \geq (v+b)\,x(\theta') - t(\theta')$. Since $x(\theta)$ and $t(\theta)$ are the same in both auctions, this implies that type v does not benefit from sending message $m_{\mathcal{A}}(v') \neq m_{\mathcal{A}}(v)$. Second, consider m' such that there is no type v' for whom $m' = m_{\mathcal{A}}(v')$. Specify the beliefs of the bidder following such message m' as the beliefs following some message $m_{\mathcal{A}}(v')$ for some v' (i.e., specify that any off-path message is interpreted as one of on-path messages). Then, a deviation to such m' is equivalent to a deviation to $m_{\mathcal{A}}(v')$ for some v'. Since the latter does not benefit type v, the former also does not. Hence, $m_{\mathcal{A}} : [\underline{v}, \overline{v}] \mapsto M$ is also an equilibrium communication strategy in the second-price auction.

Thus, we have constructed an equilibrium in the second-price auction with the same communication strategy $m_{\mathcal{A}}$ as in $\mathcal{A}$. Moreover, we have shown that given that bidders' types are drawn i.i.d. from $F_{\theta,\mathcal{A}}$, the two auctions exhibit payoff equivalence (functions $x(\theta)$ and $t(\theta)$ are the same) and thus, yield the same expected revenues. Moreover, the two auctions allocate the asset to the bidder with the highest type θ.

Part 2: By Part 1, it is without loss of generality to focus on the second-price auction. Fix the strategies of all bidder-advisor pairs but one, and consider the cheap-talk game in the remaining bidder-advisor pair. In equilibrium, to each bid corresponds a pair (q, t), where $q \in [0, 1]$ is the expected probability of winning the asset and $t \in \mathbb{R}_+$ is the expected payment. The strategy of the bidder maps messages into distributions over bids, hence, into distributions over pairs (q, t). The utility of advisor type v from (q, t) is $(v + b)q - t$. We can rewrite this utility in the form $g_1(q, t)f_1(v) + g_2(q, t)f_2(v)$, where $g_1(q, t) = q$, $f_1(v) = v + b$, $g_2(q, t) = -t$, $f_2(v) = 1$. Since f_1 is strictly increasing in v and f_2 is constant, f_1 strictly ratio dominates f_2. Hence, by Theorem 2 in Kartik et al. (2017) the advisor's utility function satisfies the strict single-crossing expectational differences. Then, Claim 1 in Kartik et al. (2017) implies that every equilibrium in the communication game is connected: if $v_l < v_m < v_h$ and $m(v_l) = m(v_h)$, then $m(v_m) = m(v_l)$.

By Lemma 3 in Che and Gale (2006), in any symmetric equilibrium of the second-price auction, each bidder submits her updated expected valuation with $F_\theta-$probability one.[16] Consider set $\tilde{\Theta}$ of bidder types that submit their updated expected valuations in equilibrium. Let $\tilde{V} \equiv \{v \in [\underline{v}, \overline{v}] : \mathbb{E}[v|m(v)] \in \tilde{\Theta}\}$ be the set of advisor types who induce one of the bidder types in $\tilde{\Theta}$. Since $\tilde{\Theta}$ occurs with $F_\theta-$probability one, $\tilde{V}$ occurs with $F-$probability one. It is convenient and without loss of generality, to refer to messages to such types as bid recommendations and denote equilibrium messages by $\tilde{m} = \mathbb{E}[v|m(v) = \tilde{m}]$. Since bidder types in $\tilde{\Theta}$ bid their updated expected valuation and $m(\cdot)$ is connected on $[\underline{v}, \overline{v}]$, $m(\cdot)$ is weakly increasing on $\tilde{V}$.

Further, it is not possible that $m(\cdot)$ is strictly increasing on some interval of advisor types $(v', v'') \cap \tilde{V}$. By contradiction, if this were the case, then the message would be fully revealing of the advisor type in (v', v''), and the bidder would bid the message. But this would imply that the advisor of type $\frac{1}{2}(v' + v'')$ would prefer to deviate to sending message (and inducing bid) $\frac{1}{2}(v' + v'') + \varepsilon$, which is a contradiction. Therefore, we have shown that $m(\cdot)$ is weakly increasing on $\tilde{V}$ and cannot be stricly increasing on any $(v', v'') \cap \tilde{V}$. This implies that the communication strategy takes an interval partition form on set $\tilde{V}$. Call the partition cutoffs $(\omega_k)_{k=0}^K$.

Since $m(\cdot)$ is connected on $[\underline{v}, \overline{v}]$, the set $[\underline{v}, \overline{v}] \backslash \tilde{V}$ is a subset of $\{\omega_k\}_{k=0}^K$. Further, if $\omega_k \in [\underline{v}, \overline{v}] \backslash \tilde{V}$, then advisor type ω_k perfectly reveals himself. This would induce bid ω_k, which equals to the updated expected valuation of the bidder, and hence, contradicts $\omega_k \in [\underline{v}, \overline{v}] \backslash \tilde{V}$. Therefore, the communication strategy takes an interval partition form on the whole set $[\underline{v}, \overline{v}]$. This concludes the proof.

□

Proof of Theorem 1. First, we show that the capped delegation equilibrium with $v^* = \infty$ is indeed an equilibrium. The argument after the theorem verifies the advisor's optimality and bidder's

[16] Lemma 3 in Che and Gale (2006) assumes that Θ is bounded from above. The proof of Lemma 3 can be modified to allow for set Θ unbounded from above.

optimality after message "quit". To verify that the bidder has incentives to follow the advisor's recommendation to "stay" at any p, consider the option value to the bidder of following the advisor's recommendation. The bidder infers from the fact that the auction reaches price p that her valuation is in $[p-b,\infty)$, and that there is at least one rival whose valuation is also in $[p-b,\infty)$. Denoting the bidder's posterior probability that n rival bidders have valuations in $[p-b,\infty)$ by $q_n(p)$ and the c.d.f. of the maximum of n i.i.d. random variables distributed according to F by $G_n(\cdot)$, the bidder's option value of following the advisor's recommendation is

$$V(p)=\int_{p-b}^{\infty}\frac{1-F(s)}{1-F(p-b)}\left(\mathbb{E}[v|v\geq s]-s-b\right)\left(\sum_{n=1}^{N-1}q_n(p)\,dG_n(s|s\geq p-b)\right). \tag{3}$$

Intuitively, if the bidder wins when the strongest rival's valuation is s, she pays $s+b$ and gets, on average, $\mathbb{E}[v|v\geq s]$. Under Assumption A, $\mathbb{E}[v|v\geq s]-s-b>0$ for any $s\geq \underline{v}$. Thus, the bidder prefers to follow recommendation "stay" at any p. Hence, this is indeed an equilibrium. In this equilibrium, the auction is won by the bidder with the highest valuation. Therefore, it is fully efficient.

Second, we prove the statement about revenues. Consider an equilibrium of the second-price auction. Let $p_{SPA}(v)$ and $t_{SPA}(v)$ be the associated expected probability of winning and expected payment conditional on winning, conditional on a bidder's valuation being v. The implied equilibrium payoff of the advisor is $U_{A,SPA}(v)=p_{SPA}(v)\,(v+b-t_{SPA}(v))$. If advisor type v mimics the equilibrium communication strategy of advisor type $\hat{v}$, his expected payoff would be $p_{SPA}(\hat{v})\,(v+b-t_{SPA}(\hat{v}))$. For $p_{SPA}(v)$ and $t_{SPA}(v)$ to be supported in equilibrium, it must be that

$$U_{A,SPA}(v)=\max_{\hat{v}}p_{SPA}(\hat{v})\,(v+b-t_{SPA}(\hat{v}))\,,$$

which by the generalized envelope theorem (Milgrom and Segal (2002)) implies $U_{A,SPA}(v)=U_{A,SPA}(\underline{v})+$

$\int_{\underline{v}}^{v} p_{SPA}(x)\, dx$. Integrating by parts, the expected revenues can be written as

$$
\begin{aligned}
& N\mathbb{E}\left[p_{SPA}(v)\, t_{SPA}(v)\right] \\
=& N\mathbb{E}\left[p_{SPA}(v)\,(v+b) - U_{A,SPA}(v)\right] \\
=& N\mathbb{E}\left[p_{SPA}(v)\,(v+b) - U_{A,SPA}(\underline{v}) - \int_{\underline{v}}^{v} p_{SPA}(x)\, dx\right] \\
=& N\left[\int_{\underline{v}}^{\overline{v}} \left(p_{SPA}(v)\,(v+b) - \int_{\underline{v}}^{v} p_{PSA}(x)\, dx\right) dF(v) - U_{A,SPA}(\underline{v})\right] \\
=& N\left[\int_{\underline{v}}^{\overline{v}} p_{SPA}(v)\,(v+b)\, dF(v) + \int_{\underline{v}}^{\overline{v}} \left(\int_{\underline{v}}^{v} p_{PSA}(x)\, dx\right) d(1-F(v)) - U_{A,SPA}(\underline{v})\right] \\
=& N\left[\int_{\underline{v}}^{\overline{v}} \left(p_{SPA}(v)\,(v+b)\right) dF(v) - \int_{\underline{v}}^{\overline{v}} p_{PSA}(v)\, \frac{1-F(v)}{f(v)} dF(v) - U_{A,SPA}(\underline{v})\right] \\
=& N\left[\int_{\underline{v}}^{\overline{v}} p_{SPA}(v) \left(v+b-\frac{1-F(v)}{f(v)}\right) dF(v) - U_{A,SPA}(\underline{v})\right] \\
=& N\left[\mathbb{E}\left[p_{SPA}(v)\, \varphi(v)\right] - U_{A,SPA}(\underline{v})\right].
\end{aligned}
$$

After this, the statement about revenues follows from the text after the statement of the theorem.

Finally, we show that there is no capped delegation equilibrium with $v^* < \infty$. By contradiction, suppose that such an equilibrium exists, and consider the auction at price $v^* + b$. The equilibrium prescribes that every remaining bidder should exit the auction at this price. However, because $\mathbb{E}[v|v \geq v^*] > v^* + b$, a bidder unilaterally benefits from deviating and waiting until the price is just above $v^* + b$. This deviation leads to a jump in the probability of winning to one and only an infinitesimal increase in the payment. Hence, there is no caped delegation equilibrium with $v^* < \infty$. □

Proof of Theorem 2. Consider a symmetric mechanism Γ comprised of a bidder's strategy set and an outcome function. Because the mechanism can potentially be dynamic, the strategy set is a set of contingent plans of bids in each round. The outcome function is a mapping from bids in all rounds to the allocation rule and the transfer rule. Consider a symmetric equilibrium in this mechanism. Let $p_\Gamma(v)$ denote the equilibrium probability (evaluated at the start of the auction) of a bidder obtaining the asset, conditional on her valuation (known by her advisor) being v. Similarly, let $p_\Gamma(v)\, t_\Gamma(v)$ be the equilibrium expected transfer of a bidder, conditional on her valuation being v. If the advisor with type v adopted the equilibrium communication strategy of the advisor with type $\hat{v} \neq v$, her bidder would win with probability $p_\Gamma(\hat{v})$ and the expected transfer would be $p_\Gamma(\hat{v})\, t_\Gamma(\hat{v})$. The fact that this should not be optimal implies that the equilibrium expected payoff of the advisor, $U_{A,\Gamma}(v)$, must satisfy:

$$U_{A,\Gamma}(v) = \max_{\hat{v}} p_\Gamma(\hat{v})\,(v + b - t_\Gamma(\hat{v})).$$

Applying the generalized envelope theorem and integration by parts, we can write the expected revenues as $N\mathbb{E}\left[p_\Gamma(v)\varphi(v)\right] - NU_{A,\Gamma}(\underline{v})$.

