Online Appendix for "Optimal Dynamic Capital Budgeting"

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This appendix provides the summary of the solution of the basic model, solutions for extensions of the basic model discussed in Section 3, and two supplementary results: the result that it is without loss of generality to assume that the division manager consumes monetary transfers immediately and the result about existence of the solution to the basic model.

I. Basic Model: Summary of the Solution

Given value function $P(W)$, the optimal investment $k^*(\theta, W)$ and audit threshold $\theta^*(W)$ are given by:

- investment $k^*(\theta, W)$ solves:

$$\begin{aligned} V_k(k,\theta) &= 1+\gamma P'(W-\gamma k), \quad &\text{if } \theta<\theta^*(W), \\ V_k(k,\theta) &= 1+\gamma P'(W), \quad &\text{if } \theta>\theta^*(W); \end{aligned}$$

- audit threshold $\theta^*(W)$ solves:

$$F^n(\theta, W) = F^a(\theta, W) - c,$$

where

$$\begin{aligned} F^n(\theta,W) &= \max_{k\in\mathbb{R}_+}\{V(k,\theta)-k+P(W-\gamma k)-P(W)\}, \\ F^a(\theta,W) &= \max_{k\in\mathbb{R}_+}\{V(k,\theta)-k+P(W)-\gamma k P'(W)\}. \end{aligned}$$

If the solution does not exist, then $\theta^*(W)=\overline{\theta}\ (\underline{\theta})$, if the left-hand side is higher (lower) than the right-hand side for all $\theta\in\left[\underline{\theta},\overline{\theta}\right]$.

The value function $P(W)$ is the maximal solution to the integro-differential equation

$$\begin{aligned} rP(W) &= \rho W P'(W)+\lambda\int_{\underline{\theta}}^{\bar{\theta}}\max\{F^n(\theta,W), F^a(\theta,W)-c\}\,dF(\theta), \text{ if } W\in(0,W^c], \\ P(W) &= P(W^c)+W^c-W, \text{ if } W\in(W^c,\infty), \end{aligned}$$

with initial value condition $P(0) = 0$ and where threshold W^c is determined by $P'(W^c) = -1$. Function $P(W)$ is weakly concave on $W \in (0, \infty)$ and strictly concave on $W \in (0, W^c)$.

II. Multiple Audit Technologies

Because there are two audit technologies, the stochastic process describing audit decisions of headquarters needs to be redefined. Let $(dA_t)_{t \geq 0}$ be such that $A_0 = 0$, $dA_t = 1$ if headquarters audit at time 1 using technology 1, $dA_t = 2$ if headquarters audit at time t using technology 2, and $dA_t = 0$ if headquarters do not audit at time t. First, I solve for the optimal direct mechanism. Second, I show how the policies implied by the optimal direct mechanism can be implemented using a budgeting mechanism with two thresholds.

Let $s_t \in \{0, 1\}$ denote the success of the audit at time t. If headquarters audit the report using technology 2, then $s_t = 1$ with probability 1. If headquarters audit the report using technology 1, then $s_t = 1$ with probability p. If the report is not audited, then $s_t = 0$ with probability 1. By analogy with Lemma 1, the evolution of the division manager's promised utility to her report $d\hat{X}_t$ from message space $\{0\} \cup \Theta$ as

$$dW_t = \rho W_{t-} dt - \gamma dK_t - dC_t + H_{ma,t}\left(d\hat{X}_t, s_t\right) - \lambda \mathbb{E}_{\theta,s}\left[H_{ma,t}(\theta, s)\right] dt, \tag{A1}$$

where $H_{ma,t}\left(d\hat{X}_t, 0\right)$ is the sensitivity of the division manager's utility to her report when audit was not informative or did not occur and $H_{ma,t}\left(d\hat{X}_t, 1\right)$ is the sensitivity of the division manager's utility to her report when audit was informative and confirmed the division manager's report. As before, a standard argument implies that if audit reveals that the division manager lied, then it is optimal to decrease her utility to zero: $dW_t = -W_t$.

Let $D_t^{A1} = \left\{d\hat{X}_t | dA_t = 1\right\}$, $D_t^{A2} = \left\{d\hat{X}_t | dA_t = 2\right\}$, and $D_t^{N} = \left\{d\hat{X}_t | dA_t = 0\right\}$ be the "audit using technology 1," "audit using technology 2," and "no audit" regions of reports at time t, respectively. Because $c_1 > 0$, $\{0\} \in D_t^N$. By analogy with Lemma 2, truth-telling is incentive compatible if and only if $H_{ma,t}\left(d\hat{X}_t, 0\right)$ and $H_{ma,t}\left(d\hat{X}_t, 1\right)$ satisfy the following restrictions:

Lemma 1. *At any time $t \geq 0$, truth-telling is incentive compatible if and only if:*

1. $\forall d\hat{X}_t \in D_t^N : H_{ma,t}\left(d\hat{X}_t, 0\right) = 0$;
2. $\forall d\hat{X}_t \in D_t^{A1} : pH_{ma,t}\left(d\hat{X}_t, 1\right) + (1-p) H_{ma,t}\left(d\hat{X}_t, 0\right) \geq 0$ and $(1-p) H_{ma,t}\left(d\hat{X}_t, 0\right) \leq pW_t$;
3. $\forall d\hat{X}_t \in D_t^{A2}$: $H_{ma,t}\left(d\hat{X}_t, 1\right) \geq 0$.

Proof of Lemma 3. Consider any $dX_t \in D_t^N$. Report dX_t dominates any report $d\hat{X}_t \in D_t^{A2}$, because the latter leads to zero expected utility with certainty. Report dX_t dominates any report $d\hat{X}_t \in D_t^{A1}$ if and only if

$$H_{ma,t}(dX_t, 0) \geq -pW_t + (1-p) H_{ma,t}\left(d\hat{X}_t, 0\right). \tag{A2}$$

Finally, report dX_t dominates any report $d\hat{X}_t \in D_t^N$, $d\hat{X}_t \neq dX_t$ if and only if $H_{ma,t}(dX_t, 0) \geq H_{ma,t}\left(d\hat{X}_t, 0\right)$. Because this inequality must hold for any $dX_t, d\hat{X}_t \in D_t^D$ and $H_{ma,t}(0,0) = 0$, truth-telling is incentive compatible for all $dX_t \in D_t^N$ if and only if

$$\begin{aligned} H_{ma,t}(dX_t, 0) &= 0 \ \forall dX_t \in D_t^N, & \text{(A3)} \\ 0 &\geq -pW_t + (1-p) H_{ma,t}\left(d\hat{X}_t, 0\right) \ \forall d\hat{X}_t \in D_t^{A1}, & \text{(A4)} \end{aligned}$$

where the second inequality follows from (A2) - (A3). Consider any $dX_t \in D_t^{A1}$. Again, report dX_t dominates any report $d\hat{X}_t \in D_t^{A2}$, because the latter leads to zero expected utility with certainty. By analogy with (A2), report dX_t dominates report $d\hat{X}_t \in D_t^{A1}$, $d\hat{X}_t \neq dX_t$ if and only if

$$pH_{ma,t}(dX_t, 1) + (1-p) H_{ma,t}(dX_t, 0) \geq -pW_t + (1-p) H_{ma,t}\left(d\hat{X}_t, 0\right). \tag{A5}$$

Finally, report dX_t dominates report $d\hat{X}_t \in D_t^N$ if and only if

$$pH_{ma,t}(dX_t, 1) + (1-p) H_{ma,t}(dX_t, 0) \geq H_{ma,t}\left(d\hat{X}_t, 0\right) = 0. \tag{A6}$$

Notice that constraint (A5) is implied by conditions (A4) and (A6). Therefore, truth-telling is incentive compatible for all $dX_t \in D_t^{A1}$ if and only if (A6) is satisfied for all $dX_t \in D_t^{A1}$. Finally, consider any $dX_t \in D_t^{A2}$. Again, report dX_t dominates any report $d\hat{X}_t \in D_t^{A2}$, $d\hat{X}_t \neq dX_t$, because the latter leads to zero expected utility with certainty. Report dX_t dominates report $d\hat{X}_t \in D_t^{A1}$ if and only if

$$H_{ma,t}(dX_t, 1) \geq -pW_t + (1-p) H_{ma,t}\left(d\hat{X}_t, 0\right). \tag{A7}$$

Finally, report dX_t dominates report $d\hat{X}_t \in D_t^N$ if and only if

$$H_{ma,t}(dX_t, 1) \geq H_{ma,t}\left(d\hat{X}_t, 0\right) = 0. \tag{A8}$$

Constraint (A7) is implied by constraints (A4) and (A8). Therefore, truth-telling is incentive compatible for all $dX_t \in D_t^{A2}$ if and only if (A8) is satisfied for all $dX_t \in D_t^{A2}$. Combining the three cases yields the conditions in the lemma.

