Online Appendix for "The Timing and Method of Payment in Mergers when Acquirers Are Financially Constrained"

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Detailed proof of steps in Proposition 1.

Step 1: Consider $p \leq p_i^*(v_l)$. If $(b_i^*(p, v_l), \alpha_i^*(p, v_l)) \neq (b_i^*(p, v_h), \alpha_i^*(p, v_h))$, then $(b_i^*(p, v_l), \alpha_i^*(p, v_l)) = \left(0, \frac{p}{V(v_l)}\right)$. By contradiction, suppose otherwise. Since $(b_i^*(p, v_l), \alpha_i^*(p, v_l)) \neq (b_i^*(p, v_h), \alpha_i^*(p, v_h))$, offer $(b_i^*(p, v_l), \alpha_i^*(p, v_l))$ reveals that the bidder's type is v_l. Therefore, it must satisfy:

$$\alpha_i^*(p, v_l) V(v_l) + b_i^*(p, v_l) \geq p.$$

Consider a deviation by type v_l from offer $(b_i^*(p, v_l), \alpha_i^*(p, v_l))$ to offer $\left(0, \frac{p}{V(v_l)}\right)$. The value of this offer is p, if perceived as coming from type v_l, and above p, if perceived as coming from type v_h with positive probability. Thus, it satisfies the "no default" condition that its value, evaluated according to the beliefs of the seller, is at least p. However, the payoff from this offer to type v_l is strictly higher:

$$\begin{aligned} V(v_l) - p &\geq V(v_l) - \alpha_i^*(p, v_l) V(v_l) - b_i^*(p, v_l) \\ &> (1 - \alpha_i^*(p, v_l)) V(v_l) - \lambda_i b_i^*(p, v_l). \end{aligned}$$

since $\lambda_i > 1$. Therefore, $(b_i^*(p, v_l), \alpha_i^*(p, v_l)) = \left(0, \frac{p}{V(v_l)}\right)$.

Assumption 1 (CKIC). According to Assumption 1 (CKIC), if bidder i submits offer (b, α) satisfying

$$(1-\alpha) V(v_h) - \lambda_i b \geq \max\{(1 - \alpha_i^*(p, v_h)) V(v_h) - \lambda_i b_i^*(p, v_h), V_o\}, \quad \text{(A1)}$$

$$(1-\alpha) V(v_l) - \lambda_i b < \max\{(1 - \alpha_i^*(p, v_l)) V(v_l) - \lambda_i b_i^*(p, v_l), V_o\}, \quad \text{(A2)}$$

then the seller must believe that bidder i's synergy is v_h. The intuition is as follows. The left-hand sides of (A1) and (A2) are the payoffs of bidder i, if it acquires the target for (b,α), if its synergy is v_h and v_l, respectively. The right-hand sides of (A1) and (A2) are the payoffs of of bidder i with synergy v_h and v_l, respectively, if it follows the equilibrium strategy. Thus, conditions (A2)–(A1) mean that the low-synergy bidder is strictly worse off deviating to offer (b,α), while the high-synergy bidder is potentially better off. According to CKIC, it is unreasonable for the seller to believe that such an offer comes from type v_l, so the seller must believe that it comes from type v_h.

Step 2: Consider $p \leq p_i^*(v_l)$. *If* $(b_i^*(p,v_l), \alpha_i^*(p,v_l)) \neq (b_i^*(p,v_h), \alpha_i^*(p,v_h))$, *then* $(b_i^*(p,v_h), \alpha_i^*(p,v_h)) = \left((1-\gamma_i)p, \frac{p}{V(v_h)}\gamma_i\right)$, *where* $\gamma_i = \left(1 + \frac{1}{\lambda_i - 1}\left(1 - \frac{V(v_l)}{V(v_h)}\right)\right)^{-1}$. Since $(b_i^*(p,v_l), \alpha_i^*(p,v_l)) \neq (b_i^*(p,v_h), \alpha_i^*(p,v_h))$, offer $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$ must satisfy:

$$\alpha_i^*(p,v_h)V(v_h) + b_i^*(p,v_h) \geq p \tag{A3}$$

$$(1-\alpha_i^*(p,v_h))V(v_l) - \lambda_i b_i^*(p,v_h) \leq V(v_l) - p \tag{A4}$$