From Myerson (1981), the mechanism that maximizes $\mathbb{E}\left[\sum_i p(v_i, \boldsymbol{v}_{-i})\varphi(v_i)\right] - NU_A(\underline{v})$ subject to constraints $\sum_i p(v_i, \boldsymbol{v}_i) \leq 1$, $p(v_i, \boldsymbol{v}_{-i}) \geq 0$, and $U_A(\underline{v}) \geq 0$ is to allocate the asset to the agent with the highest virtual valuation, provided that it is non-negative, and set $U_A(\underline{v}) = 0$. Since $\varphi(\cdot)$ is increasing, English auction with reserve price r implicitly defined by $\varphi(r - b) = 0$ (equivalently, $r = \varphi^{-1}(0) + b$) is optimal. By Assumption A, the same expected revenues are also achieved in the English auction with reserve price $r = \varphi^{-1}(0) + b$ if the seller sells to advised bidders. Therefore, any mechanism Γ that in equilibrium generates $U_{A,\Gamma}(\underline{v}) \geq 0$ cannot yield strictly higher expected revenues. □

Proof of Theorem 3. First, we show that under Assumption B, there is unique capped delegation equilibrium, and v^* is given by $MRL^{-1}(b)$. Consider any capped delegation equilibrium. Suppose the game has reached price $p < v^* + b$. Generalizing (3), the bidder's option value of following the advisor's recommendation until price $v^* + b$ and quitting the auction then is

$$V(p) = \int_{p-b}^{v^*} \frac{1 - F(s)}{1 - F(p - b)} \left(\mathbb{E}[v|v \geq s] - s - b\right) \left(\sum_{n=1}^{N-1} q_n(p)\, dG_n(s|s \geq p - b)\right). \tag{4}$$

Note that compared to (3), (4) could also include the term, corresponding to the case of winning at a tie at price $v^* + b$, but because $MRL(v^*) = b$, it equals zero, so we can omit it. If $MRL(v^*) < b$, then (4) implies $V(p) < 0$ for p sufficiently close to v^*. Therefore, there cannot be a capped delegation equilibrium with $v^* > MRL^{-1}(b)$. If $MRL(v^*) > b$, then consider the auction reaching price $v^* + b$. The candidate equilibrium prescribes the bidder to exit immediately. However, the bidder would prefer to wait until the price just above $v^* + b$ instead of exiting at price $v^* + b$. By doing this, she would ensure that she wins the auction with probability one and pays below her estimated valuation of $\mathbb{E}[v|v \geq v^*]$. Since this strategy results in a discontinuous upward jump in the expected utility of the bidder, she is better off deviating. Hence, it must be that $MRL(v^*) = b$. Next, we show that this is indeed an equilibrium. Since $MRL(\cdot)$ is strictly decreasing, $s + b > \mathbb{E}[v|v \geq s]$ for any $s < v^*$, so $V(p) > 0$ for any $p < v^* + b$. Thus, the bidder prefers to follow the advisor's recommendation for any $p < v^* + b$. When $p = v^* + b$, the bidder is indifferent between winning and losing, so leaving the auction for any recommendation of the advisor is optimal for the bidder. Finally, the strategy of communicating "stay" until price $v + b$ and "quit" after that is also optimal for the advisor given expected reaction from the bidder: Any advisor type $v \leq v^*$ implements his unconstrained optimal bidding policy this way, while any advisor type $v > v^*$ implements his constrained optimal bidding policy, since it is impossible to induce bidder bidding above $v^* + b$. Therefore, the capped delegation with cap $v^* = MRL^{-1}(b)$ is the unique capped delegation equilibrium.

Efficiency: By Lemma 3 that preceds the proof of Proposition 5, these is $\tilde{K}$ such that advisor types in $[\underline{v}, v^*]$ induce at most $\tilde{K}$ different bids in the second-price auction. Denote by $[\omega_{\tilde{K}-1}, \omega_{\tilde{K}})$ the highest interval such that $\omega_{\tilde{K}-1} \leq v^*$. Since $\omega_{\tilde{K}-1}$ satisfies equation (15), $\omega_{\tilde{K}-1} + b - \mathbb{E}[v|v \in [\omega_{\tilde{K}-1}, \omega_{\tilde{K}})] < 0$. Hence, since $\mathbb{E}[v|v \geq \omega_{\tilde{K}-1}] \geq \mathbb{E}[v|v \in [\omega_{\tilde{K}-1}, \omega_{\tilde{K}})]$, we have that $b < MRL(\omega_{\tilde{K}-1})$. On the other hand, $b = MRL(v^*)$. Since $MRL(\cdot)$ is strictly decreasing, $v^* > \omega_{K-1}$. Hence, in the capped delegation equilibrium, the pooling region $[v^*, \overline{v}]$ is smaller than $[\omega_{\tilde{K}-1}, \overline{v}]$ in the second-price auction.

Now, we can compare the efficiency of two auction formats. Denote by $v_{(i)}$ the i-th largest element in $\{v_i, i = 1, \ldots, N\}$, and by $F_{(i)}$ the c.d.f. of $v_{(i)}$. Fix some realization of $(v_{(1)}, \ldots, v_{(N)})$. If $v_{(1)}$ and $v_{(2)}$ are both below v^*, then the English auction is fully efficient, while the second-price auction is inefficient, because of ties. If $v_{(1)} \geq v^* > v_{(2)}$, then again the English auction is fully efficient, while the second-price auction is inefficient, because of ties. If $v_{(1)} \geq v_{(2)} \geq v^*$, then let $j \in \{2, \ldots, N\}$ be such that $v_{(j)} \geq v^* > v_{(j+1)}$, and let $k \in \{2, \ldots, N\}$ be such that $v_{(j)} \geq \omega_{K-1} > v_{(j+1)}$. We have that $j \leq k$. Conditional on the realization of $(v_{(1)}, \ldots, v_{(N)})$, the difference between the expected value of the winning bidder in the English auction and in the second-price auction equals

$$\begin{aligned}
&\frac{1}{j}\sum_{i=1}^{j} v_{(j)} - \frac{1}{k}\sum_{i=1}^{k} v_{(k)} \\
=&\sum_{i=1}^{j} v_{(j)}\left(\frac{1}{j} - \frac{1}{k}\right) - \frac{1}{k}\sum_{i=j+1}^{k} v_{(k)} \\
=&\sum_{i=1}^{j} v_{(j)}\frac{k-j}{jk} - \frac{1}{k}\sum_{i=j+1}^{k} v_{(k)} \\
=&\frac{k-j}{k}\left(\frac{1}{j}\sum_{i=1}^{j} v_{(j)} - \frac{1}{k-j}\sum_{i=j+1}^{k} v_{(k)}\right) \geq 0.
\end{aligned}$$

We have shown that for any realization of $(v_{(1)}, \ldots, v_{(N)})$, the capped delegation equilibrium in the English auction is more efficient than the equilibrium in the second-price auction. Thus, it is also more efficient when we integrate over $(v_{(1)}, \ldots, v_{(N)})$.

Expected Revenue: The revenue comparison of the capped delegation equilibrium with NITS equilibria of static auctions follows by the same argument as in Theorem 1. We next show that for sufficiently large N the revenue comparison holds for any equilibrium of any static auction, not necessarily NITS equilibrium. By Proposition 1, we can focus on the second-price auction among all static auctions (in the class we consider, i.e., standard auctions with continuous payments). We need to show that for all sufficiently large N, $\mathbb{E}[\min\{v_{(2)}, v^*\} + b] \geq \sum_{k=1}^{K} m_k \mathbb{P}(v_{(2)} \in [\omega_{k-1}, \omega_k))$, or equivalently,

$$\sum_{k=1}^{K} \mathbb{E}\left[\min\{v_{(2)}, v^*\} + b | v_{(2)} \in [\omega_{k-1}, \omega_k)\right] \mathbb{P}\left(v_{(2)} \in [\omega_{k-1}, \omega_k)\right) > \sum_{k=1}^{K} m_k \mathbb{P}\left(v_{(2)} \in [\omega_{k-1}, \omega_k)\right).$$

Thus, it is sufficient to show that for all sufficiently large N,

$$\mathbb{E}\left[\min\{v_{(2)}, v^*\} + b | v_{(2)} \in [\omega_{k-1}, \omega_k)\right] \geq m_k, \tag{5}$$

for all $k = 1, \ldots, K$ with a strict inequality for at least one k. This proof is somewhat technical and lengthy, so we relegate it to Online Appendix A. □

Proof of Proposition 2. The first statement follows directly from the fact that $MRL(v^*) = b$ and MRL is strictly decreasing. In the range $b \in (\lim_{v\to\bar{v}} MRL(v), MRL(\underline{v}))$, the unique equilibrium has $v^*(b) = MRL^{-1}(b) \in (\underline{v}, \bar{v})$. Since $MRL(v)$ is strictly decreasing, so the expected valuation of the winning bidder is strictly decreasing in b.

Consider the second statement. Consider $b > 0$ in the neighborhood of $b = 0$. If $\bar{v} = \infty$ and $\lim_{v\to\infty} MRL(v) > 0$, we have $v^*(b) = \infty$, so the expected revenues are $b + \int_{\underline{v}}^{\infty} v dH(v)$, where $H(\cdot)$ is the c.d.f. of the second-highest order statistic of N i.i.d. random variables with c.d.f. $F(\cdot)$. Therefore, the expected revenues are strictly increasing in b in this case. If $\bar{v} < \infty$ or $\bar{v} = \infty$ and $\lim_{v\to\infty} MRL(v) = 0$, $v^*(b) \in (\underline{v}, \bar{v})$, so the expected revenues can be written as

$$b + \int_{\underline{v}}^{v^*(b)} v dH(v) + (1 - H(v^*(b))) v^*(b). \tag{6}$$

The derivative of (6) with respect to b equals $1 + (1 - H(v^*(b))) \frac{dv^*}{db}$. Applying the implicit function theorem to $MRL(v^*(b)) = b$ yields

$$MRL'(v^*(b)) = \frac{f(v^*(b))}{1 - F(v^*(b))} MRL(v^*(b)) - 1.$$

Therefore, $\frac{dv^*}{db} = -\left(1 - b\frac{f(v^*(b))}{1-F(v^*(b))}\right)^{-1}$, which is negative by assumption B. Hence, the derivative of (6) with respect to b is

$$\frac{H(v^*(b)) - b\frac{f(v^*(b))}{1-F(v^*(b))}}{1 - b\frac{f(v^*(b))}{1-F(v^*(b))}}. \tag{7}$$

When $b \to 0$, $v^*(b) \to \bar{v}$, so the derivative equals one. Thus, the expected revenues are increasing in b around $b = 0$. When $b \to MRL(\underline{v})$, $v^*(b) \to \underline{v}$. Hence, (7) converges to $-MRL(\underline{v}) f(\underline{v}) / (1 - MRL(\underline{v}) f(\underline{v})) < 0$. Hence, (7) is negative for a sufficiently high b, so the expected revenues are decreasing in b around $b = MRL(\underline{v})$.