Given Lemma 3, I can solve for the optimal direct mechanism using the dynamic programming approach. Let $P_{ma}(W)$ denote the value to headquarters under the optimal direct mechanism in this extension as a function of W. Let W_{ma}^c and W_{ma}^* denote the lowest W at which $P'_{ma}(W) = -1$ and W at which $P_{ma}(W)$ is maximized, respectively.[1] As in Section 2.2, the optimal monetary compensation of the division manager is given by the same threshold rule with threshold W_{ma}^c. The same argument as in Section 2.2 leads to the following HJB

[1] $W_{ma}^c = \infty$ if $P'_{ma}(W) > -1$ for all W.

equation in the range $W < W^c_{ma}$:

$$\begin{aligned} rP_{ma}(W) = \max_{\left\{a_\theta, k^1_\theta, k^0_\theta, h^1_\theta, h^0_\theta\right\}_{\theta\in\Theta}} \Big\{ \lambda \int_{\underline{\theta}}^{\bar{\theta}} \left(V(k_\theta,\theta) - k_\theta - c_1 1_{\{a_\theta=1\}} - c_2 1_{\{a_\theta=2\}} \right) dF(\theta) \\ + \left[\rho W - \lambda \int_{\underline{\theta}}^{\bar{\theta}} \left[h^0_\theta \left(1_{\{a_\theta=0\}} + (1-p) 1_{\{a_\theta=1\}} \right) + h^1_\theta \left(p1_{\{a_\theta=1\}} + 1_{\{a_\theta=2\}} \right) \right] dF(\theta) \right] P'_{ma}(W) \\ + \lambda \int_{\underline{\theta}}^{\bar{\theta}} \left[P_{ma}\left(W + h^0_\theta - \gamma k^0_\theta\right) \left(1_{\{a_\theta=0\}} + (1-p) 1_{\{a_\theta=1\}} \right) \right. \\ \left. + P_{ma}\left(W + h^1_\theta - \gamma k^1_\theta\right) \left(p1_{\{a_\theta=1\}} + 1_{\{a_\theta=2\}} \right) - P_{ma}(W) \right] dF(\theta) \Big\}, \end{aligned} \tag{A9}$$

where the maximization is subject to constraints $a_\theta \in \{0,1,2\}$, $k^1_\theta \geq 0$, $k^0_\theta \geq 0$, and the incentive compatibility constraints

$$h^0_\theta = 0, \text{ if } a_\theta = 0, \tag{A10}$$

$$ph^1_\theta + (1-p) h^0_\theta \geq 0, \text{ if } a_\theta = 1, \tag{A11}$$

$$(1-p) h^0_\theta \leq W, \text{ if } a_\theta = 1, \tag{A12}$$

$$h^1_\theta \geq 0, \text{ if } a_\theta = 2. \tag{A13}$$

Taking the first-order condition of (A9) with respect to h^i_θ, $i \in \{0,1\}$ yields $h^1_\theta = \gamma k^1_\theta$ and $h^0_\theta = \min\{W/(1-p), \gamma k^0_\theta\}$.

Using this, I solve for the optimal investment. Taking the derivative of (A9) with respect to k_θ yields:

$$\frac{\partial V(k_\theta,\theta)}{\partial\theta} - 1 - \gamma P'_{ma}(W - \gamma k_\theta) = 0, \text{ if } a_\theta = 0, \tag{A14}$$

$$\frac{\partial V(k_\theta,\theta)}{\partial\theta} - 1 - \gamma P'_{ma}\left(\min\left\{W\frac{2-p}{1-p} - \gamma k_\theta, W\right\}\right) = 0, \text{ if } a_\theta = 1,\ s_t = 0, \tag{A15}$$

$$\frac{\partial V(k_\theta,\theta)}{\partial\theta} - 1 - \gamma P'_{ma}(W) = 0, \text{ if } a_\theta = 1,\ s_t = 1 \text{ or } a_\theta = 2, \tag{A16}$$

Let $k^{na}(\theta,W)$, $k^{ua}(\theta,W)$, and $k^{ia}(\theta,W)$ denote the solutions of (A14), (A15), and (A16), respectively. By concavity of $P_{ma}(W)$[2], $k^{na}(\theta,W) < k^{ua}(\theta,W) \leq k^{ia}(\theta,W)$. Finally, the next lemma, which is the analogue of Property 4, solves for the optimal audit strategies:

Lemma 2. There exist points $\theta^*_a(W) \in \Theta$ and $\theta^{**}_a(W) \in \Theta$, $\theta^{**}_a(W) > \theta^*_a(W)$, defined below, such that the optimal audit strategy is

$$a^*(\theta,W) = \begin{cases} 0, & \text{if } \theta \leq \theta^*_a(W), \\ 1, & \text{if } \theta \in (\theta^*_a(W), \theta^{**}_a(W)), \\ 2, & \text{if } \theta \geq \theta^{**}_a(W). \end{cases} \tag{A17}$$

[2]In this and other extensions, the value function is weakly concave, because the mechanism can specify randomization between any two levels of the division manager's promised utility. Furthermore, the value function is strictly concave in the range in which no monetary compensation is paid by the argument similar to Proposition 2 in the base model. I omit repeating it for each extension for brevity.

Proof of Lemma 4. Let

$$
\begin{aligned}
F^{ia}(\theta, W) &= \max_{k\in\mathbb{R}_+}\left\{V(k,\theta) - k + P_{ma}(W) - \gamma k P'_{ma}(W)\right\}, && \text{(A18)}\\
F^{ua}(\theta, W) &= \max_{k\in\mathbb{R}_+}\left\{V(k,\theta) - k + P_{ma}\left(\min\left\{W\frac{2-p}{1-p} - \gamma k, W\right\}\right)\right. && \text{(A19)}\\
&\quad \left. - \min\left\{\gamma k, \frac{\gamma W}{1-p}\right\} P'_{ma}(W)\right\},\\
F^{na}(\theta, W) &= \max_{k\in\mathbb{R}_+}\left\{V(k,\theta) - k + P_{ma}(W - \gamma k) - P_{ma}(W)\right\}. && \text{(A20)}
\end{aligned}
$$

Strategy $dA_t = 1$ is weakly better than $dA_t = 0$ if and only if

$$
pF^{ia}(\theta, W) + (1-p) F^{ua}(\theta, W) - F^{na}(\theta, W) \geq c_1. \tag{A21}
$$

By the envelope theorem, the derivative of the left-hand side of (A21) with respect to θ is

$$
\begin{aligned}
& p\frac{\partial V(k^{ia}(\theta, W), \theta)}{\partial\theta} + (1-p)\frac{\partial V(k^{ua}(\theta, W), \theta)}{\partial\theta} - \frac{\partial V(k^{na}(\theta, W), \theta)}{\partial\theta}\\
= \; & p\int_{k^{na}(\theta,W)}^{k^{ia}(\theta,W)} \frac{\partial^2 V(k,\theta)}{\partial k\partial\theta} dk + (1-p)\int_{k^{na}(\theta,W)}^{k^{ua}(\theta,W)} \frac{\partial^2 V(k,\theta)}{\partial k\partial\theta} dk \geq 0, && \text{(A22)}
\end{aligned}
$$

because $k^{ia}(\theta, W) \geq k^{na}(\theta, W)$, $k^{ua}(\theta, W) \geq k^{na}(\theta, W)$, and $\partial^2 V(k,\theta)/\partial k\partial\theta > 0$ by Assumption 1. Therefore, the left-hand side of (A21) is increasing in θ. Let $\theta_1(W) \in \Theta$ denote a point at which (A21) holds as equality, if it exists. If the left-hand side of (A21) is below c_1 for all $\theta \in \Theta$, let $\theta_1(W)$ be any point above $\bar{\theta}$. If the left-hand side of (A21) is above c_1 for all $\theta \in \Theta$, let $\theta_1(W)$ be any point below $\underline{\theta}$.

Next, strategy $dA_t = 2$ is weakly better than $dA_t = 1$ if and only if

$$
(1-p)\left(F^{ia}(\theta, W) - F^{ua}(\theta, W)\right) \geq c_2 - c_1. \tag{A23}
$$

By the envelope theorem, the derivative of the left-hand side of (A23) with respect to θ has the same sign as

$$
\begin{aligned}
& \frac{\partial V(k^{ia}(\theta, W), \theta)}{\partial\theta} - \frac{\partial V(k^{ua}(\theta, W), \theta)}{\partial\theta}\\
= \; & \int_{k^{ua}(\theta,W)}^{k^{ia}(\theta,W)} \frac{\partial^2 V(k,\theta)}{\partial k\partial\theta} dk \geq 0. && \text{(A24)}
\end{aligned}
$$

Therefore, the left-hand side of (A23) is increasing in θ. Let $\theta_2(W) \in \Theta$ denote a point at which (A23) holds as equality, if it exists. If the left-hand side of (A23) is below $c_2 - c_1$ for all $\theta \in \Theta$, let $\theta_2(W)$ be any point above $\bar{\theta}$. If the left-hand side of (A23) is above $c_2 - c_1$ for all $\theta \in \Theta$, let $\theta_2(W)$ be any point below $\underline{\theta}$.

Finally, let $\theta_a^*(W) = \min\{\theta_1(W), \theta_2(W)\}$. Then, in the range $\theta \leq \theta_a^*(W)$ strategy $dA_t = 0$ is more optimal than $dA_t = 1$ and $dA_t = 2$. If $\theta_2(W) > \theta_1(W)$, then let $\theta_a^{**}(W) = \theta_2(W)$. Then, in the range $\theta \in (\theta_a^*(W), \theta_a^{**}(W)]$ strategy $dA_t = 2$ is better than $dA_t = 0$ by the argument in the first paragraph of the proof and than $dA_t = 1$, because it is dominated by $dA_t = 0$ by the argument in the second paragraph of the proof. Similarly, in the range $\theta > \theta_a^{**}(W)$ strategy $dA_t = 2$ is better than $dA_t = 1$ by the argument in the second paragraph of the proof and than $dA_t = 0$, because it is dominated by $dA_t = 1$ by the argument in the first paragraph of the proof. If $\theta_1(W) > \theta_2(W)$, let $\theta_a^{**}(W) = \theta_a^{*}(W) = \theta_2(W)$. Then, in the range $\theta > \theta_a^{**}(W) = \theta_a^{*}(W)$ strategy $dA_t = 2$ is better than $dA_t = 1$ by the argument in the second paragraph of the proof and than strategy $dA_t = 0$, because it is dominated by strategy $dA_t = 1$ by the argument in the first paragraph of the proof.