The first inequality is the condition that the value of offer $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$ is at least p. The second inequality is the condition that type v_l is not better off deviating from $\left(0, \frac{p}{V(v_l)}\right)$ to $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$. Let us show that $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$ must be such that both (A3) and (A4) bind. First, by contradiction suppose that (A3) is slack. If $\alpha_i^*(p,v_h) = 0$, then type v_h is better off deviating to offer $(b_i^*(p,v_h) - \varepsilon, \alpha_i^*(p,v_h))$ for an infinitesimal $\varepsilon > 0$. If $\alpha_i^*(p,v_h) > 0$, consider a deviation by type v_h to $(b_i^*(p,v_h) + \varepsilon_1, \alpha_i^*(p,v_h) - \varepsilon_2)$ for infinitesimal $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, satisfying

$$(1-\alpha_i^*(p,v_h)+\varepsilon_2)V(v_h) - \lambda_i(b_i^*(p,v_h) - \varepsilon_1) > (1-\alpha_i^*(p,v_h))V(v_h) - \lambda_i b_i^*(p,v_h), \tag{A5}$$

$$(1-\alpha_i^*(p,v_h)+\varepsilon_2)V(v_l) - \lambda_i(b_i^*(p,v_h) - \varepsilon_1) < V(v_l) - p. \tag{A6}$$

For example, for an arbitrary infinitesimal $\varepsilon_2 > 0$, let $\varepsilon_1 = \frac{\varepsilon_2}{2\lambda_i}(V(v_h) + V(v_l))$. Then, according to CKIC, the target believes that offer $(b_i^*(p,v_h) + \varepsilon_1, \alpha_i^*(p,v_h) - \varepsilon_2)$ is submitted by type v_h. Since (A3) is slack and ε_1 and ε_2 are infinitesimal, the value of offer $(b_i^*(p,v_h) + \varepsilon_1, \alpha_i^*(p,v_h) - \varepsilon_2)$, as perceived by the target, exceeds p. Furthermore, since (A5) holds, type v_h is better off buying the target for $(b_i^*(p,v_h) + \varepsilon_1, \alpha_i^*(p,v_h) - \varepsilon_2)$ than for $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$, which is a contradiction with the statement that $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$ is the optimal offer for type v_h. Hence, (A3) binds. Second, by contradiction suppose that (A4) is slack. Since $(b_i^*(p,v_l), \alpha_i^*(p,v_l)) \neq (b_i^*(p,v_h), \alpha_i^*(p,v_h))$ and $b_i^*(p,v_l) = 0$, it must be that $b_i^*(p,v_h) > 0$. Consider a deviation to $(b_i^*(p,v_h) - \varepsilon V(v_h), \alpha_i^*(p,v_h) + \varepsilon)$ for an infinitesimal $\varepsilon > 0$. Since (A4) is slack and ε is infinitesimal, $(b,\alpha) = (b_i^*(p,v_h) - \varepsilon V(v_h), \alpha_i^*(p,v_h) + \varepsilon)$ satisfies (A2). Therefore, it is perceived as coming from type v_h. Hence, the seller values it the same as offer $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$. Hence, since $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$ satisfies (A3), so does $(b_i^*(p,v_h) - \varepsilon V(v_h), \alpha_i^*(p,v_h) + \varepsilon)$. However, type

v_h is better off buying the target for $(b_i^*(p,v_h) - \varepsilon V(v_h), \alpha_i^*(p,v_h)+\varepsilon)$ than for $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$:

$$\begin{aligned}(1-\alpha_i^*(p,v_h)-\varepsilon)V(v_h) - \lambda_i(b_i^*(p,v_h)-\varepsilon V(v_h)) &= (1-\alpha_i^*(p,v_h))V(v_h) - \lambda_i b_i^*(p,v_h) + (\lambda_i - 1)\varepsilon V(v_h) \\ &> (1-\alpha_i^*(p,v_h))V(v_h) - \lambda_i b_i^*(p,v_h),\end{aligned}$$

since $\lambda_i > 1$ and $\varepsilon > 0$. Therefore, both (A3) and (A4) bind. Solving this system of two equations yields $(b_i^*(p,v_h), \alpha_i^*(p,v_h)) = \left((1-\gamma_i)p, \frac{p}{V(v_h)}\gamma_i\right)$, where $\gamma_i = \left(1 + \frac{1}{\lambda_i - 1}\frac{v_h - v_l}{\Pi_T + \Pi_B + v_h}\right)^{-1}$.

Step 3: Consider $p \le p_i^*(v_l)$. It cannot be that $(b_i^*(p,v_l), \alpha_i^*(p,v_l)) = (b_i^*(p,v_h), \alpha_i^*(p,v_h))$. By contradiction, suppose there is such offer $(b_i(p), \alpha_i(p))$. If $\alpha_i(p) > 0$, consider a deviation by type v_h to $(b_i(p) + \varepsilon_1, \alpha_i(p) - \varepsilon_2)$ for infinitesimal $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying

$$\varepsilon_2 V(v_h) - \lambda_i \varepsilon_1 > 0 > \varepsilon_2 V(v_l) - \lambda_i \varepsilon_1$$

For example, for an arbitrary infinitesimal $\varepsilon_2 > 0$, let $\varepsilon_1 \equiv \frac{\varepsilon_2}{2\lambda_i}(V(v_l) + V(v_h))$. Then, according to CKIC, the seller must believe that offer $(b_i(p) + \varepsilon_1, \alpha_i(p) - \varepsilon_2)$ comes from type v_h. Since $(b_i(p), \alpha_i(p))$ is valued by the seller at least at p, ε_2 and ε_1 are infinitesimal, and v_h exceeds the average of v_h and v_l, offer $(b_i(p) + \varepsilon_1, \alpha_i(p) - \varepsilon_2)$ is valued by the seller at more than p. Furthermore, since $\varepsilon_2 V(v_h) - \lambda_i \varepsilon_1 > 0$, type v_h is strictly better off acquiring the target for $(b_i(p) + \varepsilon_1, \alpha_i(p) - \varepsilon_2)$ than for $(b_i(p), \alpha_i(p))$. Hence, there is a profitable deviation for type v_h, which is a contradiction.

Step 4: $p_i^*(v_l) = \frac{\Pi_T + v_l}{r-\mu}X_t$. Since $(b_i^*(p,v_l), \alpha_i^*(p,v_l)) = \left(0, \frac{p}{V(v_l)}\right)$, the bidder with synergy v_l bids up to the price $p_i^*(v_l)$ at which it is indifferent between acquiring the target for $\left(0, \frac{p}{V(v_l)}\right)$ and losing the auction:

$$\left(1 - \frac{p_i^*(v_l)}{V(v_l)}\right)V(v_l) = V_o,$$

which yields $p_i^*(v_l) = V(v_l) - V_o = \frac{\Pi_T + v_l}{r-\mu}X_t$.