Finally, consider the third statement. Notice that for any $v < \bar{v}$, $\lim_{N\to\infty} H(v) = 0$. Indeed, by definition of $H(\cdot)$, $H(v) = NF(v)^{N-1} - (N-1)F(v)^N$. Therefore,

$$\lim_{N\to\infty} H(v) = \lim_{N\to\infty}\left((N-1)F(v)^N\right) \times \left(\lim_{N\to\infty}\frac{NF(v)^{N-1}}{(N-1)F(v)^N} - 1\right)$$
$$= \frac{\lim_{N\to\infty} F(v)^N}{-\ln F(v)} \times \left(\frac{1}{F(v)} - 1\right) = 0$$

for any $v < \bar{v}$, where we used l'Hospital's rule. Also, notice that for any $b > 0$, the cut-off type $v^*(b)$ does not depend on N. Therefore, for any $b > 0$, there exists $N(b)$ such that $H(v^*(b)) - \frac{f(v^*(b))}{1-F(v^*(b))} b < 0$ for all $N > N(b)$. Therefore, for any $b > 0$, (7) is negative for any $N > N(b)$.

□

Proof of Proposition 3. We only overview the key steps of the proof leaving the details for Online Appendix A. On the first step, we derive the expected payoffs of the bidder and her advisor for any incentive-compatible contract. On the second step, we show that any incentive-compatible contract $\theta(v)$ must be continuous and consisting of flat regions and regions $\theta(v) = v + b$. Next, we show the optimality of capped delegation. Finally, we show that the optimal cap is $v^* + b$. All these statements are strict and thus the optimal contract is unique in the class of direct revelation contracts, if $x(\cdot)$ is strictly increasing in the range $[\underline{v} + b, \overline{v}]$. In contrast, if $x(\cdot)$ is only weakly increasing, then the proof shows that this contract leads to a weakly higher payoff to the bidder than any other contract. □

A Online Appendix A for "Selling to Advised Buyers" Andrey Malenko, Anton Tsoy: Omitted Proofs and Additional Results

A.1 Omitted Proofs

Omitted details for Theorem 3: Proof of (5). We first show that for sufficiently large N, a strict inequality holds in (5) for all $k = 1, \dots, K-1$. Denote

$$h(x|N) \equiv Nx^{N-1} - (N-1)x^N. \tag{8}$$

Let

$$H_k(v) \equiv \frac{h\left(F(v)|N\right) - h\left(F(\omega_{k-1})|N\right)}{h\left(F(\omega_k)|N\right) - h\left(F(\omega_{k-1})|N\right)}$$

be the c.d.f. of the distribution of the second-highest order statistic $v_{(2)}$ conditional on $v_{(2)} \in [\omega_{k-1}, \omega_k)$, and let

$$H_*(v) \equiv \frac{h\left(F(v)|N\right) - h\left(F(\omega_{K-1})|N\right)}{h\left(F(v^*)|N\right) - h\left(F(\omega_{K-1})|N\right)}$$

be the c.d.f. of the distribution of $v_{(2)}$ conditional on $v_{(2)} \in [\omega_{K-1}, v^*)$. We start with two auxiliary claims:

Claim 1. Function $h(x|N)$ is strictly convex on $[0, 1 - \frac{1}{N-1})$. Further, for any $\tilde{x} \in [0,1)$ and $\varepsilon > 0$, it holds for all N sufficiently large that $h(\tilde{x}|N) \leq h(x|N) + \varepsilon(\tilde{x} - x)$ for all $x \in [0, \tilde{x}]$.

Proof. The first statement follows from

$$h''(x|N) = N(N-1)x^{N-3}((N-2) - (N-1)x)$$

is greater than 0 whenever $x \in [0, 1 - \frac{1}{N-1})$. To prove the second statement, note that since

$$h'(x|N) = N(N-1)(x^{N-2} - x^{N-1}) \leq N(N-1)x^{N-2} \text{ for } x \in [0,1],$$

and $N\left(N-1\right)x^{N-2}$ decreases in N to zero for sufficiently large N, for all N sufficiently large it holds that $h'(x|N) \leq \varepsilon$ for all $x \in [0, \tilde{x}]$. Hence, $h(\tilde{x}|N) \leq h(x|N) + \varepsilon(\tilde{x} - x)$ for all $x \in [0, \tilde{x}]$. *q.e.d.* □

Claim 2. For all sufficiently high N, $H_k(v)$ first-order stochastically dominates $F(v|v \in [\omega_{k-1}, \omega_k))$ for all $k = 1, \dots, K-1$, and $H_*(v)$ first-order stochastically dominates $F(v|v \in [\omega_{K-1}, v^*))$.

Proof. Fix $k = 1, \dots, K-1$. It is sufficient to show that for all $v \in (\omega_{k-1}, \omega_k)$, $H_k(v) < F(v|v \in [\omega_{k-1}, \omega_k))$, or equivalently,

$$h(F(v)|N) < \frac{F(v) - F(\omega_{k-1})}{F(\omega_k) - F(\omega_{k-1})} h(F(\omega_k)|N) + \frac{F(\omega_k) - F(v)}{F(\omega_k) - F(\omega_{k-1})} h(F(\omega_{k-1})|N).$$

This inequality indeed holds for sufficiently large N, because $h(\cdot|N)$ is strictly convex on $[0, F^*(v))$ for sufficiently large N by Claim 1. The proof of the fact that $H_*(v)$ first-order stochastically dominates $F(v|v \in [\omega_{K-1}, v^*))$ is analogous. *q.e.d.* □

We proved earlier that $\omega_{K-1} < v^*$. Hence, by Claim 2, for all sufficiently large N it holds that

$$\begin{aligned}
\mathbb{E}[\min\{v_{(2)}, v^*\} + b|v_{(2)} \in [\omega_{k-1}, \omega_k)] &= \mathbb{E}[v_{(2)} + b|v_{(2)} \in [\omega_{k-1}, \omega_k)] \\
&> \mathbb{E}[v_{(2)}|v_{(2)} \in [\omega_{k-1}, \omega_k)] \\
&\geq \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] = m_k.
\end{aligned}$$

for all $k = 1, \dots, K-1$. Thus, we have strict inequality in (5) for all $k = 1, \dots, K-1$. We are left to show that $\mathbb{E}[\min\{v_{(2)}, v^*\} + b|v_{(2)} \geq \omega_{K-1}] \geq m_K$. We start with the following auxiliary claim.

Claim 3. For sufficiently large N, $\mathbb{P}(v \geq v^*|v \geq \omega_{K-1}) < \mathbb{P}\left(v_{(2)} \geq v^*|v_{(2)} \geq \omega_{K-1}\right)$.

Proof. Note that

$$\frac{1 - h(F(\omega_{K-1})|N)}{1 - F(\omega_{K-1})} \geq 1 - h(F(v^*)|N),$$

with $\lim_{N\to\infty} h(F(v^*)|N) = 0$, and so $(1 - h(F(\omega_{K-1})|N)) / (1 - F(\omega_{K-1})) > \frac{1}{2}$ for sufficiently high N. Applying Claim 1 with $\tilde{x} = F(v^*)$, $x = F(\omega_{K-1})$, and $\varepsilon = \frac{1}{2}$, we obtain that for sufficiently high N

$$\begin{aligned}
h(F(v^*)|N) &< h(F(\omega_{K-1})|N) + (F(v^*) - F(\omega_{K-1})) \frac{1}{2} \\
&< h(F(\omega_{K-1})|N) + (F(v^*) - F(\omega_{K-1})) \frac{1 - h(F(\omega_{K-1})|N)}{1 - F(\omega_{K-1})}.
\end{aligned}$$

The last inequality implies that

$$\frac{1 - F(v^*)}{1 - F(\omega_{K-1})} < \frac{1 - h(F(v^*)|N)}{1 - h(F(\omega_{K-1})|N)},$$

which proves the desired inequality. *q.e.d.* □

Combining Claims 2 and 3 implies that for sufficiently high N:

$$\begin{aligned}
&\mathbb{E}\left[\min\{v_{(2)}, v^*\} + b | v_{(2)} \geq \omega_{K-1}\right] \\
=&\mathbb{E}\left[v_{(2)} + b | v_{(2)} \in [\omega_{K-1}, v^*)\right] \mathbb{P}\left(v_{(2)} \in [\omega_{K-1}, v^*) | v_{(2)} \geq \omega_{K-1}\right) + (v^* + b)\mathbb{P}\left(v_{(2)} \geq v^* | v_{(2)} \geq \omega_{K-1}\right) \\
\geq&\mathbb{E}\left[v | v \in [\omega_{K-1}, v^*)\right] \mathbb{P}\left(v_{(2)} \in [\omega_{K-1}, v^*) | v_{(2)} \geq \omega_{K-1}\right) + \mathbb{E}\left[v | v \geq v^*\right] \mathbb{P}\left(v_{(2)} \geq v^* | v_{(2)} \geq \omega_{K-1}\right) \\
\geq&\mathbb{E}\left[v | v \in [\omega_{K-1}, v^*)\right] \mathbb{P}\left(v \in [\omega_{K-1}, v^*) | v \geq \omega_{K-1}\right) + \mathbb{E}\left[v | v \geq v^*\right] \mathbb{P}\left(v \geq v^* | v \geq \omega_{K-1}\right) = \mathbb{E}\left[v | v \geq \omega_{K-1}\right]
\end{aligned}$$

The first inequality follows from Claim 2 and $b > 0$; the second inequality follows from Claim 3. Since expected revenues in the second-price auction cannot exceed $\mathbb{E}\left[v | v \geq \omega_{K-1}\right]$, the capped delegation equilibrium in the English auction yields higher expected revenues than any equilibrium of the second-price auction if N is sufficiently high.

Details of the Proof of Proposition 3. In this proof we assume that $y(\theta)$ is strictly increasing in the range $[\underline{v} + b, \overline{v}]$ and show that contract $\theta_i(\omega_i) = b + \min\{\omega_i, v^*\}$ strictly dominates any other contract $\theta_i(\cdot)$. In contrast, if $y(\theta)$ is only weakly increasing in the range $[\underline{v} + b, \overline{v}]$, then the arguments in this proof show that bidder i does at least as well with contract $\theta_i(\omega_i)$ as with any other contract $\theta_i(\cdot)$, but need not do strictly better. Since $y(\cdot)$ may have mass points, we need to account for the possibility of winning at ties. Let $y(\theta_-)$ and $y(\theta_+)$ denote the left and right limits of $y(\cdot)$ at θ, respectively, and let ρ_θ denote the probability of winning conditional on bidding θ and tying. Let $x(\theta) \equiv y(\theta_-) + (y(\theta_+) - y(\theta_-))\rho_\theta$ denote the probability of winning if bidder i submits bid θ. Let $t(\theta) \equiv \int_0^{\theta-} c\,dy(c) + (y(\theta_+) - y(\theta_-))\rho_\theta \theta$ denote the expected transfer of bidder i to the seller in this case. Note that we can re-write it as $t(\theta) = \int_0^{\theta} c\,dx(c)$. Then, bidder i's problem is to maximize $\mathbb{E}[x(\theta(v))v - t(\theta(v))]$ over all contracts $\theta(v)$ that satisfy the incentive compatibility condition for the advisor:

$$x(\theta(v))(v + b) - t(\theta(v)) \geq x(\theta(w))(v + b) - t(\theta(w)), \ \forall v, w \in [\underline{v}, \overline{v}].$$

We prove the proposition in three steps.