To finish characterization of the optimal direct mechanism, I need to define the evolution of W_t when $d\hat{X}_t = 0$. Using (A1), I get

$$dW_t = g_{ma}(W_t)\, W_t dt, \tag{A25}$$

where

$$\begin{aligned} g_{ma}(W) &= \rho - \lambda \int_{\theta_a^*(W)}^{\bar{\theta}} \frac{\gamma k^{ia}(\theta, W)}{W} dF(\theta) \\ &\quad + \lambda \int_{\theta_a^*(W)}^{\theta_a^{**}(W)} \frac{\gamma(1-p)\left(k^{ia}(\theta, W) - k^{ua}(\theta, W)\right)}{W} dF(\theta). \end{aligned} \tag{A26}$$

We next formulate the proposition, analogous to Proposition 3, that the policies implied by the optimal direct mechanism can be implemented using a simple mechanism, similar to the budgeting mechanism with threshold separation of financing in the basic model:

Proposition 1. *Consider the following mechanism. At the initial date, headquarters allocate a spending account with balance B_0 to the division manager. The division manager is allowed to draw on it for investment at her discretion. At time $t \geq 0$, the account is replenished with rate $g_{ma}(\gamma B_t)$: $dB_t = g_{ma}(\gamma B_t) B_t dt$, if $B_t < W_{ma}^c/\gamma$, and with rate zero, otherwise, where $g_{ma}(\cdot)$ and W_{ma}^c are defined in the online appendix. In addition, there are two thresholds on the size of individual investment projects, k_t^* and k_t^{**}, given by $k_t^* = k^{ia}(\theta_a^*(\gamma B_t), \gamma B_t)$, $k_t^{**} = k^{ia}(\theta_a^{**}(\gamma B_t), \gamma B_t)$, where $k^{ia}(\cdot,\cdot)$, $\theta_a^*(\cdot)$, and $\theta_a^{**}(\cdot)$ are defined in the online appendix. At any time t the division manager can pass the project to headquarters claiming that it requires an investment of k, where $k > k_t^*$:*

1. If $k > k_t^{**}$, the project is audited using technology 2. If the audit reveals that $k^{ia}(\theta, \gamma B_t) \geq k_t^{**}$, headquarters invest $k^{ia}(\theta, \gamma B_t)$ and do not alter the spending account balance. If the audit reveals that $k^{ia}(\theta, \gamma B_t) < k_t^{**}$, headquarters do not invest and punish the division manager by reducing the account balance to zero.

2. If $k \in (k_t^*, k_t^{**})$, the project is audited using technology 1. If the audit is informative and reveals $k^{ia}(\theta, \gamma B_t) = k$, headquarters invest $k^{ia}(\theta, \gamma B_t)$ and do not alter the spending account balance. If the audit is informative and reveals $k^{ia}(\theta, \gamma B_t) \neq k$, headquarters do not invest and punish the division manager by reducing the account balance to zero. If the audit is uninformative and $k \leq k_t^{***} \equiv B_t/(1-p)$, where $k_t^{***} \in [k_t^*, k_t^{**}]$ is defined in the online appendix, headquarters invest k and do not alter the spending account balance. If the audit is uninformative and $k > k_t^{***}$, headquarters invest k_t^{***} and do not alter the spending account *balance, and the division manager adds investment $k - k_t^{***}$ out of her account balance.*

Suppose that $dC_t = 0$ if $B_t < W_{ma}^c/\gamma$, $dC_t = g\left(W_{ma}^c\right) W_{ma}^c dt$ if $B_t = W_{ma}^c/\gamma$, and that if $R > W_{ma}^c$, an immediate payment of $R - W_{ma}^c$ is made to the division manager. Then, the manager finds it optimal to (i) invest out of her spending account in the way that maximizes headquarters' value; (ii) pass a project to headquarters if and only if $k^{ia}\left(\theta, \gamma B_t\right) > k_t^*$; (iii) if she passes a project to headquarters, claim $k = k^{ia}\left(\theta, \gamma B_t\right)$. If, in addition, the initial account balance is $B_0 = W_0/\gamma$, then this mechanism is optimal.

Proof of Proposition 5. First, using the argument of Proposition 3, it is easy to show that the evolution of γB_t is the same as the evolution of W_t in the optimal direct mechanism. The starting point is equal to $\gamma B_0 = W_0$ and the increments of γB_t and W_t are the same if investment policies are the same. Because in region D_t^N the change in the division manager's utility does not depend on dK_t, allocating the spending account between the current and future investment opportunities in the way that maximizes headquarters' value, $V\left(dK_t, \theta\right) + P_{ma}\left(\gamma\left(B_t - \gamma dK_t\right)\right)$, is incentive compatible. The implied investment policy is $k^{na}\left(\theta, \gamma B_t\right) = k^{na}\left(\theta, W_t\right)$. Similarly, consider region D_t^{A1} and suppose that audit is uninformative. If headquarters provide only k_t^{***} of capital, then additional investment of the division manager does not affect her expected payoff. Hence, the division manager has incentives to co-finance the project in a way that maximizes headquarters' value. The implied investment policy is $k^{ua}\left(\theta, \gamma B_t\right) = k^{ua}\left(\theta, W_t\right)$. The same argument as in Proposition 3 applies here to show that the division manager has incentives to pass the project to the headquarters and state the optimal investment truthfully.

III. Observable Realized Values

This section contains the proof of Proposition 4. First, I establish the incentive compatibility conditions for the division manager and set up the optimization problem. Second, I characterize the solution of the optimization problem. Slightly abusing notations, I use the same letters as in the basic model to denote all stochastic processes.

Incentive Compatibility

Since there is no audit, the past history can be summarized using the report process $\hat{X} = \left(d\hat{X}_t\right)_{t\geq 0}$ and the project verifiable success process $Y = \left(dY_t\right)_{t\geq 0}$, defined by $Y_0 = 0$, $dY_t = 1$, if the project that the firm invests in at time t pays off immediately, and $dY_t = 0$, otherwise. By analogy with the basic model, let $W_t\left(\hat{X}, Y\right)$ denote the continuation utility of the division manager at time t after a history of reports $\left\{d\hat{X}_s, s \leq t\right\}$ and project successes $\left\{dY_s, s \leq t\right\}$, conditional on reporting truthfully in the future:

$$W_t\left(\hat{X}, Y\right) = \mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(s-t)}\left(\gamma dK_s + dC_s\right)\right]. \tag{A27}$$

The dynamics of $W_t\left(\hat{X}, Y\right)$ can be expressed using the martingale representation theorem. The lifetime expected utility of the division manager $U_t(X, Y)$, evaluated as of time t, is

$$U_t(X, Y) = \int_0^t e^{-\rho s}(\gamma dK_s + dC_s) + e^{-\rho t} W_t(X, Y). \tag{A28}$$

Process $U(X, Y) = \{U_t(X, Y)\}_{t \geq 0}$ is a right-continuous $\mathcal{F}$-martingale by definition. Applying the martingale representation theorem for marked point processes, we can write

$$dU_t = \begin{cases} -\left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} (p\theta h_t(\theta, 1) + (1 - p\theta) h_t(\theta, 0)) \, dF(\theta)\right) dt, & \text{if } t \neq T_n \text{ for any } n \geq 1, \\ h_t(\theta_n, dY_t) - \left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} (p\theta h_t(\theta, 1) + (1 - p\theta) h_t(\theta, 0)) \, dF(\theta)\right) dt, & \text{if } t = T_n \text{ for some } n \geq 1. \end{cases} \tag{A29}$$

for some function $h_t(\theta, y)$, which is $\mathcal{F}$-predictable for any fixed $\theta \in \Theta$ and $y \in \{0, 1\}$. Defining $H_t(\cdot)$ to be scaled (by factor $e^{\rho t}$) $h_t(\cdot)$ and writing it as a function of $dX_t \in \{0\} \cup \Theta$ and $dY_t \in \{0, 1\}$, defining it to be zero if $dX_t = 0$, we can write:

$$dU_t = e^{-\rho t}\left(H_t(dX_t, dY_t) - \left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} \bar{H}_t(\theta) \, dF(\theta)\right) dt\right), \tag{A30}$$

where $\bar{H}_t(\theta) \equiv p\theta H_t(\theta, 1) + (1 - p\theta) H_t(\theta, 0)$ is the expected change in the promised utility of the division manager with project of quality θ.

Equating (A30) with the increment of $U_t(X, Y)$, given by

$$dU_t = e^{-\rho t}(\gamma dK_t + dC_t) - \rho e^{-\rho t} W_{t-}(X, Y) + e^{-\rho t} dW_t(X, Y), \tag{A31}$$

yields

$$dW_t = \rho W_{t-} dt - \gamma dK_t - dC_t + H_t(dX_t, dY_t) - \left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} \bar{H}_t(\theta) \, dF(\theta)\right) dt. \tag{A32}$$

Consider the division manager with project of quality θ at time t. For now, consider only reports $\hat{\theta} \in \Theta$, i.e., ignore the report "no project available." Given (A32), truth-telling is incentive-compatible if and only if

$$\theta \in \arg\max_{\hat{\theta} \in \Theta} \left\{p\theta H_t\left(\hat{\theta}, 1\right) + (1 - p\theta) H_t\left(\hat{\theta}, 0\right)\right\}. \tag{A33}$$

Equivalently, we can write (A33) as

$$\hat{\theta} \in \arg\max_{\theta \in \Theta} \left\{p\theta H_t\left(\hat{\theta}, 1\right) + (1 - p\theta) H_t\left(\hat{\theta}, 0\right) - \bar{H}_t(\theta)\right\}. \tag{A34}$$

Differentiating the objective in θ, we obtain that at any point at which $\bar{H}_t'(\theta)$ exists, it is given by

$$\bar{H}_t'(\theta) = p(H_t(\theta, 1) - H_t(\theta, 0)) \equiv p\Delta_t(\theta)$$

I refer to $\Delta_t(\theta)$ as the pay-performance sensitivity for the project of quality θ. Assuming that $\Delta_t(\theta)$ is uniformly bounded, $\bar{H}_t(\theta)$ is Lipshitz continuous. Hence, it is differentiable, so

$$\bar{H}_t(\theta) = \bar{H}_t(\underline{\theta}) + p\int_{\underline{\theta}}^{\theta} \Delta_t(\tau)\, d\tau. \tag{A35}$$

This is the first-order condition for the optimality of truth-telling. The second-order condition for the optimization problem (A34) is that $\Delta_t(\theta)$ is nondecreasing. Finally, consider the possibility of reporting that no project is available. Reporting the project truthfully is better for the division manager than concealing the project if and only if $\bar{H}_t(\underline{\theta}) \geq 0$. When the division manager has no project available, truthful reporting is optimal if and only if $0 \geq H_t(\theta, 0)$ for all $\theta \in \Theta$, which can be re-written as $p\theta\Delta_t(\theta) \geq \bar{H}_t(\theta)$. Since $\Delta_t(\theta)$ is non-decreasing, if $p\theta\Delta_t(\theta) \geq \bar{H}_t(\theta)$ holds for $\theta = \underline{\theta}$, then it also holds for any $\theta > \underline{\theta}$. Therefore, truthful reporting that no project is available is optimal if and only if $p\underline{\theta}\Delta_t(\underline{\theta}) \geq \bar{H}_t(\underline{\theta})$.