Step 5: Consider $p \in (p_i^*(v_h) - \varepsilon, p_i^*(v_h)]$ for an infinitesimal $\varepsilon > 0$. It must be that $(b_i^*(p,v_h), \alpha_i^*(p,v_h)) = \left(0, \frac{p}{V(v_h)}\right)$. By the argument in step 2, (A3) binds. By contradiction, suppose that $b_i^*(p,v_h) > 0$. Since $p \in (p_i^*(v_h) - \varepsilon, p_i^*(v_h)]$, $(b_i^*(p,v_h), \alpha_i^*(p,v_h))$ is such that the payoff of the bidder with synergy v_h, $(1-\alpha_i^*(p,v_h))V(v_h) - \lambda_i b_i^*(p,v_h)$, is in the neighborhood of V_o. Consider a deviation to $(b', \alpha') = (b_i^*(p,v_h) - \varepsilon V(v_h), \alpha_i^*(p,v_h) + \varepsilon)$ for an infinitesimal $\varepsilon > 0$. Since ε is infinitesimal, $(1-\alpha')V(v_h) - \lambda_i b'$ is in the neighborhood of V_o. Since $v_l < v_h$, we conclude that

$$(1-\alpha')V(v_l) - \lambda_i b' < V_o.$$

Therefore, (b', α') satisfies (A2). In addition, (b', α') satisfies (A1):

$$\begin{aligned}(1-\alpha') V(v_h) - \lambda_i b' &= (1-\alpha_i^*(p, v_h)) V(v_h) - \lambda_i b_i^*(p, v_h) + (\lambda_i - 1)\varepsilon V(v_h) \\ &> (1-\alpha_i^*(p, v_h)) V(v_h) - \lambda_i b_i^*(p, v_h),\end{aligned}$$

since $\lambda_i > 1$. Therefore, offer (b', α') is perceived as coming from type v_h, and type v_h is better off deviating to (b', α') from $(b_i^*(p, v_h), \alpha_i^*(p, v_h))$, which is a contradiction. Hence, $b_i^*(p, v_h) = 0$. Since (A3) binds, $\alpha_i^*(p, v_h) = \frac{p}{V(v_h)}$. Finally, notice that $\left(0, \frac{p}{V(v_h)}\right)$ satisfies CKIC. Indeed, there exists no deviation that benefits type v_h and satisfies (A3), since (A3) binds and $b_i^*(p, v_h) = 0$.

Step 6: $p_i^*(v_h) = \frac{\Pi_T + v_h}{r-\mu} X_t$. Since $(b_i^*(p, v_h), \alpha_i^*(p, v_h)) = \left(0, \frac{p}{V(v_h)}\right)$ for p close to $p_i^*(v_h)$, type v_h bids up to the price $p_i^*(v_h)$ at which it is indifferent between acquiring the target for $\left(0, \frac{p}{V(v_h)}\right)$ and losing the auction:

$$\left(1 - \frac{p_i^*(v_h)}{V(v_h)}\right) V(v_h) = V_o,$$

which yields $p_i^*(v_h) = V(v_h) - V_o = \frac{\Pi_T + v_h}{r-\mu} X_t$.

Step 7: $(b_i^*(p_i^*(v), v), \alpha_i^*(p_i^*(v), v)) = \left(0, \frac{\Pi_T + v}{\Pi_T + \Pi_B + v}\right)$, $v \in \{v_l, v_h\}$, *and* $(b_i^*(p_i^*(v_l), v_h), \alpha_i^*(p_i^*(v_l), v_h)) = \left((1-\gamma_i) p_i^*(v_l), \frac{\Pi_T + v_l}{\Pi_T + \Pi_B + v_h}\gamma_i\right)$, *where* $\gamma_i = \left(1 + \frac{1}{\lambda_i - 1}\frac{v_h - v_l}{\Pi_T + \Pi_B + v_h}\right)^{-1}$. Plugging $p_i^*(v) = \frac{\Pi_T + v}{r-\mu} X_t$ into $\left(0, \frac{p_i^*(v)}{V(v)}\right)$ yields $\left(\frac{\Pi_T + v}{\Pi_T + \Pi_B + v}, 0\right)$. Plugging $p_i^*(v_l) = \frac{\Pi_T + v_l}{r-\mu} X_t$ into the expression for $(b_i^*(p_i^*(v_l), v_h), \alpha_i^*(p_i^*(v_l), v_h))$ from step 2 yields the last expression.