Step 1: Deriving Expected Payoffs of the Bidder and the Advisor Let $u(\theta, v) \equiv x(\theta)v - t(\theta)$ denote the expected utility of a bidder with valuation v who bids θ in the second-price auction (follows the strategy of bidding up to price θ in the English auction). Then, $u(\theta, v + b)$ is the expected payoff of the advisor of type v if he induces the bidder to bid θ.

The next lemma shows that the advisor's expected payoff is inverted U-shaped in bid θ with the maximum at $\theta = v + b$:

Lemma 1. *$u(\theta, v + b)$ is weakly increasing in θ for $\theta < v + b$ (strictly for $\theta \in [\underline{v} + b, v + b)$), and weakly decreasing in θ for $\theta > v + b$ (strictly for $\theta \in (v + b, \overline{v}]$). For any $v \in (\underline{v}, \overline{v} - b)$, $u(\theta, v + b)$*

reaches a unique maximum at $\theta = v + b$.

Proof. Using the expression for $t(\theta)$, we have $u(\theta, v+b) = x(\theta)(v+b) - \int_0^\theta c dx(c)$. Thus, for any θ_2 and $\theta_1 > \theta_2$,

$$u(\theta_1, v+b) - u(\theta_2, v+b) = (x(\theta_1) - x(\theta_2))(v + b - \mathbb{E}[c | c \in [\theta_2, \theta_1]]),$$

where c is distributed according to $x(\cdot)$. Since $x(\theta)$ is weakly increasing, it follows that $u(\theta, v+b)$ is weakly increasing in the range $\theta < v+b$ and weakly decreasing in the range $\theta > v+b$. Both relationships are strict wherever $x(\theta)$ is strictly increasing, i.e., for all $\theta \in [\underline{v} + b, \overline{v}]$. Thus, for any $v \in (\underline{v}, \overline{v} + b)$, $u(\theta, v+b)$ reaches a unique maximum in θ at $\theta = v+b$. $\square$

Since contract $\theta(v)$ is incentive compatible for the advisor, $u(\theta(v), v+b) = \max_{\hat{v}} x(\theta(\hat{v}))(v+b) - t(\theta(\hat{v}))$ and by the generalized envelope theorem (Milgrom and Segal (2002)) the payoff of the advisor of type v can be written as

$$u(\theta(v), v+b) = \int_{\underline{v}}^{v} x(\theta(u)) du + u(\theta(\underline{v}), \underline{v} + b), \tag{9}$$

where the expected payoff of the lowest type of the advisor is $u(\theta(\underline{v}), \underline{v}+b) = x(\theta(\underline{v}))(\underline{v}+b) - t(\theta(\underline{v}))$.

Conditional on the advisor's type v, the bidder's expected payoff equals $u(\theta(v), v) = u(\theta(v), v+b) - bx(\theta(v))$. Using (9) and integrating over the distribution of v by parts, we obtain the bidder's expected payoff from contract $\theta(v)$:

$$\begin{aligned} & \int_{\underline{v}}^{\overline{v}} (u(\theta(v), v+b) - bx(\theta(v))) dF(v) \\ = & \int_{\underline{v}}^{\overline{v}} (1 - F(v) - bf(v)) x(\theta(v)) dv + u(\theta(\underline{v}), \underline{v}+b). \end{aligned} \tag{10}$$

Step 2: Shape of Incentive Compatible Contracts For any $\tilde{v} \in [\underline{v}, \overline{v}]$, define $\theta^+(\tilde{v}) \equiv \lim_{v \to \tilde{v}+} \theta(v)$ and $\theta^-(\tilde{v}) \equiv \lim_{v \to \tilde{v}-} \theta(v)$. Since the bidder's valuation cannot exceed $\overline{v}$, the optimal contract cannot have $\theta(v) > \overline{v}$, so we restrict attention to contracts with $\theta(v) \leq \overline{v}$. We next show that any incentive-compatible contract must have a rather specific form. This result is a counter-part of Proposition 1 in Melumad and Shibano (1991).

Lemma 2 (**Melumad and Shibano, 1991**). *An incentive-compatible $\theta(v)$ must satisfy the following:*

1. *$\theta(\cdot)$ is weakly increasing;*

2. *If $\theta(\cdot)$ is strictly increasing and continuous on an open interval (v_1, v_2), then $\theta(v) = v + b$ on it;*

3. If $\theta(\cdot)$ is discontinuous at $\tilde{v}$, the discontinuity must be a jump discontinuity that satisfies

$$x\left(\theta^{-}(\tilde{v})\right)(\tilde{v}+b)-t\left(\theta^{-}(\tilde{v})\right)=x\left(\theta^{+}(\tilde{v})\right)(\tilde{v}+b)-t\left(\theta^{+}(\tilde{v})\right); \tag{11}$$

$$\begin{aligned}\theta(v)&=\theta^{-}(\tilde{v}),\forall v\in[\theta^{-}(\tilde{v})-b,\tilde{v});\\ \theta(v)&=\theta^{+}(\tilde{v}),\forall v\in(\tilde{v},\theta^{+}(\tilde{v})-b].\end{aligned} \tag{12}$$

Proof. The proof follows the argument in Melumad and Shibano (1991) with the adjustment to the fact that we do not assume differentiability or strict concavity of the advisor's utility function. Weak monotonicity of $\theta(\cdot)$ follows by the same argument as the weak monotonicity of $m(\cdot)$ in the proof of Part 2 of Proposition 1. Consider the second statement. By contradiction, suppose that $\theta(v)<v+b$ for some $v\in(v_1,v_2)$ (case $\theta(v)>v+b$ is analogous). Since $\theta(\cdot)$ is continuous and strictly increasing, there exists $\varepsilon>0$: $\theta(v)<\theta(v+\varepsilon)<v+b$. Then, by Lemma 1, $u(\theta(v+\varepsilon),v+b)>u(\theta(v),v+b)$, i.e., type v is better off deviating to reporting $v+\varepsilon$ rather than ε, which is a contradiction. Finally, consider the last statement. Equality (11) follows from the incentive compatibility. We next show the first line in (12). The proof of the second line is symmetric. Consider $v\in[\theta^{-}(\tilde{v})-b,\tilde{v})$. By construction, $\theta^{-}(\tilde{v})\leq v+b$. By the first statement of the lemma, $\theta(v)\leq\theta^{-}(\tilde{v})$. If $\theta(v)<\theta^{-}(\tilde{v})\leq v+b$, then by Lemma 1 $u(\theta(\tilde{v}),v+b)>u(\theta(v),v+b)$, implying that type v would be better off reporting $\tilde{v}$ than v, which contradicts incentive compatibility of $\theta(v)$. Thus, $\theta(v)=\theta^{-}(\tilde{v})$. □

Step 3: Optimality of the Capped Delegation We first rule out discontinuous jumps in the optimal $\theta(\cdot)$. By contradiction, suppose that there is a discontinuous jump in $\theta(\cdot)$ at some $\tilde{v}\in(\underline{v},\overline{v})$. Consider new contract $\tilde{\theta}(v)$, identical to $\theta(v)$ for all types but $v\in[\theta^{-}(\tilde{v})-b,\theta^{+}(\tilde{v})-b]$, where $\tilde{\theta}(v)=v+b$. Let $v^{j}\equiv\theta^{j}(\tilde{v})-b$ and $x^{j}\equiv x\left(\theta^{j}(\tilde{v})\right)$ for $j\in\{+,-\}$. Since contract $\theta(\cdot)$ is incentive-compatible, the advisor with type $\tilde{v}$ is indifferent between inducing bids $\theta^{-}(\tilde{v})$ and $\theta^{+}(\tilde{v})$. Using $t(\theta)=\int_0^{\theta}c\,dx(c)$, (11) implies

$$\int_{v^{-}}^{\tilde{v}}\left(x(v+b)-x^{-}\right)dv=\int_{\tilde{v}}^{v^{+}}\left(x^{+}-x(v+b)\right)dv. \tag{13}$$

Subtracting the bidder's expected payoff (10) under contract $\theta(v)$ from the expected payoff under contract $\tilde{\theta}(v)$:

$$\int_{v^{-}}^{\tilde{v}}(1-F(v)-bf(v))\left(x(v+b)-x^{-}\right)dv-\int_{\tilde{v}}^{v^{+}}(1-F(v)-bf(v))\left(x^{+}-x(v+b)\right)dv>0,$$

from (13) and the fact that $F(v)+bf(v)$ is strictly increasing. Therefore, contract $\theta(\cdot)$ is dominated by contract $\tilde{\theta}(\cdot)$, which contradicts optimality of $\theta(\cdot)$.

By Lemma 2 and continuity of the optimal contract, it is sufficient to consider contracts $\theta(\cdot)$ that are comprised of, at most, one upward sloping part $\theta(v)=v+b$ for $v\in[v_1,v_2]$ and two flat parts, $\theta(v) = v_1 + b$ for $v \in [\underline{v}, v_1]$ and $\theta(v) = v_2 + b$ for $v \in [v_2, \overline{v}]$. Any other contract satisfying Lemma 2 exhibits a discontinuous jump in $\theta(v)$ at some v, which is inconsistent with the optimality of $\theta(\cdot)$ by the previous paragraph. For any pair of $v_1 \geq \underline{v}$ and $v_2 \in [v_1, \overline{v}]$, letting $x_1 \equiv x(v_1 + b)$ and $x_2 \equiv x(v_2 + b)$ and using (10), the expected payoff of the bidder can be written as:

$$\begin{aligned} & x_1 \int_{\underline{v}}^{v_1} (1 - F(v) - bf(v))\, dv + \int_{v_1}^{v_2} (1 - F(v) - bf(v))\, x(v+b)\, dv \\ & + x_2 \int_{v_2}^{\overline{v}} (1 - F(v) - bf(v))\, dv + x_1 (\underline{v} + b) - \int_0^{v_1+b} v dx(v) \end{aligned} \tag{14}$$

Integrating by parts, $\int_0^{v_1+b} v dx(v) = x_1(v_1+b) - \int_0^{v_1+b} x(v)\, dv$. Taking the difference of (14) for $v_1' \in (v_1, v_2)$ and v_1 and simplifying the expression, we obtain:

$$-\left(x_1' - x_1\right) \int_{\underline{v}}^{v_1} (F(v) + bf(v))\, dv - \int_{v_1}^{v_1'} \left(x_1' - x(v+b)\right)(F(v) + bf(v))\, dv,$$

where we denote $x_1' \equiv x(v_1' + b)$. This expression is strictly negative, because $x(\cdot)$ is strictly increasing on $[\underline{v} + b, \overline{v}]$. herefore, for any $v_1 > \underline{v}$, contract $\theta(v)$ is dominated by contract $\tilde{\theta}(v)$ that coincides with $\theta(v)$ but lowers v_1 by an infinitesimal amount. Hence, the optimal contract has no flat region at the bottom: $v_1 = \underline{v}$.

Step 4: Optimal Pooling Region at the Top The previous steps imply that the optimal contract $\theta(v)$ take form of capped delegation: $\theta(v) = b + \min\{v, v_2\}$ for some v_2. We next show that $v_2 = v^*$. Taking the difference of the bidder's expected payoff (14) for $v_2 + \varepsilon$ and v_2, we obtain:

$$\int_{v_2}^{v_2+\varepsilon} (1 - F(v) - bf(v))(x(v+b) - x_2)\, dv + \left(x_2' - x_2\right) \int_{v_2+\varepsilon}^{\overline{v}} (1 - F(v) - bf(v))\, dv,$$

where we denote $x_2' \equiv x(v_2 + b + \varepsilon)$. Taking the limit as $\varepsilon \to 0$ and using the fact that $x(\cdot)$ is strictly increasing, we obtain that the difference has the same sign as $\int_{v_2}^{\overline{v}} (1 - F(v) - bf(v))\, dv$. Since $F(v) + bf(v)$ is strictly increasing in v, the bidder's expected payoff (14) is inverted U-shaped in v_2 with the maximum at $v_2 = \hat{v}_2$, which is the unique solution to $\int_{\hat{v}_2}^{\overline{v}} (1 - F(v) - bf(v))\, dv = 0$. Note that this equation can be re-written as $MRL(\hat{v}_2) = b$, and thus $\hat{v}_2 = v^*$. Therefore, the unique optimal cap is $v^* + b$.