To sum up, truthful reporting is incentive compatible if and only if the sensitivity of the division manager's utility to her report satisfies (A35) with the initial condition $\bar{H}_t(\underline{\theta}) \in [0, p\underline{\theta}\Delta_t(\underline{\theta})]$ and the pay-performance sensitivity $\Delta_t(\theta)$ is non-decreasing.

Solution to the Optimization Problem

As in the basic model, let $P(W)$ denote the value that headquarters obtain in the optimal mechanism that delivers expected utility W to the division manager. By the same argument, the optimal payment to the division manager at time t is given by zero, if $W_t < W_{ov}^c$, and $W_t - W_{ov}^c$, if $W > W_{ov}^c$, where W_{ov}^c is the lowest W at which $P'(W) = -1$. In particular, this implies that $P(W) = P(W_{ov}^c) + W - W_{ov}^c$ in the region $W > W^c$.

Consider region $W < W_{ov}^c$. The expected flow of value to headquarters over the next instant is $\lambda dt \int_{\underline{\theta}}^{\bar{\theta}} (\theta V(dK_t) - dK_t)\, dF(\theta)$. To evaluate the expected change in $P(W_t)$, I apply Itô's lemma:

$$\mathbb{E}[dP(W_t)] = \left(\rho W_{t-} - \left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} \bar{H}_t(\theta)\, dF(\theta)\right)\right) P'(W_{t-})\, dt$$
$$+\lambda dt \int_{\underline{\theta}}^{\bar{\theta}} [p\theta P(W_{t-} + H_t(\theta,1) - \gamma dK_t) + (1-p\theta) P(W_{t-} + H_t(\theta,0) - \gamma dK_t) - P(W_{t-})]\, dF(\theta).$$

The difference compared to (9) is in the second term on the right side of the equation. For each project type θ, it contains two terms, one corresponding to the case in which the immediate success is realized and the other corresponding to the case in which it is not realized. Equating the sum of $\mathbb{E}[dP(W_t)]$ and the expected flow of value with $rP(W_t)$ and using $H_t(\theta,1) = \overline{H}_t(\theta) + (1-p\theta)\Delta_t(\theta)$ and $H_t(\theta,0) = \overline{H}_t(\theta) - p\theta\Delta_t(\theta)$, I obtain the HJB equation for the headquarters' value function $P(W)$:

$$(r+\lambda) P(W) = \max_{\{k_\theta, h_\theta, \delta_\theta\}} \Big\{ \lambda \int_{\underline{\theta}}^{\bar{\theta}} (\theta v(k_\theta) - k_\theta)\, dF(\theta)$$
$$+ \left(\rho W - \lambda \int_{\underline{\theta}}^{\bar{\theta}} h_\theta dF(\theta)\right) P'(W)$$
$$+ \lambda \int_{\underline{\theta}}^{\bar{\theta}} [p\theta P(W + h_\theta + (1-p\theta)\delta_\theta - \gamma k_\theta) + (1-p\theta) P(W + h_\theta - p\theta\delta_\theta - \gamma k_\theta)]\, dF(\theta) \Big\}, \tag{A36}$$

where the maximization is subject to the four constraints: (1) $k_\theta \geq 0$; (2) $h_\theta = \underline{h} + p\int_{\underline{\theta}}^{\theta} \delta_\tau d\tau$;

(3) δ_θ is non-decreasing in θ; (4) $0 \leq \underline{h} \leq p\underline{\theta}\delta_{\underline{\theta}}$.

This problem is solved using techniques from the optimal control theory. Intuitively, think of $\theta \in [\underline{\theta}, \overline{\theta}]$ as time, h_θ and δ_θ as state variables, and $y_\theta = \delta'_\theta$ and k_θ as the control variables. I focus on piecewise continuously differentiable δ_θ, as typical in control theory. Moving $\rho W P'(W)$ from (A36) on the left-hand side and dividing by λ, the optimization program can be written as:[3]

$$\max_{\{k_\theta, h_\theta, \delta_\theta, y_\theta\}} \int_{\underline{\theta}}^{\overline{\theta}} J(k_\theta, h_\theta, \delta_\theta, \theta, W) f(\theta) \, d\theta \tag{A37}$$

subject to

$$h'_\theta = p\delta_\theta \, \forall \theta \tag{A38}$$

$$\delta'_\theta = y_\theta \, \forall \theta \tag{A39}$$

$$y_\theta \geq 0 \, \forall \theta, \tag{A40}$$

where

$$J(k, h, \delta, \theta, W) = \theta v(k) - k - hP'(W) + p\theta P(W + h + (1 - p\theta)\delta - \gamma k) + (1 - p\theta) P(W + h - p\theta\delta - \gamma k)$$

is the virtual surplus of headquarters for the type-θ project. This is a bounded control problem (e.g., see Kamien and Schwartz (1991)). Note that W is a parameter in this problem, and since $P(\cdot)$ is concave, $J(\cdot)$ is pseudo-concave in the policy variables (k, h, δ). Let μ_θ and ν_θ denote the shadow values of the transition equations for h_θ and δ_θ, respectively. Then, the Hamiltonian is

$$\mathcal{H}(\theta, h_\theta, \delta_\theta, y_\theta, k_\theta, \mu_\theta, \nu_\theta, W) = f(\theta) J(k_\theta, h_\theta, \delta_\theta, \theta, W) + \mu_\theta \delta_\theta + \nu_\theta y_\theta.$$

The first-order conditions with respect to h_θ and δ_θ are:

$$-\mu'_\theta = f(\theta) \begin{pmatrix} p\theta P'(W + h_\theta + (1 - p\theta)\delta_\theta - \gamma k_\theta) \\ + (1 - p\theta) P'(W + h_\theta - p\theta\delta_\theta - \gamma k_\theta) - P'(W) \end{pmatrix}, \tag{A41}$$

$$-\nu'_\theta = \mu_\theta + f(\theta) p\theta (1 - p\theta) \begin{pmatrix} P'(W + h_\theta + (1 - p\theta)\delta_\theta - \gamma k_\theta) \\ -P'(W + h_\theta - p\theta\delta_\theta - \gamma k_\theta) \end{pmatrix}. \tag{A42}$$

In addition, according to Portryagin's maximum principle, $\mathcal{H}(\cdot)$ should be maximized with respect to control variables y_θ and k_θ. Therefore

$$\nu_\theta \leq 0, \text{ with } \nu_\theta y_\theta = 0,$$

$$\theta v'(k_\theta) = 1 + \gamma \begin{pmatrix} p\theta P'(W + h_\theta + (1 - p\theta)\delta_\theta - \gamma k_\theta) \\ + (1 - p\theta) P'(W + h_\theta - p\theta\delta_\theta - \gamma k_\theta) \end{pmatrix}. \tag{A43}$$

Consider, first, types at which the monotonicity constraint does not bind ($\theta : y_\theta > 0$). The complementary slackness condition $v_\theta y_\theta = 0$ implies that $v_\theta = 0$. Therefore, when

[3] W is a parameter in this optimization problem, so I do not write it as one of the arguments of $J(\cdot)$.

$y_\theta > 0$ over an interval, the first-order condition with respect to δ_θ becomes:

$$f(\theta)\, p\theta(1-p\theta)\left(P'(W+h_\theta+(1-p\theta)\delta_\theta-\gamma k_\theta)-P'(W+h_\theta-p\theta\delta_\theta-\gamma k_\theta)\right)=-\mu_\theta. \tag{A44}$$

Therefore, if the monotonicity constraint does not bind, optimal policies k_θ and δ_θ are given by (A43) and (A44), given h_θ and μ_θ. Transition equations (A38) and (A41) pin down the increments to h_θ and μ_θ.

Second, consider types at which the monotonicity constraint binds ($\theta : y_\theta = 0$). In this case, optimal policies are obtained via the "ironing" procedure. Consider a (maximal) interval of types $[\theta_1, \theta_2]$ over which the monotonicity constraint binds. Then, $\delta_\theta = \delta_{\theta_1}$ $\forall\theta \in [\theta_1, \theta_2]$. The transition equation (A38) implies $h_\theta = h_{\theta_1} + p(\theta-\theta_1)\delta_{\theta_1}$ $\forall\theta \in [\theta_1, \theta_2]$. Thus, the equation for k_θ simplifies to

$$\theta v'(k_\theta) = 1+\gamma\left(p\theta P'(W+h_{\theta_1}+(1-p\theta_1)\delta_{\theta_1}-\gamma k_\theta)+(1-p\theta)P'(W+h_{\theta_1}-p\theta_1\delta_{\theta_1}-\gamma k_\theta)\right). \tag{A45}$$

Since the monotonicity constraint does not bind at the ends of the interval, we have $\nu_{\theta_1} = \nu_{\theta_2} = 0$. Using (A42), this implies

$$\int_{\theta_1}^{\theta_2}\left(\mu_\theta + f(\theta)\, p\theta(1-p\theta)\left(P'(W+h_{\theta_1}+(1-p\theta_1)\delta_{\theta_1}-\gamma k_\theta)-P'(W+h_{\theta_1}-p\theta_1\delta_\theta-\gamma k_\theta)\right)\right)d\theta = 0. \tag{A46}$$

Therefore, for an interval of types $[\theta_1, \theta_2]$ at which the monotonicity constraint does not bind, investment k_θ is given by (A45), $\delta_\theta = \delta_{\theta_1}$, $h_\theta = h_{\theta_1} + p(\theta-\theta_1)\delta_{\theta_1}$. Transition equations (A38) and (A42) pin down the dynamics of μ_θ and ν_θ over $[\theta_1, \theta_2]$ with the initial value conditions $\nu_{\theta_1} = 0$ and μ_{θ_1}, coming from the range at which the monotonicity constraint does not bind. The bounds of the interval $[\theta_1, \theta_2]$ are pinned down by $\delta_{\theta_1} = \delta_{\theta_2}$ and (A46).