Proof of Lemma 1. By contradiction, suppose that $\bar{X}_i(s)$ is not decreasing for some $i \in \{1, 2\}$. Without loss of generality, suppose this is for $i = 1$. Since $\bar{X}_i(s)$ is monotone, there can be two cases: (1) $\bar{X}_i(s)$ is increasing in s for both $i \in \{1, 2\}$; (2) $\bar{X}_1(s)$ is increasing in s, but $\bar{X}_2(s)$ is decreasing in s. For each case, consider bidder 1 with signal s_1 playing the strategy of initiating the auction at threshold $\hat{X}$, if it has not been initiated yet.

First, consider case (1). If s_2 is low enough so that $\bar{X}_2(s_2) < \hat{X}$, the auction is initiated by bidder 2 at threshold $\bar{X}_2(s_2)$. The expected surplus of bidder 1 from the auction in this case is $s_1(1-s_2)\psi_1(v_h, v_l)\frac{\bar{X}_2(s_2)}{r-\mu}$. Otherwise, the auction is initiated by bidder 1 at threshold $\hat{X}$. Its expected surplus from the auction in this case is $s_1(1-s_2)\psi_1(v_h, v_l)\frac{\hat{X}}{r-\mu}$. Thus, the expected value to bidder 1 at any time t prior to initiation of the auction is

$$\Pi_B \frac{X_t}{r-\mu} + \left(\frac{X_t}{\hat{X}}\right)^{\beta} \int_{\bar{X}_2^{-1}(\hat{X})}^{\bar{s}} \left(s_1(1-s_2)\psi_1(v_h, v_l) \frac{\hat{X}}{r-\mu} - I \right) \frac{dF\left(s_2\right)}{1 - F\left(\bar{X}_2^{-1}(\max_{u\in[0,t]} X_u)\right)}$$
$$+ \int_{\bar{X}_2^{-1}\left(\max_{u\in[0,t]} X_u\right)}^{\bar{X}_2^{-1}(\hat{X})} \left(\frac{X_t}{\bar{X}_2\left(s_2\right)}\right)^{\beta} \left(s_1(1-s_2)\psi_1(v_h, v_l) \frac{\bar{X}_2(s_2)}{r-\mu} - I \right) \frac{dF\left(s_2\right)}{1 - F\left(\bar{X}_2^{-1}(\max_{u\in[0,t]} X_u)\right)}.$$

The optimal threshold $\hat{X}$ maximizes this expected value. Differentiating in $\hat{X}$ and s_1 yields

$$-\left(\beta-1\right) X_t^{\beta} \hat{X}^{-\beta-1} \int_{\bar{X}_2^{-1}(\hat{X})}^{\bar{s}} (1-s_2)\psi_1(v_h, v_l) \frac{\hat{X}}{r-\mu} \frac{dF\left(s_2\right)}{1 - F\left(\bar{X}_2^{-1}(\max_{u\in[0,t]} X_u)\right)} < 0.$$

Second, consider case (2). Since $\bar{X}_2\left(\cdot\right)$ is decreasing, the argument of Section 3.2 applies and the payoff to bidder 1 is given by (12). Differentiating in $\hat{X}$ and s_1 yields

$$-\left(\beta-1\right) X_t^{\beta} \hat{X}^{-\beta-1} \int_{\underline{s}}^{\bar{X}_2^{-1}(\hat{X})} \left(1-z_2\right) \psi_1(v_h, v_l) \frac{\hat{X}}{r-\mu} \frac{dF\left(s_2\right)}{F\left(\bar{X}_2^{-1}(\max_{u\in[0,t]} X_u)\right)} < 0.$$

By Topkis's theorem (Topkis 1978), the optimal initiation threshold of bidder 1 is decreasing in s_1 in both cases, which is a contradiction.