A.2 Additional Results for Subsection 4.1

Here, we derive several additional results. First, Proposition 4 provides explicit formulas for the cut-off advisor types in any equilibrium communication partition.

Proposition 4. *In any standard auction with continuous payments, in any equilibrium with K partition intervals, thresholds $(\omega_k)_{k=0}^K$ satisfy $\omega_0 = \underline{v}$, $\omega_K = \overline{v}$, and*

$$G(\omega_{k-1}, \omega_k)(1 - \Lambda_k)(\omega_k + b - m_k) = -G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}), \tag{15}$$

where

$$m_k \equiv \mathbb{E}\left[v | v \in [\omega_{k-1}, \omega_k)\right], \tag{16}$$

and

$$\Lambda_k \equiv \frac{1}{G(\omega_{k-1}, \omega_k)} \left(\sum_{n=1}^{N-1} \binom{N-1}{n} \frac{F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n}}{n+1} \right) \tag{17}$$

is the probability of winning conditional on a tie at bid m_k. Further if $(\omega_k)_{k=0}^K$ satisfy $\omega_0 = \underline{v}$, $\omega_K = \overline{v}$, and (15)*, then they are part of equilibrium communication partition.*

Proof. By Proposition 1 it is without loss to restrict attention to communication in equilibria of the second-price auction. By the interval partition form of communication, we need to determine incentives of threshold types of the advisor ω_k. Consider any such type ω_k. In the second-price auction, a message is simply an expected value of the bidder m_k given by (16).

As an auxiliary step, let us derive the probability that a bidder with bid $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$ wins a tie, conditional on the tie taking place at bid $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$. Without loss of generality, we refer to this bidder as bidder N and to her rivals as bidders $i \in \{1, 2, ..., N-1\}$. Denote such a probability by Λ_k. Since ties are broken randomly,

$$\Lambda_k = \mathbb{E}\left[\frac{1}{\tilde{n}_k + 1} | \hat{v} \in [\omega_{k-1}, \omega_k)\right],$$

where $\tilde{n}_k = \sum_{i=1}^{N-1} 1\{v_i \in [\omega_{k-1}, \omega_k)\}$ is a random variable, denoting the number of rival bidders with the same bid m_k. Re-writing,

$$\begin{aligned} \Lambda_k &= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\Pr\left[\tilde{n}_k = n, \hat{v} \in [\omega_{k-1}, \omega_k)\right]}{G(\omega_{k-1}, \omega_k)} \\ &= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\Pr\left[\tilde{n}_k = n, \sum_{i=1}^{N-1} 1\{v_i < \omega_{k-1}\} = N-1-n\right]}{G(\omega_{k-1}, \omega_k)} \\ &= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n}}{G(\omega_{k-1}, \omega_k)}, \end{aligned}$$

which coincides with expression (17).

Now, we can show (15). Let $\hat{m}$ be the message of the highest bidder among $N-1$ opponents of the bidder. From submitting a message m_k, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)\Lambda_k(\omega_k + b - m_k),$$

where the expected utility from bidding m_k when the other bidders submit bids below m_k and when some bidders tie with the bidder is captured by the first and second terms, respectively. Analogously, from submitting a message m_{k+1}, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) + G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}).$$

Type ω_k should be indifferent between the two which gives the eq. (15). Thus, any PBE communication strategy can be described by a solution $(\omega_k)_{k=0}^K$ to recursion (15) where $\omega_0 = \underline{v}$ and $\omega_K = \overline{v}$.

The last statement of the proposition is immediate from the specification of m_k in (16) and the fact that cut-off types ω_k are indifferent between the two adjacent equilibrium bids m_k and m_{k+1}, and the single-crossing property of advisor's payoffs. □

The next proposition verifies that there indeed exist equilibria that satisfy NITS.

Proposition 5. *In any static auction, there is an equilibrium that satisfies the NITS condition.*

We first show the following auxiliary lemma:

Lemma 3. *If $\omega_{k+1} = \omega_k$, then $k = 0$. Furthermore, there exists $\varepsilon > 0$ such that $\omega_{k+1} - \omega_k > \varepsilon$ for $k = 1, \ldots, K$.*

Proof. To prove the first statement, suppose by contradiction that $\omega_{k+1} = \omega_k$ for some $0 < k \leq K$. This implies that $G(\omega_k, \omega_{k+1}) = 0$ and so, from (15), $G(\omega_{k-1}, \omega_k)(1 - \Lambda_{k-1})(\omega_k + b - m_k) = 0$ which in turn implies that $\omega_k + b = m_k$ or $\omega_{k-1} = \omega_k$. If $\omega_{k-1} < \omega_k$, then $m_k < \omega_k < \omega_k + b$ which is a contradiction. If $\omega_{k-1} = \omega_k$, then choose j so that $\omega_{k-j-1} < \omega_{k-j} = \cdots = \omega_{k-1} = \omega_k = \omega_{k+1}$ and the argument proceeds as in the case $\omega_{k-1} < \omega_k$.

To prove the second statement, suppose $\omega_{k+1} - \omega_k > 0$ for some k. By the first statement of the lemma, $k > 0$. Since $\omega_k + b - m_k > \omega_k + b - m_{k+1}$, it follows from (15) and $\omega_{k+1} - \omega_k > 0$ that

$$\omega_k + b \leq \mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]. \tag{18}$$

If to contradiction for any $\varepsilon > 0$, there were an equilibrium such that $\omega_{k+1} - \omega_k < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \leq \omega_k + \varepsilon < \omega_k + b$ which would contradict (18) for $\varepsilon < b$. Thus, there is $\varepsilon > 0$ such that $\omega_{k+1} - \omega_k > \varepsilon$ for $0 < k \leq K$.

□

We can now proceed to the proof of Proposition 5.

Proof of Proposition 5. We consider separately cases $\overline{v} < \infty$ and $\overline{v} = \infty$.

Case 1: $\overline{v} < \infty$

Lemma 3 implies that there is an upper bound $\bar{K}$ on the number of intervals in the communication strategy. To prove there is an equilibrium that satisfies NITS, we adapt the proof of Proposition 1 from Chen et al. (2008) for our problem. It is useful to introduce the following notations:

$$\begin{aligned}\Psi(\omega_{k-1},\omega_k) = \quad & G(\omega_{k-1},\omega_k)(1-\Lambda_k) = & \sum_{n=1}^{N-1}\binom{N-1}{n}F(\omega_{k-1},\omega_k)^n F(\omega_{k-1})^{N-1-n}\frac{n}{n+1},\\ \Phi(\omega_k,\omega_{k+1}) = \quad & G(\omega_k,\omega_{k+1})\Lambda_{k+1} = & \sum_{n=1}^{N-1}\binom{N-1}{n}F(\omega_k,\omega_{k+1})^n F(\omega_k)^{N-1-n}\frac{1}{n+1}.\end{aligned}$$

Denote $m(\omega_{k-1},\omega_k) = \mathbb{E}[v|v \in (\omega_{k-1},\omega_k)]$ and

$$H(\omega_{k-1},\omega_k,\omega_{k+1}) \equiv \Psi(\omega_{k-1},\omega_k)(\omega_k + b - m(\omega_{k-1},\omega_k)) + \Phi(\omega_k,\omega_{k+1})(\omega_k + b - m(\omega_k,\omega_{k+1})). \quad (19)$$

Note that an equilibrium with K intervals is given by recursion $H(\omega_{k-1},\omega_k,\omega_{k+1}) = 0$, with $\omega_0 = \underline{v}$ and $\omega_K = \overline{v}$. To prove the statement, we will show that if an equilibrium with K intervals $\omega = (\omega_0,\omega_1,...,\omega_K)$, violates the NITS condition, then for all $k = 1,...,K$, there exists a solution to recursion $H(\omega_{k-1},\omega_k,\omega_{k+1}) = 0$, with $k+1$ intervals, ω^k, that satisfies $\omega_0^k = \underline{v}$, $\omega_k^k > \omega_{k-1}$, and $\omega_{k+1}^k = \omega_k$. After this result is established, the statement of the proposition follows from the following argument. By contradiction, suppose that the most informative equilibrium (i.e., one with $\bar{K}$ intervals) violates the NITS condition. Applying the result above for $k = \bar{K}$, there must be a solution to $H(\omega_{k-1},\omega_k,\omega_{k+1}) = 0$ with $\bar{K}+1$ intervals satisfying boundary conditions $\omega_0^{\bar{K}} = \underline{v}$ and $\omega_{\bar{K}+1}^{\bar{K}} = \omega_{\bar{K}} = \bar{v}$. By Propositions 1 and 4, this is an equilibrium, which contradicts the statement that $\bar{K}$ is the highest number of equilibrium intervals.

We show the result by induction on k. As an induction base, consider $k = 1$. If the equilibrium with K intervals $\omega = (\omega_0,\omega_1,...,\omega_K)$ violates the NITS, it must be that $\underline{v} + b < m(\underline{v},\omega_1)$, and hence, $H(\underline{v},\underline{v},\omega_1) < 0$. At the same time, $H(\underline{v},\omega_1,\omega_1) > 0$, since $\omega_1 > m(\underline{v},\omega_1)$ and $b > 0$. By continuity, there exists $x \in (\underline{v},\omega_1)$ at which $H(\underline{v},x,\omega_1) = 0$. Hence, the claim holds for $k = 1$: $\omega^1 = \left(\omega_0^1,\omega_1^1,\omega_2^1\right) = (\underline{v},x,\omega_1)$ solves $H(\omega_{k-1},\omega_k,\omega_{k+1}) = 0$ with $\omega_0^1 = \underline{v}$, $\omega_1^1 > \omega_0 = \underline{v}$, and $\omega_2^1 = \omega_1$. Consider the difference $H\left(\omega_k^k,\omega_k,\omega_{k+1}\right) - H(\omega_{k-1},\omega_k,\omega_{k+1})$:

$$\begin{aligned}&\Psi\left(\omega_k^k,\omega_k\right)\left(\omega_k + b - m\left(\omega_k^k,\omega_k\right)\right) - \Psi(\omega_{k-1},\omega_k)(\omega_k + b - m(\omega_{k-1},\omega_k))\\ =&F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F\left(\omega_k^k,\omega_k\right)}{F(\omega_k)}\right)^n\left(\frac{F\left(\omega_k^k\right)}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)\left(\omega_k + b - m\left(\omega_k^k,\omega_k\right)\right)\\ &- F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F(\omega_{k-1},\omega_k)}{F(\omega_k)}\right)^n\left(\frac{F(\omega_{k-1})}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)(\omega_k + b - m(\omega_{k-1},\omega_k)).\end{aligned}$$

Since $\omega_k^k > \omega_{k-1}$, we have two implications. First, $m\left(\omega_k^k, \omega_k\right) > m\left(\omega_{k-1}, \omega_k\right)$, implying $\omega_k + b - m\left(\omega_k^k, \omega_k\right) < \omega_k + b - m\left(\omega_{k-1}, \omega_k\right)$. Second, binomial distribution with success probability $\frac{F(\omega_{k-1},\omega_k)}{F(\omega_k)}$ dominates binomial distribution with success probability $\frac{F\left(\omega_k^k,\omega_k\right)}{F(\omega_k)}$ in the sense of first-order stochastic dominance, implying $\Psi\left(\omega_k^k, \omega_k\right) < \Psi\left(\omega_{k-1}, \omega_k\right)$. Therefore, $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) - H\left(\omega_{k-1}, \omega_k, \omega_{k+1}\right) < 0$. Since $H\left(\omega_{k-1}, \omega_k, \omega_{k+1}\right) = 0$, we conclude that $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) < 0$.