Having characterized the solution to the optimal control problem (A37)-(A40), denote it by $k_{ov}(\theta, W)$, $h_{ov}(\theta, W)$, and $\delta_{ov}(\theta, W)$. Eq. (A32) implies that if no project is reported and $W_t < W^c$, the evolution of W_t is $dW_t = g_{ov}(W_t)\, W_t dt$, where

$$g_{ov}(W) \equiv \rho - \lambda \int_{\underline{\theta}}^{\bar{\theta}} \frac{h_{ov}(\theta, W)}{W} dF(\theta). \tag{A47}$$

Implementation

Define the bonus and fine parameters in the performance-sensitive budgeting mechanism:

$$B_t^+ = B^+(k, B_{t-}) = \frac{h_{ov}\left(k_{ov}^{-1}(k,\gamma B_{t-}),\gamma B_{t-}\right)+\left(1-pk_{ov}^{-1}(k,\gamma B_{t-})\right)\delta_{ov}\left(k_{ov}^{-1}(k,\gamma B_{t-}),\gamma B_{t-}\right)}{\gamma} \tag{A48}$$

$$B_t^- = B^-(k, B_{t-}) = \frac{pk_{ov}^{-1}(k,\gamma B_{t-})\,\delta_{ov}\left(k_{ov}^{-1}(k,\gamma B_{t-}),\gamma B_{t-}\right)-h_{ov}\left(k_{ov}^{-1}(k,\gamma B_{t-}),\gamma B_{t-}\right)}{\gamma}. \tag{A49}$$

Also, define the accumulation limit B_{ov}^c as:

$$B_{ov}^c \equiv \frac{W_{ov}^c}{\gamma}. \tag{A50}$$

First, I show that the evolution of γB_t is the same as the evolution of W_t in the optimal direct mechanism if the investment policies are the same. The starting point is $\gamma B_0 = W_0$ and the evolution of γB_t if $B_t < B_{ov}^c$ and there is no arrival of the investment project is the same as W_t. Consider the evolution of γB_t if $B_t < B_{ov}^c$ and the project of type θ arrives. If the project results in immediate verifiable success,

$$d\left(\gamma B_t\right) = g_{ov}\left(\gamma B_t\right) B_t dt + \gamma \frac{h_{ov}\left(\theta, W\right) + \left(1 - p\theta\right) \delta_{ov}\left(\theta, W\right)}{\gamma}.$$

Otherwise,

$$d\left(\gamma B_t\right) = g_{ov}\left(\gamma B_t\right) B_t dt + \gamma \frac{h_{ov}\left(\theta, W\right) - p\theta \delta_{ov}\left(\theta, W\right)}{\gamma}.$$

Hence, the evolutions of γB_t and W_t are the same if investment policies are the same. Second, I show that the implied investment policy in the budgeting mechanism is the same as the investment policy in the optimal direct mechanism. Suppose that a project of type θ arrives to the division manager. The investment amount satisfies:

$$\begin{aligned}\max\nolimits_k\{&\theta v\left(k\right) + p\theta P\left(\gamma\left(B_t - k\right) + h_{ov}\left(k_{ov}^{-1}\left(k, \gamma B_t\right), \gamma B_t\right) + \left(1 - p k_{ov}^{-1}\left(k, \gamma B_{t-}\right)\right) \delta_{ov}\left(k_{ov}^{-1}\left(k, \gamma B_t\right), \gamma B_t\right)\right) \\ &+ \left(1 - p\theta\right) P\left(\gamma\left(B_t - k\right) + h_{ov}\left(k_{ov}^{-1}\left(k, \gamma B_t\right), \gamma B_t\right) - p k_{ov}^{-1}\left(k, \gamma B_{t-}\right) \delta_{ov}\left(k_{ov}^{-1}\left(k, \gamma B_t\right), \gamma B_t\right)\right)\}.\end{aligned}$$

Since truth-telling is optimal in the direct problem, $k = k_{ov}\left(\theta, W\right)$ solves this problem.

Special cases

It can be interesting to consider two special cases, $p = 1$ and $p \to 0$. First, consider the case of $p = 1$, i.e., when the project of quality θ results in immediate verifiable success with probability θ. In this case, we can plug $p = 1$ in the optimization problem above, and the solution is conceptually similar to the case of $p \in (0, 1)$. Second, consider the case of $p \to 0$, i.e., when the project results in immediate verifiable success with an infinitesimal probability. In this limit case, the transition equation (A38) implies that $h_\theta = \underline{h}$ for any θ. In addition, recall that truthful reporting that no project is available requires $\underline{h} \in (0, p\underline{\theta}\delta_{\underline{\theta}})$, which in this limit case implies $\underline{h} = 0$. Thus, the HJB equation simplifies to

$$\left(r + \lambda\right) P\left(W\right) = \max_{\{k_\theta\}} \left\{ \lambda \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta v\left(k_\theta\right) - k_\theta\right) dF\left(\theta\right) + \rho W P'\left(W\right) + \lambda \int_{\underline{\theta}}^{\bar{\theta}} P\left(W - \gamma k_\theta\right) dF\left(\theta\right) \right\}, \tag{A51}$$

which coincides with the HJB equation (10) in the basic model in the case of a prohibitively expensive audit. Also notice that B_t^- converges to zero, implying that the "fine" for lack of verifiable immediate success, which in the limit case occurs with certainty, becomes non-existent. Thus, in the limit case of project values being almost unobservable, the solution of the model with observable realized values approximates the solution of the basic model.

IV. Random Auditing

In the basic model, I assumed that headquarters can only commit to pure audit strategies. This assumption follows classic costly state verification models (Townsend, 1979; Gale and

Hellwig, 1985), but it is with loss of generality: headquarters could do better by committing to random audit strategies. In this section, I consider the basic model with the additional change that headquarters can commit to any random audit strategy.

Consider time t. Depending on report $d\hat{X}_t$, the mechanism prescribes headquarters to audit with probability $q_t\left(d\hat{X}_t\right) \in [0,1]$, where function $q_t(\cdot)$ is measurable with respect to $\left\{d\hat{X}_s, s \leq t, dX_s, s < t : dA_s = 1\right\}$. As in the basic model, because audit is costly, it is not optimal to audit if the division manager reports that there is no project available: $q_t(0) = 0$. The analysis is unchanged up to Lemma 1 specifying the evolution of the division manager's promised utility W. Since for each report $d\hat{X}_t$, audit may or may not occur, the sensitivity of the division manager's promised utility to her report may be contingent not only on the report itself, but also on whether it is audited. Let $H_t^N\left(d\hat{X}_t\right)$ denote the sensitivity of the division manager's utility to her report if it does not get audited. Similarly, let $H_t^A\left(dX_t, d\hat{X}_t\right)$ denote the sensitivity of the division manager's utility to her report if it gets audited and headquarters learn the true project quality $dX_t \in \{0\} \cup \Theta$. Since audit reveals project quality with certainty and lying does not occur on equilibrium path, it is without loss of generality to impose maximum punishment if the audit reveals that the division manager's report is not truthful. Therefore, $H_t^A\left(dX_t, d\hat{X}_t\right) = -W_{t-}$ for $dX_t \neq d\hat{X}_t$. The following lemma is an analogue of Lemma 2 for a model with random audit:

Lemma 3. *At any time $t \geq 0$, the evolution of the division manager's promised utility W_t following report $d\hat{X}_t \in \{0\} \cup \Theta$ is*

$$\begin{aligned} dW_t &= \rho W_{t-}dt - \gamma dK_t - dC_t + H_t^N\left(d\hat{X}_t\right)(1 - dA_t) + H_t^A\left(dX_t, d\hat{X}_t\right) dA_t \\ &\quad - \left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} \left(H_t^N(\theta) + q_t(\theta)\left(H_t^A(\theta,\theta) - H_t^N(\theta)\right)\right) dF(\theta)\right) dt. \end{aligned} \tag{A52}$$

Functions $H_t^N(\cdot)$ and $H_t^A(\cdot)$ satisfy: (i) $H_t^N(0) = 0$ and $H_t^A(0,0) = 0$; (ii) for any fixed $\theta \in \{0\} \cup \Theta$, $H_t^N(\theta)$ is $\mathcal{F}$-predictable; (iii) for any fixed $\hat{\theta} \in \{0\} \cup \Theta$ and $\theta \in \{0\} \cup \Theta$, $H_t^A\left(\theta, \hat{\theta}\right)$ is $\mathcal{F}$-predictable.

The proof of Lemma 5 is identical to the proof of Lemma 2 with the change that the change in the lifetime expected utility of the division manager is driven by two functions, one corresponding to the case of an audited report and the other corresponding to the case of an unaudited report.