Proof of Lemma 2. Take the full derivative of equations $\delta_i\left(x, \zeta_i\left(x\right), \zeta_{-i}\left(x\right)\right) = 0$ in x:

$$\frac{\partial \delta_i}{\partial x} + \frac{\partial \delta_i}{\partial \zeta_i} \zeta_i'\left(x\right) + \frac{\partial \delta_i}{\partial \zeta_j} \zeta_j'\left(x\right) = 0, i \in \left\{1, 2\right\}.$$

Multiply the equation for δ_i by $\frac{\partial \delta_j}{\partial \zeta_j}$, the equation for δ_j by $\frac{\partial \delta_i}{\partial \zeta_j}$:

$$\begin{aligned} \frac{\partial \delta_i}{\partial x}\frac{\partial \delta_j}{\partial \zeta_j} + \frac{\partial \delta_i}{\partial \zeta_i}\frac{\partial \delta_j}{\partial \zeta_j}\zeta_i'\left(x\right) + \frac{\partial \delta_i}{\partial \zeta_j}\frac{\partial \delta_j}{\partial \zeta_j}\zeta_j'\left(x\right) &= 0; \\ \frac{\partial \delta_j}{\partial x}\frac{\partial \delta_i}{\partial \zeta_j} + \frac{\partial \delta_j}{\partial \zeta_j}\frac{\partial \delta_i}{\partial \zeta_j}\zeta_j'\left(x\right) + \frac{\partial \delta_j}{\partial \zeta_i}\frac{\partial \delta_i}{\partial \zeta_j}\zeta_i'\left(x\right) &= 0. \end{aligned}$$

Subtract the latter equation from the former one, observing that the third term in the first equation and the second term in the second equation cancel out:

$$\left(\frac{\partial \delta_i}{\partial \zeta_i}\frac{\partial \delta_j}{\partial \zeta_j} - \frac{\partial \delta_j}{\partial \zeta_i}\frac{\partial \delta_i}{\partial \zeta_j}\right)\zeta_i'\left(x\right) = \frac{\partial \delta_j}{\partial x}\frac{\partial \delta_i}{\partial \zeta_j} - \frac{\partial \delta_i}{\partial x}\frac{\partial \delta_j}{\partial \zeta_j}. \tag{A7}$$

In our case, $\frac{\partial \delta_i}{\partial x} = \zeta_i\left(x\right) m(\zeta_j(x)) > 0$; similarly, $\frac{\partial \delta_j}{\partial x} > 0$. Also, $\frac{\partial \delta_i}{\partial \zeta_j} = x\zeta_i\left(x\right)\frac{\partial m(\zeta_j(x))}{\partial \zeta_j} \leq 0$ and $\frac{\partial \delta_j}{\partial \zeta_j} = xm(\zeta_i(x)) > 0$. Therefore, the right-hand side of (A7) is strictly negative. Because $\bar{X}_i\left(s\right)$ is strictly decreasing in $s \in [\underline{s}, \bar{s}]$, $\zeta_i'(x) < 0$ for any $x \in \left[\bar{X}_i(\bar{s}), \bar{X}(\underline{s})\right]$. Therefore, $\frac{\partial \delta_i}{\partial \zeta_i}\frac{\partial \delta_j}{\partial \zeta_j} - \frac{\partial \delta_j}{\partial \zeta_i}\frac{\partial \delta_i}{\partial \zeta_j} > 0$ for any $x \in \left[\bar{X}_i(\bar{s}), \bar{X}_i(\underline{s})\right]$. Because the derivations apply for any $i \in \{1,2\}$, we conclude that $\frac{\partial \delta_1}{\partial \zeta_1}\frac{\partial \delta_2}{\partial \zeta_2} - \frac{\partial \delta_1}{\partial \zeta_2}\frac{\partial \delta_2}{\partial \zeta_1} > 0$ for all $x \in \left\{\min_{i\in\{1,2\}} \bar{X}_i(\bar{s}), \max_{i\in\{1,2\}} \bar{X}_i(\underline{s})\right\}$, that is, for all x, at which the auction could occur in equilibrium.