On the other hand, since $\omega_k > m\left(\omega_k^k, \omega_k\right)$ and $b > 0$, we have $\omega_k + b > m\left(\omega_k^k, \omega_k\right)$, implying $H\left(\omega_k^k, \omega_k, \omega_k\right) > 0$. This, $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) < 0$, and continuity imply that there exists $x \in (\omega_k, \omega_{k+1})$ at which $H\left(\omega_k^k, \omega_k, x\right) = 0$. Since $\omega_{k+1}^k = \omega_k$, the same x satisfies $H\left(\omega_k^k, \omega_{k+1}^k, x\right) = 0$. That is, there exists a solution in which the $(k+1)$st intervals ends at $\omega_{k+1}^k = \omega_k$ and the $(k+2)$nd interval ends at $x < \omega_{k+1}$. By continuity, there exists a solution to the recursion in which the $(k+2)$nd interval ends at any $\omega \in \left(\omega_{k+1}^k, \omega_{k+1}\right)$. By continuity, for one such ω, denoted ω_{k+1}^{k+1}, the $(k+2)$nd interval ends exactly at ω_{k+1}, i.e., $\omega_{k+2}^{k+1} = \omega_{k+1}$. Hence, there exists a solution to recursion $H\left(\omega_{k-1}, \omega_k, \omega_{k+1}\right) = 0$, with $k+2$ intervals, ω^{k+1}, that satisfies $\omega_0^{k+1} = \underline{v}$, $\omega_{k+1}^{k+1} > \omega_k$, and $\omega_{k+2}^{k+1} = \omega_{k+1}$. This completes the proof of the inductive step.

Case 2: $\overline{v} = \infty$

Consider a sequence of $\overline{v}_j = j, j = 1, 2, \ldots$. By case 1, there exists an equilibrium partition $(\omega_k(j))_{k=0}^{K_j}$ in the game with distribution of types truncated from above at $\overline{v}_j$ that satisfies the NITS condition, i.e.,

$$\underline{v} + b \geq \mathbb{E}[v|v \leq \omega_1(j)]. \tag{20}$$

Let us construct partition $(\omega_k^*)_{k=0}^K$ as follows. Consider a sequence $(\omega_1(j))_{j=1,2,\ldots}$. If there is a divergent subsequence in $(\omega_1(j))_{j=1,2,\ldots}$, then set $K = 1$ and $\omega_1^* = \infty$. If there is no divergent subsequence in $(\omega_1(j))_{j=1,2,\ldots}$, then consider an arbitrary convergent subsequence $(\omega_1(j(i)))_{i=1,2,\ldots}$ of $(\omega_1(j))_{j=1,2,\ldots}$ and set $\omega_1^* = \lim_{i\to\infty} \omega_1\left(j(i)\right)$. We next move to $k = 2$. Again, if there is a divergent subsequence in $(\omega_1(j(i)))_{i=1,2,\ldots}$, then set $K = 2$ and $\omega_2^* = \infty$. Otherwise, consider a convergent subsequence of $(\omega_1(j(i)))_{i=1,2,\ldots}$ and set ω_2^* to its limits. We proceed this way to construct $(\omega_k^*)_{k=0}^K$.

For any k, $\omega_k(j)$ satisfies

$$H\left(\omega_{k-1}(j), \omega_k(j), \omega_{k+1}(j)|\overline{v}_j\right) = 0, \tag{21}$$

where function $H(\cdot|\overline{v}_j)$ is defined as in (19) with F replaced by $F(\cdot|v \leq \overline{v}_j)$. By taking the limit of (21) over the subsequence used to obtain ω_{k+1}^*, we get that each ω_k^* satisfies (19) or equivalently (15). By Proposition 4, $(\omega_k^*)_{k=0}^K$ are part of the equilibrium communication partition.

Finally, by construction, $\omega_1^* = \lim_{i\to\infty} \omega_1\left(j(i)\right)$ and it holds (20). By taking the limit (20) as $i \to \infty$ we get that $\underline{v} + b \geq \mathbb{E}[v|v \leq \omega_1^*]$. Thus, the constructed partition satisfies NITS, which completes the proof. $\square$

A.3 Additional Result for Section 4.2

In the somewhat unnatural case when neither Assumption A nor B are satisfied, the English auction can have multiple capped delegation equilibria. However, for the case of two bidders, we can generalize our comparison results:

Proposition 6. *Suppose that $b > 0$ and $N = 2$. Then, the pooling region (if it is not empty) in any capped delegation equilibrium in the English auction is finer than the top interval in any equilibrium of the second-price auction: $v^* \geq \omega_{K-1}$. If, in addition, $\varphi(\cdot)$ is strictly increasing, then any capped delegation equilibrium in the English auction brings higher expected revenues than any equilibrium satisfying NITS in the second-price auction. Both comparisons are strict if $v^* > \underline{v}$.*

Proof. We prove the first statement by contradiction. Suppose there exists a capped delegation equilibrium in the English auction and an equilibrium in the second-price auction satisfying $v^* < \omega_{K-1}$. Then, in the set of intervals in equilibrium of the second-price auction, there exists an interval (ω_{k-1}, ω_k) satisfying $\omega_{k-1} \leq v^* < \omega_k$. Consider the indifference condition of type ω_k in the second-price auction. When $N = 2$, $G(\omega_k, \omega_{k+1}) = F(\omega_k, \omega_{k+1})$ and $\Lambda_k = \Lambda_{k+1} = \frac{1}{2}$. Therefore, (15) can be simplified to

$$\omega_k + b = \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})} m_k + \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})} m_{k+1} = \mathbb{E}[v | v \in (\omega_{k-1}, \omega_{k+1})]. \tag{22}$$

Since $\omega_{k+1} \leq \bar{v}$ and $\omega_{k-1} \leq v^*$, the right-hand side of (22) satisfies:

$$\mathbb{E}[v | v \in (\omega_{k-1}, \omega_{k+1})] \leq \mathbb{E}[v | v \geq \omega_{k-1}] \leq \mathbb{E}[v | v \geq v^*].$$

On the other hand, since $v^* < \omega_k$, the left-hand side of (22) satisfies $\omega_k + b > v^* + b$. Hence, $v^* + b < \mathbb{E}[v | v \geq v^*]$. On the other hand, by the argument in Theorem 3, whenever $v^* < \overline{v}$ it is necessary that $\mathbb{E}[v | v \geq v^*] = v^* + b$, which gives us the contradiction. Therefore, $v^* \geq \omega_{K-1}$. If $v^* > \underline{v}$, then $v^* > \omega_{K-1}$, since $v^* = \omega_{K-1}$ cannot be by contradiction. Indeed, in this case, equation (22) implies $v^* + b = \mathbb{E}[v | v \geq \omega_{K-2}] < \mathbb{E}[v | v \geq v^*]$, which again contradicts $\mathbb{E}[v | v \geq v^*] = v^* + b$. The second statement of the theorem follows from the same argument as Theorem 1, since it relies only on higher efficiency of an equilibrium of the English auction than an equilibrium of the second-price auction and on the NITS condition.

□

B Online Appendix B for "Selling to Advised Buyers" Andrey Malenko, Anton Tsoy: Selection of the Capped Delegation Equilibria

In this appendix, we introduce the dynamic version of NITS and show that it selects capped delegation equilibria.

B.1 Dynamic Version of NITS

We require that the NITS condition holds in every round of the game. For any history h, let $\mu(h)$ be the bidder's posterior after history h in the beginning of the current round (before the advisor sends a message in the current round). Let

$$v_w(h) \equiv \min\{v | v \in supp(\mu(h))\} \tag{23}$$

be the weakest remaining (according to the bidder's beliefs) type of the advisor after history h. Similarly to Chen et al. (2008), an equilibrium violates the dynamic version of NITS condition if after some history h, the advisor of type $v_w(h)$ is better off claiming that he is the weakest remaining type than playing his equilibrium strategy. To capture this condition, we require that any unexpected message is interpreted as a signal of the weakest type (then the advisor's sequential rationality implies that after any history, the equilibrium strategy is weakly preferred to signaling that you are the weakest type). Formally, the dynamic version of NITS that we impose is stated as follows:

Definition 3. *An equilibrium of the English auction $(m, a, \tilde{\mu})$ satisfies the NITS condition if the following holds. Consider any p-round history h, in which the advisor deviates in round p' for the first time and sends $\tilde{m} \notin \bigcup_{v \in supp(\mu(h'))} m(v, p', \mu(h'))$, where h' is a truncation at round p' of history h. Then $\mu(h)$ assigns probability one to $v_w(h')$.*[17]

Several observations are in order. First, Definition 3 states that after the first unexpected message, the bidder assigns probability one to the weakest type in the round when the deviation happened and never updates her belief after that. Second, in dynamic auctions the weakest type can (and will) change as the auction progresses. Third, the condition in Definition 3 requires that any unsent message is perceived as a signal of the weakest type which is slightly stronger than assuming that the lowest type of advisor does not want to reveal itself in equilibrium.

[17] We implicitly assume that the set of messages is rich enough so that there is always an "unused" message in any equilibrium.

B.2 Online Strategies

We first show that it is sufficient to look for equilibria of the English auction in which the advisor gives a real-time recommendation of the action ("quit" or "stay") to the bidder, both advisors' and bidders' strategies are of the threshold form, and bidders follow the recommendations of their advisors on equilibrium path. We refer to these equilibria as equilibria in online threshold strategies.

Definition 4. *An equilibrium in the English auction is* in online threshold strategies *if the strategies of each advisor and bidder satisfy:*

$$m(v,p,\mu)=\begin{cases}1, & \text{if } p\geq \hat{p}(v,\mu)\,,\\ 0, & \text{if } p<\hat{p}(v,\mu)\,,\end{cases}\quad a(p,\tilde{\mu})=\begin{cases}1, & \text{if } p\geq \bar{p}(\tilde{\mu})\,,\\ 0, & \text{if } p<\bar{p}(\tilde{\mu})\,,\end{cases}\tag{24}$$

for some $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$, where $\tilde{\mu}$ denotes the posterior belief of the bidder at price p, having observed her advisor's message in this round. Functions $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$ are such that on equilibrium path the bidder exits the auction the first time her advisor sends message $\tilde{m}=1$.