In the optimal mechanism, the division manager finds it optimal to send a truthful report: $d\hat{X}_t = dX_t$. Depending on report $d\hat{X}_t$, headquarters audit it with probability $q_t\left(d\hat{X}_t\right)$. Because audit is costly, it is never optimal to audit if the division manager reports that there is no project: $q_t(0) = 0$. Sending report $d\hat{X}_t \neq dX_t$ is suboptimal if and only if

$$H_t^N\left(d\hat{X}_t\right)\left(1 - q_t\left(d\hat{X}_t\right)\right) - q_t\left(d\hat{X}_t\right) W_{t-} \leq H_t^N(dX_t)(1 - q_t(dX_t)) + q_t(dX_t) H_t^A(dX_t, dX_t). \tag{A53}$$

This inequality must hold for all dX_t and $d\hat{X}_t \neq dX_t$ in $\{0\} \cup \Theta$. The intuition for (A53) is as follows. The right-hand side is the expected change in the utility if the division manager sends a truthful report dX_t. If the report is not audited (probability $1 - q_t(dX_t)$), the division manager's expected utility changes by $H_t^N(dX_t)$; if the report is audited (probability $q_t(dX_t)$), the division manager's expected utility changes by $H_t^N(dX_t, dX_t)$. The left-hand side of the inequality denotes the expected change in the division manager's utility if she sends report $d\hat{X}_t \neq dX_t$. This report gets audited with probability $q_t\left(d\hat{X}_t\right)$. If the report is not audited, the division manager's expected utility changes by $H_t^N\left(d\hat{X}_t\right)$; if the report is audited, headquarters learn dX_t, so the division manager's expected utility drops to zero.

Let $P_{ra}(W)$ denote the value that headquarters obtain in the optimal mechanism in the model with random auditing. Becase $P_{ra}(W)$ must be concave, the optimal compensation policy is given by threshold W_{ra}^c, defined by the lowest point at which $P'(W_{ra}^c) = -1$. Consider region $W < W_{ra}^c$. The expected instantaneous change in headquarters' value function is $rP_{ra}(W_{t-})\,dt$. It must be equal to the sum of the expected flow of value over the next instant and the change in $P_{ra}(W_t)$ due to the evolution of W_t. Since zero investment is optimal if the division manager reports that no project arrives, the expected flow of value of an instant is

$$\lambda\left(\int_{\underline{\theta}}^{\bar{\theta}} \left(V(dK_t, \theta) - dK_t - cq_t(\theta)\right) dF(\theta)\right) dt. \tag{A54}$$

To evaluate the expected instantaneous change in $P_{ra}(W)$, I apply Itô's lemma and use (A52):

$$\begin{aligned}\mathbb{E}\left[dP_{ra}(W_t)\right] = &\left[\begin{array}{c}\rho W_{t-} dt - \\ \left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} \left(H_t^N(\theta) + q_t(\theta)\left(H_t^A(\theta,\theta) - H_t^N(\theta)\right)\right)\right) dF(\theta)\end{array}\right] P'_{ra}(W_{t-}) \\ &+\lambda dt \int_{\underline{\theta}}^{\bar{\theta}} \left[\begin{array}{c}(1 - q_t(\theta))\, P_{ra}\left(W_{t-} + H_t^N(\theta) - \gamma dK_t\right) \\ +q_t(\theta)\, P_{ra}\left(W_{t-} + H_t^A(\theta,\theta) - \gamma dK_t\right) - P_{ra}(W_{t-})\end{array}\right] dF(\theta).\end{aligned} \tag{A55}$$

Combining (A54) with (A55) and equating their sum to $rP_{ra}(W_t)\,dt$ yields the following HJB equation:

$$\begin{aligned}rP_{ra}(W) = \max_{\left\{q_\theta, k_\theta^n, k_\theta^a, h_\theta^n, h_\theta^a\right\}} &\left\{\lambda \int_{\underline{\theta}}^{\bar{\theta}} \left((1 - q_\theta)\left(V(k_\theta^n, \theta) - k_\theta^n\right) + q_\theta\left(V(k_\theta^a, \theta) - k_\theta^a - c\right)\right) dF(\theta)\right. \\ &+ \left[\rho W - \lambda \int_{\underline{\theta}}^{\bar{\theta}} \left((1 - q_\theta) h_\theta^n + q_\theta h_\theta^a\right) dF(\theta)\right] P'_{ra}(W) \\ &\left. + \lambda \int_{\underline{\theta}}^{\bar{\theta}} \left[(1 - q_\theta) P_{ra}(W + h_\theta^n - \gamma k_\theta^n) + q_\theta P_{ra}(W + h_\theta^a - \gamma k_\theta^a) - P_{ra}(W)\right] dF(\theta)\right\},\end{aligned} \tag{A56}$$

where the maximization is subject to $k_\theta^n \geq 0$, $k_\theta^n \geq 0$, $q_\theta \in [0, 1]$, and the incentive compatibility constraints

$$(1 - q_\theta) h_\theta^n + q_\theta h_\theta^a \geq (1 - q_{\hat{\theta}}) h_{\hat{\theta}}^n - q_{\hat{\theta}} W \,\forall \left(\theta, \hat{\theta}\right) \in (\{0\} \cup \Theta)^2. \tag{A57}$$

Compared to the HJB equation in the basic model without random audit, (A56) has one difference. There are two investment levels (k_θ^n and k_θ^a) and two changes in the promised utility (h_θ^n and h_θ^a) for each project type θ, corresponding to cases in which the report does not get audited and gets audited and confirmed to be truthful, respectively. In contrast, in the basic model, each project type is either audited or not, so it is sufficient to specify one investment and one change in the promised utility for each project type θ.

Denote the optimal policies that solve this optimization problem by $k_{ra}^n(\theta, W)$, $k_{ra}^a(\theta, W)$, $h_{ra}^n(\theta, W)$, $h_{ra}^a(\theta, W)$, and $q_{ra}(\theta, W)$. Let $I \equiv \max_{\theta\in\{0\}\cup\Theta}\{(1-q_\theta)h_\theta^n - q_\theta W\}$. Then, (A57) can be written as $(1-q_\theta)h_\theta^n + q_\theta h_\theta^a \geq I$. Let $\tilde{\Theta}$ denote the subset of $\{0\}\cup\Theta$ at which $(1-q_\theta)h_\theta^n - q_\theta W < I$. The derivatives of (A56) with respect to h_θ^a and h_θ^n are proportional to $P'_{ra}(W + h_\theta^a - \gamma k_\theta^a) - P'_{ra}(W)$ and $P'_{ra}(W + h_\theta^n - \gamma k_\theta^n) - P'_{ra}(W)$, respectively. Consider type $\theta \in \tilde{\Theta}$. Since (A57) is slack, (A56) is maximized at $h_\theta^a = \gamma k_\theta^a$ and $h_\theta^n = \gamma k_\theta^n$. Differentiating (A56) with respect to k_θ^n and k_θ^a yields $k_\theta^n = k_\theta^a = k_{ra}^*(\theta, W)$, implicitly defined as

$$\frac{\partial V(k_{ra}^*(\theta, W), \theta)}{\partial k} = 1 + \gamma P'_{ra}(W). \tag{A58}$$

This implies $h_\theta^a = h_\theta^n = \gamma k_{ra}^*(\theta, W)$. The derivative with respect to q_θ is proportional to $-c$. Therefore, for any $\theta \in \tilde{\Theta}$, it must be that $q_\theta = 0$.

Thus, if $q_\theta > 0$, then either $(1-q_\theta)h_\theta^n + q_\theta h_\theta^a = I$ or $(1-q_\theta)h_\theta^n - q_\theta W = I$. Let me show that it cannot be $(1-q_\theta)h_\theta^n + q_\theta h_\theta^a = I$. By contradiction, suppose this is the case. Then, $(1-q_\theta)h_\theta^n - q_\theta W < I$. Denoting the Lagrange multiplier of the equality by μ_θ, the maximization problem becomes

$$\begin{aligned}\max_{h_\theta^a, h_\theta^n, k_\theta^n, k_\theta^a, q_\theta} \quad & (1-q_\theta)\left(V(k_\theta^n, \theta) - k_\theta^n - h_\theta^n P'_{ra}(W) + P_{ra}(W + h_\theta^n - \gamma k_\theta^n)\right) \\ & + q_\theta\left(V(k_\theta^a, \theta) - k_\theta^a - h_\theta^a P'_{ra}(W) - c + P_{ra}(W + h_\theta^a - \gamma k_\theta^a)\right) \\ & + \mu_\theta\left((1-q_\theta)h_\theta^n + q_\theta h_\theta^a - I\right)\end{aligned}$$

This implies $k_\theta^n = k_\theta^a$ and $h_\theta^n = h_\theta^a = I$. However, if this is the case, then the objective function is strictly decreasing in q_θ, implying $q_\theta = 0$, which contradicts $q_\theta > 0$. Hence, if $q_\theta > 0$, then $(1-q_\theta)h_\theta^n - q_\theta W = I$, which implies $(1-q_\theta)h_\theta^n + q_\theta h_\theta^a > I$. Therefore, (A56) is maximized at $h_\theta^a = \gamma k_\theta^a$ and $k_\theta^a = k_{ra}^*(\theta, W)$. Denoting the Lagrange multiplier of the equality by λ_θ, the maximization problem becomes:

$$\begin{aligned}\max_{h_\theta^n, k_\theta^n, q_\theta} \quad & (1-q_\theta)\left(V(k_\theta^n, \theta) - k_\theta^n - h_\theta^n P'_{ra}(W) + P_{ra}(W + h_\theta^n - \gamma k_\theta^n)\right) \\ & + q_\theta\left(V(k_\theta^a, \theta) - (1 + \gamma P'_{ra}(W))k_\theta^a - c + P_{ra}(W)\right) \\ & + \lambda_\theta\left((1-q_\theta)h_\theta^n - q_\theta W - I\right).\end{aligned}$$