Details of the solution of the model in Section 3.2

For brevity, denote $\xi_i \equiv \frac{\beta}{\beta-1}\frac{(r-\mu)I}{\psi_i(v_h,v_l)}$. Then, (13) becomes

$$\bar{X}_i(s) = \frac{\xi_i}{sm\left(\bar{X}_j^{-1}(\bar{X}_i(s))\right)}.$$

For transparency, in what follows we unpack $m\left(\bar{X}_j^{-1}(\bar{X}_i(s))\right)$ as $1 - E\left[z|z \le \bar{X}_j^{-1}\left(\bar{X}_i(s)\right)\right]$. Without loss of generality, suppose that bidder 1 faces higher financial constraints: $\lambda_1 > \lambda_2$. By Proposition 5, $\bar{X}_1(s) > \bar{X}_2(s)$ for any s. Let $\hat{s}_1$ be the signal at which $\bar{X}_1(\hat{s}_1) = \bar{X}_2(\underline{s})$. Let $\hat{s}_2$ be the signal at which $\bar{X}_1(\bar{s}) = \bar{X}_2(\hat{s}_2)$.

First, consider bidder 2 with signal $s \ge \hat{s}_2$. When $X(t)$ is about to hit its initiation threshold $\bar{X}_2(s)$, bidder 2 did not learn anything about the signal of bidder 1, since no threshold $\bar{X}_1(s)$, $s \in [\underline{s}, \bar{s}]$ has been passed yet. Therefore, for any $s \ge \hat{s}_2$,

$$\bar{X}_2(s) = \frac{\xi_2}{s(1 - \mathbb{E}[z])}.$$

Second, consider bidder 1 with signal $\bar{s}$. When $X(t)$ is about to hit its initiation threshold $\bar{X}_1(\bar{s})$, bidder 1 believes that the signal of bidder 2 cannot exceed $\hat{s}_2$, since otherwise bidder 2 would have initiated the auction earlier. Therefore,

$$\bar{X}_1(\bar{s}) = \frac{\xi_1}{\bar{s}(1 - \mathbb{E}[z|z \le \hat{s}_2])}.$$

Since $\hat{s}_2$ is defined by $\bar{X}_1(\bar{s}) = \bar{X}_2(\hat{s}_2)$, it satisfies:

$$\frac{\xi_2}{\hat{s}_2(1 - \mathbb{E}[z])} = \frac{\xi_1}{\bar{s}(1 - \mathbb{E}[z|z \le \hat{s}_2])}. \tag{A8}$$

Third, consider bidder 1 with signal $s \le \hat{s}_1$. When $X(t)$ is about to hit its initiation threshold $\bar{X}_1(s)$, bidder 1 believes that bidder 2 with any signal would have initiated the auction before. By our belief updating rule, we assume that bidder 1 believes that bidder 2's signal is lowest possible, $\underline{s}$. Therefore, for any $s \le \hat{s}_1$,

$$\bar{X}_1(s) = \frac{\xi_1}{s(1 - \underline{s})}.$$

Fourth, consider bidder 2 with signal $\underline{s}$. When $X(t)$ is about to hit its initiation threshold $\bar{X}_2(\underline{s})$, bidder 2 believes that the signal of bidder 1 cannot exceed $\hat{s}_1$, since otherwise bidder 1 would have initiated the auction earlier. Thus,

$$\bar{X}_2(\underline{s}) = \frac{\xi_2}{\underline{s}\left(1 - \mathbb{E}\left[z|z \leq \hat{s}_1\right]\right)}.$$