Intuitively, at any price p, the advisor sends a binary message to his bidder recommending to quit the auction immediately or stay in it, and on equilibrium path, the bidder follows the advisor's recommendation. The next lemma shows that the restriction to equilibria in online threshold strategies is without loss of generality:

Lemma 4. *For any equilibrium there is also an equilibrium in online threshold strategies that results in the same bidding behavior on equilibrium path. For any equilibrium that satisfies NITS there is an equilibrium in online threshold strategies that satisfies NITS and results in the same bidding behavior on equilibrium path.*

The first statement is that any equilibrium with a general communication strategy has an equivalent in online threshold strategies. The proof is the manifestation of the sure-thing principle (Savage (1972)), stating that if an action is optimal for a decision-maker in every state, then it must be optimal if she does not know the state. Intuitively, since the advisor's information is only relevant for determining the price level at which the bidder quits the auction, any equilibrium quitting strategy can be achieved by the advisor delaying communication as much as possible, which occurs when she sends a recommendation to quit immediately when the price hits the level at which the bidder is supposed to quit. The second statement implies that the NITS condition is not stronger for equilibria in online threshold strategies than in general: If an equilibrium with a general communication strategy satisfies NITS, then an equivalent in online threshold strategies also satisfies NITS.

Proof of Lemma 4. The proof of the first statement follows the argument of the proof of Lemma IA.2 in Grenadier et al. (2016). Specifically, for any pure-strategy PBEM we construct an equilibrium in online threshold strategies that results in the same bidding behavior on equilibrium path.

Consider any pure-strategy equilibrium E with some strategies $\bar{m}(v,p,\mu)$ and $\bar{a}(p,\tilde{\mu})$. It implies an equilibrium exit price $\bar{\tau}(v)$, which is the price at which the bidder exits the auction, if the valuation is v, provided that the bidder and her advisor play the equilibrium strategies $\bar{m}(\cdot)$ and $\bar{a}(\cdot)$. Note that $\bar{\tau}(v)$ must be weakly increasing in v. To see this, suppose by contradiction that $\bar{\tau}(v_1) > \bar{\tau}(v_2)$ for some $v_1 \in [\underline{v}, \bar{v})$ and $v_2 \in (v_1, \bar{v}]$. Since the advisor's payoff from acquiring the asset at any price p is higher for type v_2 than for type v_1 ($v_2 + b - p > v_1 + b - p$), the advisor's continuation value from not exiting the auction at any price p cannot be lower for type v_2 than for type v_1. The payoff from exiting the auction at any current price p does not depend on the type and equals zero. Thus, $\bar{\tau}(v_2) \geq \bar{\tau}(v_1)$. Let $\varrho \equiv \{p : \exists v \in [\underline{v}, \bar{v}]$ such that $\bar{\tau}(v) = p\}$ be the set of prices at which the bidder exits the auction for some realization of v. It will be convenient to define $v_l(p) \equiv \inf\{v : \bar{\tau}(v) = p\}$ and $v_h(p) \equiv \sup\{v : \bar{\tau}(v) = p\}$ for any $p \in \varrho$. We extend the definition of $v_l(p)$ for any $p \notin \varrho$ by setting $v_l(p) \equiv \inf\{v : \bar{\tau}(v) \geq p\}$.

Consider an online threshold strategy of the advisor, $m(v,p,\mu)$, with $\hat{p}(v,\mu) = \bar{\tau}(v)$ and the following belief updating rule of the bidder. For any belief μ, price p, and message $\tilde{m}$ such that there is $v \in supp(\mu(h))$ with $m(v,p,\mu) = \tilde{m}$, belief μ is updated via the Bayes rule. Any other message $\tilde{m}$ (i.e., a message for which there is no $v \in supp(\mu(h))$ with $m(v,p,\mu) = \tilde{m}$) is treated as some message $\tilde{m}'$ for which there is some $v \in supp(\mu(h))$ with $m(v,p,\mu) = \tilde{m}'$, and belief μ is updated following message $\tilde{m}$ in the same way as following message $\tilde{m}'$.[18] Given this, the posterior belief of the bidder for any history h is as follows. A sequence of messages $m = 0$ for all prices $p' \leq p$ up to price p implies that the bidder's posterior belief is given by the prior distribution of valuations truncated from below at $v_l(p)$. A sequence of messages $m = 0$ for all prices $p' < p'' \in \varrho$ and message $m = 1$ at price $p'' \in \varrho$ and any history of messages after that results in the bidder's posterior belief given by the prior distribution of valuations truncated at $v_l(p'')$ from below and at $v_h(p'')$ from above. Any history involving off-equilibrium messages leads to the posterior belief equivalent to one of these two posterior beliefs by construction of the updating rule. Given this, consider an online threshold strategy of the bidder, $a(p,\tilde{\mu})$, with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v|v \geq v_l(p)]$ for the posterior belief $\tilde{\mu}$ in the history of the first kind (i.e., when the advisor never recommended quitting at one of prices $p \in \varrho$ in the past), and with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v|v \in [v_l(p''), v_h(p'')]]$ for the posterior belief $\tilde{\mu}$ in the history of the second type (i.e., when the advisor recommended to quit the auction at price $p'' \in \varrho$). Let E' denote a combination of these online threshold strategies of the advisor and the bidder and the belief updating rule. Below we show that E' is indeed an equilibrium and that it results in the same equilibrium exit price $\bar{\tau}(v)$ as equilibrium E.

For the collection of strategies and beliefs E' to be an equilibrium, we need to verify the incentive compatibility (IC) conditions of the advisor and the bidder.

[18] Intuitively, according to this updating rule, the bidder effectively ignores unexpected messages. As a consequence, it is sufficient to consider only deviations to on-path (expected) messages. Because no deviation to an off-path message can be beneficial, we do not lose any equilibria by focusing on this belief updating rule.

1 - IC of the advisor. First, we verify that the advisor is not better off deviating from (24) with $\hat{p}(v,\mu) = \bar{\tau}(v)$. Because of the above definition of the off-path beliefs, it is sufficient to consider only deviations to $m \in \{0,1\}$ at $p \in \varrho$. First, consider a deviation of type v to $m = 1$ at $p \in \varrho$ at which $p < \bar{\tau}(v)$. This deviation is equivalent to mimicking the communication strategy of type $v' : \bar{\tau}(v') = p$. Since mimicking the communication strategy of type v' is not profitable for type v in equilibrium E (otherwise, it would not be an equilibrium), it is also not profitable here. Second, consider a deviation of type v to $m = 0$ at $p = \bar{\tau}(v)$. Depending on her communication strategy at later prices, this deviation will result in exit at price $\bar{\tau}(v')$ for some $v' \geq v_h(\bar{\tau}(v))$. Hence, any such deviation is equivalent to mimicking the communication strategy of type v'. Since it is not profitable for type v in equilibrium E, it is also not profitable here.

2 - IC of the bidder after observing $m = 1$ at $p \in \varrho$ and $m = 0$ before. We argue that $\bar{p}(\tilde{\mu}) \leq p$ in this case, so the bidder's best response is to quit the auction immediately. Given this history, the bidder's posterior belief is that $v \in [v_l(p), v_h(p)]$. Because the bidder expects the advisor to follow (24) with $\hat{p}(v,\mu) = \bar{\tau}(v)$, she expects the advisor to send $m = 1$ at any later price. Since the bidder expects to not learn anything new about v, her optimal exit strategy is given by the expected valuation, i.e., $\mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. It follows that the bidder exits immediately if $p \geq \mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. Next, we show that $\bar{\tau}(p)$ in equilibrium E must satisfy this condition at any $p \in \varrho$. Since exiting at price p is optimal for the bidder for any realization $v \in [v_l(p), v_h(p)]$ of the valuation, it must be that $p \geq \mathbb{E}\left[v|\mathcal{H}_p^E\right]$ for any history $\mathcal{H}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_l(p), v_h(p)]$. It follows that $p \geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right]$, where $\mathbb{H}_p^E$ denotes the set of such histories. Using the law of iterated expectations and fact that the maximum of a random variable cannot be below its mean,

$$
\begin{aligned}
p &\geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right] \geq \mathbb{E}\left[\mathbb{E}\left[v|\mathcal{H}_p^E\right]|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] \\
&= \mathbb{E}\left[v|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] = \mathbb{E}\left[v|v \in [v_l(p), v_h(p)]\right].
\end{aligned}
$$

Therefore, when the bidder observes message $m = 1$ at $p \in \varrho$ for the first time, she finds it optimal to quit the auction immediately.

3 - IC of the bidder after observing a sequence of messages $m = 0$ up to price $p < \bar{\tau}(\bar{v})$. We argue that $\bar{p}(\tilde{\mu}) > p$ for any such history, i.e., it is optimal for the bidder to wait. Given this history, the bidder's posterior is that $v \in [v_h(p'), \bar{v}]$ for highest $p' \in \varrho$ satisfying $p' < p$. Consider equilibrium E and any history $\tilde{\mathcal{H}}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_h(p'), \bar{v}]$. Denote the set of such histories by $\tilde{\mathbb{H}}_p^E$. Since the bidder finds it optimal to wait, the payoff from waiting is weakly above the payoff from quitting the auction immediately (i.e., zero) for any such history $\tilde{\mathcal{H}}_p^E$. In strategy profile E' the bidder learns no less between price p and the exit price than in strategy profile E. Hence, the fact that waiting is optimal for any history $\tilde{\mathcal{H}}_p^E \in \tilde{\mathbb{H}}_p^E$ implies that waiting is also optimal when the bidder expects the advisor to follow (24) with $\hat{p}(v,\mu) = \bar{\tau}(v)$.

Therefore, the collection of strategies and beliefs E' is an equilibrium. Furthermore, on equilibrium path, the advisor with type v recommends to quit the auction when the price reaches $\bar{\tau}(v)$, and the bidder exits the auction immediately. Therefore, E' results in the same bidding behavior as E.

The second statement of the lemma can be proved by contradiction. Consider equilibrium E that satisfies NITS, and suppose that an equilibrium in online threshold strategies with the same bidding behavior violates NITS. Hence, there exists price p such that the advisor with type $v_l(p)$ is better off credibly revealing itself at price p than getting the expected (as of information at price p) payoff in equilibrium E'. Hence, the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself then exceeds the time-0 expected payoff of the advisor of type $v_l(p)$ in equilibrium E'. Now, consider equilibrium E, and the strategy of the advisor of type $v_l(p)$ to send equilibrium message $\bar{m}(v, p', \mu)$ for all $p' < p$ and to credibly reveal itself at price p (by definition of $v_l(p)$, type $v_l(p) = \inf\{v | v \in supp(\mu(h))\}$ for any history induced by this message strategy up to price p). In equilibrium E, bidding behavior of other bidders is the same and the bidder's reaction to the advisor credibly revealing itself at price p is the same as in equilibrium E'. Hence, the time-0 expected payoff of the advisor of type $v_l(p)$ from this strategy is the same as the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself at price p in equilibrium E', which is strictly higher than the time-0 equilibrium expected payoff of the advisor of type $v_l(p)$. Hence, equilibrium E also violates the NITS condition, which is a contradiction. □

B.3 Selection Result

Lemma 4 implies that any equilibrium of the second-price auction has an equivalent equilibrium in online threshold strategies in the English auction. Indeed, suppose that there is a single round of informative communication in the English auction, followed by babbling at any other round, where bidders believe that any message has no information content. Since there is no informative communication in later rounds, bidding in the English auction is equivalent to bidding in the second-price auction for the same distribution of bidder types induced by communication at the initial round. Thus, any equilibrium from Proposition 1 has an equivalent equilibrium in the English auction. By Lemma 4, the English auction has also an equivalent equilibrium in online threshold strategies.[19] Thus, the set of equilibria in the English auction is potentially very large. Nevertheless, it turns out that all equilibria