Taking the first-order conditions and re-arranging the terms yields:

$$\begin{gathered}\frac{\partial V(k_\theta^n, \theta)}{\partial k} = 1 + \gamma\left(P'_{ra}(W) - \lambda_\theta\right), \\ P'_{ra}(W + h_\theta^n - \gamma k_\theta^n) = P'_{ra}(W) - \lambda_\theta, \\ V(k_\theta^a, \theta) - (1 + \gamma P'_{ra}(W))k_\theta^a - c + P_{ra}(W) - \lambda_\theta W \\ = V(k_\theta^n, \theta) - k_\theta^n - h_\theta^n\left(P'_{ra}(W) - \lambda_\theta\right) + P_{ra}(W + h_\theta^n - \gamma k_\theta^n).\end{gathered}$$

These equations pin down $k_\theta^n = k_{ra}^n(\theta, W)$, $h_\theta^n = h_{ra}^n(\theta, W)$, and λ_θ. Then, $q_\theta = q_{ra}(\theta, W)$

is given by $\frac{h^n_{ra}(\theta,W)-I}{h^n_{ra}(\theta,W)+W}$. We can re-write the last equation as:

$$\begin{array}{c}\max_k \left\{V\left(k,\theta\right)-\left(1+\gamma P'_{ra}\left(W\right)\right)k\right\} \\ -\max_{k,h}\left\{V\left(k,\theta\right)-k+P_{ra}\left(W+h-\gamma k\right)-h\left(P'_{ra}\left(W\right)-\lambda_\theta\right)\right\}=c+\lambda_\theta W-P_{ra}\left(W\right).\end{array} \tag{A59}$$

By the envelope theorem, the derivative of the left-hand side in θ is

$$\int_{k^n_\theta}^{k^*_{ra}(\theta,W)}\frac{\partial^2 V\left(k,\theta\right)}{\partial k\partial\theta}dk-\frac{\partial\lambda_\theta}{\partial\theta}h^n_\theta=0.$$

Since $k^*_{ra}\left(\theta,W\right)>k^n_\theta$ and $\frac{\partial V(k,\theta)}{\partial k\partial\theta}>0$ by part (c) of Assumption 1, λ_θ is increasing in θ. By Topkis's theorem applied for the second maximization problem in (A59), h^n_θ is increasing in θ. Since $q_\theta=\frac{h^n_\theta-I}{h^n_\theta+W}$ and h^n_θ is increasing in θ, q_θ is increasing in θ too.

Next, I find I. Above, I have shown that $\left(1-q_\theta\right)h^n_\theta-q_\theta W=I$ for any $\theta:q_\theta>0$. This and (A57) imply that $h^n_\theta=I$ for any $\theta:q_\theta=0$. In particular, this must hold for $\theta=0$, which implies $I=0$. Hence, $q_\theta=\frac{h^n_{ra}(\theta,W)}{h^n_{ra}(\theta,W)+W}$.

It remains to find when $q_\theta=0$ or $q_\theta>0$. If $q_\theta=0$, then $h^n_\theta=0$ and the investment k^n_θ solves

$$\frac{\partial V\left(k^n_\theta,\theta\right)}{\partial k}=1+\gamma P'_{ra}\left(W-\gamma k^n_\theta\right).$$

Denote the solution to this equation by $k^0_{ra}\left(\theta,W\right)$. Differentiating (A56) in q_θ, we obtain that $q_\theta=0$ is optimal if and only if

$$\begin{array}{rl} & V\left(k^*_{ra}\left(\theta,W\right),\theta\right)-\left(1+\gamma P'_{ra}\left(W\right)\right)k^*_{ra}\left(\theta,W\right)-c-\left(P'_{ra}\left(W\right)-P'_{ra}\left(W-\gamma k^0_{ra}\left(\theta,W\right)\right)\right)W \\ < & V\left(k^0_{ra}\left(\theta,W\right),\theta\right)-k^0_{ra}\left(\theta,W\right)+P_{ra}\left(W-\gamma k^0_{ra}\left(\theta,W\right)\right)-P_{ra}\left(W\right),\end{array}$$

or, equivalently,

$$c>F^a_{ra}\left(\theta,W\right)-F^n_{ra}\left(\theta,W\right)+\left(P'_{ra}\left(W-\gamma k^0_{ra}\left(\theta,W\right)\right)-P'_{ra}\left(W\right)\right)W, \tag{A60}$$

where

$$\begin{array}{rcl} F^a_{ra}\left(\theta,W\right) & \equiv & \max_{k\in\mathbb{R}_+}\left\{V\left(k,\theta\right)-\left(1+\gamma P'_{ra}\left(W\right)\right)k\right\}, \\ F^n_{ra}\left(\theta,W\right) & \equiv & \max_{k\in\mathbb{R}_+}\left\{V\left(k,\theta\right)-k+P_{ra}\left(W-\gamma k\right)-P_{ra}\left(W\right)\right\},\end{array}$$

by analogy with (26)–(27). Using the envelope theorem, it is easy to see that the right-hand side of (A60) is increasing in θ:

$$\int_{k^0_{ra}(\theta,W)}^{k^*_{ra}(\theta,W)}\frac{\partial^2 V\left(k,\theta\right)}{\partial k\partial\theta}dk-\gamma P''_{ra}\left(W\right)W\frac{\partial k^0_{ra}\left(\theta,W\right)}{\partial\theta},$$

which is positive, since $\frac{\partial^2 V(k,\theta)}{\partial k\partial\theta}>0$ by Assumption 1 and $P''_{ra}\left(W\right)\leq 0$ and $\frac{\partial k^0_{ra}(\theta,W)}{\partial\theta}\geq 0$ by concavity of $P_{ra}\left(\cdot\right)$. Therefore, $q_\theta=0$ if and only if $\theta<\theta^*_{ra}\left(W\right)$, where $\theta^*\left(W\right)$ is defined as the lowest $\theta\in\Theta$ at which $c\leq F^a_{ra}\left(\theta,W\right)-F^n_{ra}\left(\theta,W\right)+\left(P'_{ra}\left(W-\gamma k^0_{ra}\left(\theta,W\right)\right)-P'_{ra}\left(W\right)\right)W$.

Plugging the solution into (A52), we obtain the evolution of W_t following the report $d\hat{X}_t = 0$:

$$dW_t = g_{ra}(W_{t-}) W_{t-}dt - dC_t,$$

where

$$g_{ra}(W) = \rho - \lambda \int_{\theta^*_{ra}(W)}^{\bar{\theta}} \frac{(\gamma k^*_{ra}(\theta) + W) h^n_{ra}(\theta, W)}{W(h^n_{ra}(\theta, W) + W)} dF(\theta). \tag{A61}$$

Combining these results characterizes the investment, audit, and the evolution of the division manager's promised utility in the optimal mechanism. If the division manager reports that no project arrives, then her promised utility accumulates continuously at rate $g_{ra}(W_{t-})$, provided that $W_{t-} < W^c_{ra}$. Once it reaches W^c_{ra}, the division manager gets paid a flow of constant bonus payments of $g_{ra}(W^c_{ra}) W^c_{ra}$ per unit time until the division manager reports an arrival of a project. If the division manager reports a project with quality $\theta < \theta^*_{ra}(W_{t-})$, headquarters do not audit the report, the firm invests $k^0_{ra}(\theta, W_{t-})$, and the post-investment promised utility of the division manager changes by $dW_t = -\gamma k^0_{ra}(\theta, W_{t-})$. If the division manager reports a project with quality $\theta \geq \theta^*_{ra}(W_{t-})$, the report is audited with probability $\frac{h^n_{ra}(\theta, W_{t-})}{h^n_{ra}(\theta, W_{t-}) + W_{t-}}$. If the report is audited and audit confirms that the project's quality is θ, amount $k^*_{ra}(\theta, W_{t-})$ is invested and the post-investment promised utility of the division manager is kept constant at W_{t-}. If the report is audited and the audit does not confirm that the quality of the project is θ (this never occurs on equilibrium path), then nothing is invested and the division manager's promised utility is set to zero. Finally, if the report is not audited, the firm invests $k^n_{ra}(\theta, W_{t-})$, and the post-investment promised utility of the division manager changes by $dW_t = -\gamma k^n_{ra}(\theta, W) + h^n_{ra}(\theta, W)$. The following proposition summarizes these findings:[4]

Proposition 2. *The following mechanism is optimal. If $R \leq W^c_{ra}$, then the initial value W_0 is $\max\{R, W^*_{ra}\}$, where W^*_{ra} is the point at which $P_{ra}(W)$ is maximized. If $R > W^c_{ra}$, then an immediate payment of $R - W^c_{ra}$ is made to the division manager and $W_0 = W^c_{ra}$. At any t, the division manager sends a report $d\hat{X}_t$ from message space $\{0\} \cup \Theta$.*

1. *If $d\hat{X}_t = 0$, then $dK_t = 0$ and $dA_t = 0$. If $W_{t-} < W^c_{ra}$, then*

$$dW_t = g_{ra}(W_{t-}) W_{t-}dt, \tag{A62}$$

and $dC_t = 0$. If $W_{t-} = W^c_{ra}$, then $dW_t = 0$ and $dC_t = g_{ra}(W^c_{ra}) W^c_{ra}dt$.

2. *If $d\hat{X}_t \in [\underline{\theta}, \theta^*_{ra}(W_{t-}))$, then $dA_t = 0$, $dK_t = k^0_{ra}(\theta, W_{t-})$, and $dW_t = -\gamma dK_t$.*

3. *If $d\hat{X}_t \in [\theta^*_{ra}(W_{t-}), \bar{\theta}]$, then $dA_t = 1$ with probability $\frac{h^n_{ra}(\theta, W_{t-})}{h^n_{ra}(\theta, W_{t-}) + W_{t-}}$ and $dA_t = 0$ with probability $\frac{W_{t-}}{h^n_{ra}(\theta, W_{t-}) + W_{t-}}$.*

   (a) If $dA_t = 1$, then $dK_t = k^*_{ra}\left(d\hat{X}_t, W_{t-}\right)$ and $dW_t = 0$, if the audit reveals that $dX_t = d\hat{X}_t$. Otherwise, $dK_t = 0$ and $dW_t = -W_{t-}$.

[4] The verification is very similar to the proof of Proposition 2, so I omit it.

(b) If $dA_t = 0$, then $dK_t = k^n_{ra}\left(d\hat{X}_t, W_{t-}\right)$ and $dW_t = h^n_{ra}(\theta, W_{t-}) - \gamma dK_t$.

The following proposition shows how the above mechanism can be implemented via a capital budgeting process, which is similar in many dimensions to the budgeting mechanism with threshold separation of financing in the basic model:

Proposition 3. *Consider the following mechanism. At the initial date, headquarters allocate a spending account $B_0 = W_0/\gamma$ to the division manager, who is allowed to use this account at her discretion to invest in projects. At time t, the spending account is replenished at rate $g_{ra}(\gamma B_t)$. In addition, headquarters specify a project size threshold $k^*_t = k^a_{ra}(\theta^*_{ra}(\gamma B_t), \gamma B_t)$, such that at any time t the division manager can pass the project to headquarters claiming that the investment should be $k^a_{ra}(\theta, \gamma B_{t-}) \geq k^*_{t-}$, where θ is the quality of the current project. If the division manager passes the project claiming the optimal investment should be $k^a_{ra}(\theta, \gamma B_{t-})$, it gets audited with probability $\frac{h^n_{ra}(\theta,\gamma B_{t-})}{h^n_{ra}(\theta,\gamma B_{t-})+\gamma B_{t-}}$. If the audit reveals that the report is truthful, then headquarters invest $k^a_{ra}(\theta, \gamma B_{t-})$ and do not alter the account balance. If the audit reveals that the report is not truthful, then headquarters punish the division manager by reducing her spending account balance to zero. If the project is not audited, headquarters invest $h^n_{ra}(\theta, \gamma B_{t-})/\gamma$ and the division manager is offered to co-invest any extra amount from her spending account. The monetary compensation of the division manager is $dC_t = 0$, if $B_t < W^c_{ra}/\gamma$, and $dC_t = g_{ra}(W^c_{ra}) W^c_{ra} dt$, if $B_t = W^c_{ra}/\gamma$. Then, this mechanism implements the optimal mechanism of Proposition 6. In particular, the division manager finds it optimal to (i) pass a project to headquarters if and only if $\theta \geq \theta^*_{ra}(\gamma B_{t-})$ and report the asked investment amount truthfully; (ii) allocate investment account between current and future investment opportunities in the way that maximizes headquarters' value. Specifically, the division manager finds it optimal to invest dK_t that maximizes $V(dK_t, \theta) + P_{ra}(\gamma(B_{t-} - dK_t))$ in projects of quality $\theta < \theta^*_{ra}(\gamma B_{t-})$ and to co-invest amount $dK_t - h^n_{ra}(\theta, \gamma B_{t-})/\gamma$ that maximizes $V(dK_t, \theta) + P_{ra}(\gamma(B_{t-} - dK_t) + h^n_{ra}(\theta, \gamma B_{t-}))$.*

This proposition can be proven in the same way as Proposition 3. The evolution of γB_t is the same as the evolution of W_t in Proposition 6. The starting point is $\gamma B_0 = W_0$ by construction and the evolution of γB_t if $B_t < W^c_{ra}/\gamma$ and the project is not passed to headquarters is

$$d(\gamma B_t) = (g_{ra}(\gamma B_t) B_t dt - dK_t)\gamma. \tag{A63}$$

Hence, the evolutions of γB_t and W_t are the same if the investment policies are the same. Because the change in the division manager's utility, $dW_t + \gamma dK_t = g_{ra}(W_t) W_t dt$, does not depend on dK_t, allocating the spending account between the current and future investment opportunities in the way that maximizes $V(dK_t, \theta) + P_{ra}(\gamma(B_{t-} - dK_t))$ is optimal for the division manager. This investment solves

$$\max_k \{V(\theta, k) + P_{ra}(\gamma(B_{t-} - k))\}, \tag{A64}$$

yielding $k^0_{ra}(\theta, \gamma B_{t-})$. Similarly, because the division manager's expected utility after headquarters invest $h^n_{ra}(\theta, \gamma B_{t-})/\gamma$ does not depend on any additional investment she adds from the spending account, the total investment in the project $\theta \geq \theta^*_{ra}(\gamma B_{t-})$ solves

$$\max_k \{V(\theta, k) + P_{ra}(\gamma(B_{t-} - k) + h^n_{ra}(\theta, \gamma B_{t-}))\},$$

yielding $k_{ra}^{n}(\theta, \gamma B_{t-})$.

V. Private Savings of the Division Manager

The basic model assumes that the agent consumes the monetary transfer immediately upon receiving it. In other words, the division manager cannot save the compensation for the future. In this section, I show that this assumption is not material provided that the division manager has no savings at the start of the game and her savings account grows at rate r, i.e., at the discount rate of headquarters, which is below the discount rate of the division manager ρ.

To show this, I introduce additional notation. Let S_t denote the division manager's balance on the savings account at time t, and let $\left(d\tilde{C}_t\right)_{t\geq 0}$ denote the stochastic process governing the division manager's consumption ($d\tilde{C}_t$ represents her consumption at time t). Then, the evolution of the division manager's savings account is

$$dS_t = rS_t dt + dC_t - d\tilde{C}_t. \tag{A65}$$

The starting value is $S_0 = 0$. The division manager must maintain a nonnegative balance on the savings account, $S_t \geq 0$.

First, consider the case in which the division manager's savings are contractible. In this case, a direct mechanism Γ is described by a quadruple (A, K, C, S) of stochastic processes (the consumption process $\left(d\tilde{C}_t\right)_{t\geq 0}$ is implied by $(dC_t)_{t\geq 0}$ and $(dS_t)_{t\geq 0}$. As before, processes $(dK_t)_{t\geq 0}$ and $\left(d\tilde{C}_t\right)_{t\geq 0}$ must satisfy $dK_t \geq 0$ and $d\tilde{C}_t \geq 0$, i.e., investment and consumption of the division manager must be non-negative. Note, however, that monetary transfers dC_t may be negative, i.e., the mechanism may specify the division manager to make transfers to headquarters from her savings account provided that the savings account balance does not go negative. Consider any incentive compatible mechanism $\Gamma = (A, K, C, S)$. Let $\tilde{C}$ denote the consumption process implies by C and S: $d\tilde{C}_t = rS_t dt + dC_t - dS_t$ for any t. Consider another mechanism $\Gamma' = (A, K, C', 0)$ with $C' = \tilde{C}$. Because mechanism Γ' results in the same investment and consumption of the division manager as mechanism Γ, it is also incentive compatible and results in the same payoff of the division manager. Because $S_0 = 0$, (A65) implies:

$$\mathbb{E}\left[\int_0^\infty e^{-rt} dC_t\right] = \mathbb{E}\left[\int_0^\infty e^{-rt} d\tilde{C}_t\right] + \mathbb{E}\left[\lim_{t\to\infty} e^{-rt} S_t\right].$$

Therefore, $\mathbb{E}\left[\int_0^\infty e^{-rt} dC_t\right] \geq \mathbb{E}\left[\int_0^\infty e^{-rt} d\tilde{C}_t\right]$, so mechanism Γ' also results in the expected payoff to the headquarters that is not lower than mechanism Γ. Therefore, it is without loss of generality to assume that the division manager has no savings if her savings are contractible.

Second, consider the case in which the division manager's savings are hidden. Consider any incentive-compatible direct mechanism $\Gamma = (A, K, C)$, and let $\left(S, \tilde{C}\right)$ be the saving and consumption policies chosen by the division manager given Γ. Since the outcome of this problem can be replicated as an outcome of the problem in which the division manager's savings are contactible and the contract is (A, K, C, S), the optimal mechanism in the problem with hidden savings results in weakly lower payoff to the headquarters than the optimal mechanism in the problem with contractible savings, which is equivalent to the optimal mechanism in the basic model by the argument in the previous paragraph. Lastly, I argue that the optimal mechanism in the basic model remains incentive compatible even if we relax the assumption that the division manager does not save monetary transfers. The proof of this point follows the argument in the proof of Proposition 2 in DeMarzo and Sannikov (2006). Consider an arbitrary feasible reporting and consumption strategy of the division manager, $\left(\hat{X}, \tilde{C}\right)$. Let $\hat{U}_t$ denote the lifetime expected utility of the division manager, evaluated as of time t, that the division manager attained if she consumed the outstanding savings account balance S_t immediately:

$$\hat{U}_t = \int_0^t e^{-\rho s} \left(\gamma dK_s + d\tilde{C}_s\right) + e^{-\rho t} \left(S_t + W_t\right).$$

I show that $\hat{U}_t$ is a supermartingale. Using (A65), we can write $d\hat{U}_t$ as:

$$e^{\rho t} d\hat{U}_t = \gamma dK_t + d\tilde{C}_t + dS_t - \rho S_{t-} dt + dW_t - \rho W_{t-} dt.$$

Plugging in (7) and (A65) and simplifying the terms,

$$e^{\rho t} d\hat{U}_t = (r - \rho) S_t dt + H_t(d\hat{X}_t) - \left(\lambda \int_{\underline{\theta}}^{\bar{\theta}} H_t(\theta) \, dF(\theta)\right) dt.$$

Because $\rho > r$ and $S_t \geq 0$, $(r - \rho) S_t \leq 0$, so $\hat{U}$ is a supermartingale. If there are no savings and the agent reports truthfully, then $\hat{U}$ is a martingale. Therefore, for any incentive-compatible mechanism from the basic model, the division manager finds it optimal to report truthfully and maintain no savings in the problem where she is able to save in a hidden way. In particular, this is also true for the optimal mechanism in the basic model. Therefore, the mechanism from Proposition 2 is also optimal in the model with hidden savings.