[19]Specifically, consider any equilibrium in the second-price auction with partition $\{\omega_k\}_{k=0}^{K}$. An equivalent equilibrium in online threshold strategies in the English auction has the following form. Type $v \in (\omega_{k-1}, \omega_k)$ plays the strategy of sending message $m = 0$, if $p < \mathbb{E}[v | v \in (\omega_{k-1}, \omega_k)]$, and message $m = 1$, otherwise. Let $\mathcal{P} \equiv \{p : \mathbb{E}[v | v \in (\omega_{k-1}, \omega_k)] = p$ for some $p\}$ denote the set of informative prices – prices at which some type switches from $m = 0$ to $m = 1$ with some probability. Let the bidder's belief updating rule for any $p \notin \mathcal{P}$ be such that any message is interpreted the same way (i.e., belief is unchanged for any message sent at such price). Finally, the bidder's strategy is to exit at the first informative price $p \in \mathcal{P}$, at which she gets message $m = 1$.

of the English auction satisfying NITS take a simple form of capped delegation strategies:

Proposition 7. *Any equilibrium in the English auction that satisfies the NITS condition is in capped delegation strategies with cutoff v^* satisfying:*

$$\begin{aligned} &\text{if } v^* \in (\underline{v}, \overline{v}), \text{ then } b = \mathbb{E}[v|v \geq v^*] - v^*; \\ &\text{if } v^* = \underline{v}, \text{ then } b \geq \mathbb{E}[v|v \geq v^*] - v^*; \\ &\text{if } v^* = \overline{v}, \text{ then } \overline{v} = \infty \text{ and } b \leq \lim_{s\to\infty} \mathbb{E}[v|v \geq s] - s. \end{aligned} \tag{25}$$

Proof of Proposition 7. By Lemma 4, it is without loss of generality to focus on equilibria in online threshold strategies. In the proof of Lemma 4, we introduced function $\bar{\tau}(v)$, which denotes the equilibrium exit price of the bidder if the advisor's type is v. In an equilibrium in online threshold strategies, $\bar{\tau}(v)$ is also the first price at which the advisor with type v sends message "quit" to the bidder.

Any equilibrium generates partition Π of $[\underline{v}, \overline{v}]$ satisfying $\bar{\tau}(v) = \bar{\tau}(v')$ for any $v, v' \in \pi$ for any element $\pi \in \Pi$. As shown in Lemma 4, $\bar{\tau}(v)$ is weakly increasing, so any $\pi \in \Pi$ is an interval (possibly consisting of one element). We say that types in $\pi \in \Pi$ *pool* if $\bar{\tau}(v)$ is constant on $v \in \pi$, i.e., these types start sending message "quit" at the same price. We say that types in $[v', v'']$ *separate*, if $\bar{\tau}(v)$ is strictly increasing on $[v', v'']$, i.e., these types start sending message "quit" at different prices. Let Π^P and $\Pi^S \equiv [\underline{v}, \overline{v}] \backslash \Pi^P$ be the sets of all types that pool with some other type and that separate, respectively. Denote by $\partial\Pi^P$ the boundary of Π^P.

Babbling ($\bar{\tau}(v) = \mathbb{E}[v]\ \forall v$) is an equilibrium of the English auction, and it satisfies NITS if and only if $\mathbb{E}[v] \leq \underline{v} + b$. This proves case $v^* = \underline{v}$ of the proposition. Hence, we can consider the case in which there is a non-trivial information transmission in equilibrium.

Claim 4. *For any $\pi, \pi' \in \Pi^P$, π and π' are not adjacent.*

Proof: By contradiction, suppose that there are two adjacent intervals of types, π and π', such that $\bar{\tau}(v) = p\ \forall v \in \pi$ and $\bar{\tau}(v) = p'\ \forall v \in \pi'$. Without loss of generality, $p' > p$. Consider the advisor with type $\tilde{v}$ on the boundary of π and π'. By continuity, the advisor with type $\tilde{v}$ is indifferent between his bidder quitting the auction at prices p and p'. The benefit of the latter is winning against types in π, while the cost is risking to win against types in π' and paying p'. The indifference of type $\tilde{v}$ implies that $p' > \tilde{v} + b$. Consider running price $\frac{p'+p}{2}$. Type $\tilde{v}$ is the weakest remaining type at this price. Since $p' > \tilde{v} + b$, following his equilibrium strategy of waiting to send recommendation $m = 1$ until price p' generates negative expected payoff to the advisor at this point. In contrast, claiming that he is the weakest remaining type at the current price of $\frac{p'+p}{2}$ will lead to the bidder quitting immediately, yielding the payoff of zero to the advisor. This contradicts the NITS condition. *q.e.d.*

We next show that whenever types within an interval separate, they start recommending to quit the auction at their most preferred time.

Claim 5. *If $\bar{\tau}(v)$ is strictly increasing on (v', v''), then $\bar{\tau}(v) = v + b$ for any $v \in (v', v'')$.*

Proof: By contradiction, suppose there is $v \in (v', v'')$ with $\bar{\tau}(v) \neq v+b$. Then, either $\bar{\tau}(v) > v+b$ or $\bar{\tau}(v) < v+b$. First, consider the former case. Since v is interior, there exists a subset of (v', v'') of types $v+\varepsilon > v$ with positive measure with $\bar{\tau}(v) > v+\varepsilon+b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v+\varepsilon) > \bar{\tau}(v) > v+\varepsilon+b$. Therefore, any such type $v+\varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v+\varepsilon)$: by doing this, the advisor ensures that the bidder does not win when the valuation of the strongest rival is in $(v, v+\varepsilon)$, in which case the bidder overpays relative to the advisor's maximum willingness to pay of $v+\varepsilon+b$. Hence, it cannot be that $\bar{\tau}(v) > v+b$. Second, consider the case $\bar{\tau}(v) < v+b$. Now, there exists a subset of (v', v'') of types $v-\varepsilon < v$ with positive measure with $\bar{\tau}(v) < v-\varepsilon+b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v-\varepsilon) < \bar{\tau}(v) < v-\varepsilon+b$. Therefore, any such type $v-\varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v-\varepsilon)$: by doing this, the advisor ensures that the bidder wins when the valuation of the strongest rival is in $(v-\varepsilon, v)$, in which case the advisor gets a positive payoff, since the bidder pays below the advisor's maximum willingness to pay. Therefore, it cannot be that $\bar{\tau}(v) < v+b$. We conclude that $\bar{\tau}(v) = v+b$. *q.e.d.*

Claim 6. *If* $\Pi^P \neq \emptyset$ *and* $\Pi^S \neq \emptyset$*, then* Π^P *contains a single interval* $\pi^P = [v^*, \overline{v}]$*, where* $v^* > \underline{v}$*.*

Proof: By contradiction, suppose that Π^P contains more than one interval or that it contains one interval that is to the left of Π^S. In the former case, Claim 4 implies that the intervals are not adjacent. Therefore, there is an interval $\pi \in \Pi^P$ that lies to the left of an interval in Π^S. Let $v \in \pi$ be the highest type in this interval. Since it must be indifferent between separation and pooling and $\bar{\tau}(v) = v+b$ in the separation region by Claim 2, we have $v+b = \bar{\tau}(w)$ for any $w \in \pi$. Therefore, $\bar{\tau}(w) > w+b$ for any $w \in \pi$, $w \neq v$. In particular, it holds for the lowest type in the interval $w' = \min_{w \in \pi} w$. However, this violates the NITS condition. Indeed, consider running price $p = \frac{\bar{\tau}(w)+w'+b}{2}$. Type w' is the weakest remaining type of the advisor at this price. Since $p > w'+b$, following his equilibrium strategy of waiting until price $\bar{\tau}(w)$, $w \in \pi$ to send recommendation $m = 1$ generates negative expected payoff to the advisor at the current point. In contrast, claiming that he is the weakest remaining type at the current price will lead to the bidder quitting immediately, yielding the payoff of zero. Therefore, Π^P contains only one interval that lies to the right of Π^S, i.e., the interval is of the form $\pi^P = [v^*, \overline{v}]$ for some $v^* > \underline{v}$. *q.e.d.*

Claim 7. *Cut-off* v^* *satisfies* (25).

Proof: Case $v^* = \underline{v}$ (the babbling equilibrium) was covered above (before Claim 4). Consider case $v^* = \overline{v}$, i.e., $\Pi^P = \emptyset$. By Claim 5, $\bar{\tau}(v) = v+b$ for any $v \in (\underline{v}, \bar{v})$. If $\bar{v} < \infty$, the upper bound on the bidder's utility in round $p = v+b$ is $\overline{v} - (v+b) \overset{v \to \overline{v}}{\to} -b$, which contradicts the optimality of the bidder to follow the advisor's recommendation. Hence, it must be that $\bar{v} = \infty$. Next, by contradiction, suppose that $b > \lim_{s \to \infty} \mathbb{E}[v | v \geq s] - s$. By continuity, there is $\overline{s} < \infty$ such that $b > \mathbb{E}[v | v \geq s] - s$ for any $s > \overline{s}$. If the bidder wins in any round $p \geq s+b$, then her expected utility equals $\mathbb{E}[v | v \geq s] - s - b < 0$ and so, the value of following the advisor's recommendations is negative, which is a contradiction.

Therefore, it must be that $b \leq \lim_{s\to\infty} \mathbb{E}[v|v \geq s] - s$.

Finally, consider case $v^* \in (\underline{v}, \bar{v})$. By contradiction, suppose that $v^* + b \neq \mathbb{E}[v|v \geq v^*]$. By indifference of type v^*, it must be that $\bar{\tau}(v) = v^* + b$ for any $v \in [v^*, \bar{v}]$. If $v^* + b < \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v)$, $v > v^*$ violates the incentive compatibility condition of the bidder. To see this, consider running price just below $v^* + b$. The equilibrium behavior prescribes the bidder to exit the auction in the next instant, which is below her maximum willingness to pay of $\mathbb{E}[v|v \geq v^*]$. By waiting a little beyond price $\bar{\tau}(v) = v^* + b$, the bidder ensures that she wins the auction with probability one and pays below her estimated valuation of $\mathbb{E}[v|v \geq v^*]$. Since this strategy results in a discontinuous upward jump in the expected utility of the bidder, she is better off deviating. Hence, it cannot be that $v^* + b < \mathbb{E}[v|v \geq v^*]$. If $v^* + b > \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v) = v^* + b$, $v \geq v^*$ violates the incentive compatibility condition of the bidder, because she would prefer to exit the auction slightly earlier. Consider the running price $p = v^* + b - \varepsilon$ for an infinitesimal positive ε and suppose that the bidder has got a sequence of recommendations $\tilde{m} = 0$. Her posterior belief is that the valuation is in the range $(v^* - \varepsilon, \overline{v}]$. Suppose that the bidder follows her equilibrium play. If $v \in (v^* - \varepsilon, v^*)$ and the bidder wins, she pays $v + b$ above her valuation v. If $v \in (v^*, \bar{v}]$ and the bidder wins, she pays $\bar{\tau}(v) = v^* + b$, which is, on average, above her valuation v ($\mathbb{E}[v|v \geq v^*]$). Since the bidder wins with positive probability, her expected payoff from following the equilibrium play is negative. In contrast, immediate exit yields zero expected payoff. Hence, the bidder is better off deviating and exiting the auction immediately. Therefore, $v^* + b = \mathbb{E}[v|v \geq v^*]$. *q.e.d.*

The proof of Proposition 7 follows from Claims 5, 6, and 7.

□