Since $\hat{s}_1$ is defined by $\bar{X}_1(\hat{s}_1) = \bar{X}_2(\underline{s})$, it satisfies:

$$\frac{\xi_1}{\hat{s}_1(1-\underline{s})} = \frac{\xi_2}{\underline{s}\left(1 - \mathbb{E}\left[z|z \leq \hat{s}_1\right]\right)}. \tag{A9}$$

Finally, consider bidder 1 with signal $s_1 \in [\hat{s}_1, \bar{s}]$ or bidder 2 with signal $s_2 \in [\underline{s}, \hat{s}_2]$. Bidders with such valuations initiate the auction at thresholds in interval $\left[\bar{X}_1(\hat{s}_1) = \bar{X}_2(\underline{s}), \bar{X}_1(\bar{s}) = \bar{X}_2(\hat{s}_2)\right]$. Take any x in this interval. It is the initiation threshold of bidder 1 with some signal in $[\hat{s}_1, \bar{s}]$, denoted $s_1^*(x) = \bar{X}_1^{-1}(x)$. It is also the initiation threshold of bidder 2 with some signal in $[\underline{s}, \hat{s}_2]$, denoted $s_2^*(x) = \bar{X}_2^{-1}(x)$. For each parameter x, equation (13) in the paper yields a system of two equations with two unknowns, s_1 and s_2:

$$x = \frac{\xi_1}{s_1(1-\mathbb{E}[z|z \leq s_2])} \tag{A10}$$

$$x = \frac{\xi_2}{s_2(1-\mathbb{E}[z|z \leq s_1])} \tag{A11}$$

For each x, the solution to this system is $s_1^*(x)$ and $s_2^*(x)$. Equililbrium initiation thresholds $\bar{X}_1(s)$ and $\bar{X}_2(s)$ are the inverses of $s_1^*(x)$ and $s_2^*(x)$, respectively.

In what follows, we specialize this solution to the case of uniform distribution of signals over $[\underline{s}, \bar{s}]$ with $0 < \underline{s} < \bar{s} \leq 1$.

Example: uniform distribution of signals

In this case, equations for $\hat{s}_1$ and $\hat{s}_2$, (A8) and (A9), simplify to

$$\frac{\xi_2}{\hat{s}_2\left(1 - \frac{\underline{s}+\bar{s}}{2}\right)} = \frac{\xi_1}{\bar{s}\left(1 - \frac{\underline{s}+\hat{s}_2}{2}\right)},$$

$$\frac{\xi_1}{\hat{s}_1(1-\underline{s})} = \frac{\xi_2}{\underline{s}\left(1 - \frac{\underline{s}+\hat{s}_1}{2}\right)}.$$

The solutions are:

$$\hat{s}_1 = \frac{\frac{\xi_1}{\xi_2}\underline{s}\left(1-\frac{\underline{s}}{2}\right)}{1-\underline{s}+\frac{\xi_1}{\xi_2}\frac{\underline{s}}{2}}, \tag{A12}$$

$$\hat{s}_2 = \frac{\bar{s}\left(1-\frac{\underline{s}}{2}\right)}{\frac{\xi_1}{\xi_2}\left(1-\frac{\underline{s}}{2}\right)-\left(\frac{\xi_1}{\xi_2}-1\right)\frac{\bar{s}}{2}}. \tag{A13}$$

For example, in the limit $\frac{\xi_1}{\xi_2} \to 1$, we have $\hat{s}_1 \to \underline{s}$ and $\hat{s}_2 \to \bar{s}$. That is, if both bidders have approximately identical financial constraints, there is approximately no range of thresholds X at which only one of the two bidders initiates the auction with positive probability.

Solutions (A12)-(A13) immediately give $\bar{X}_1(s)$ for $s \le \hat{s}_1$ and $\bar{X}_2(s)$ for $s \ge \hat{s}_2$:

$$\bar{X}_1(s) = \frac{\xi_1}{s(1-\underline{s})} \text{ for } s \le \hat{s}_1,$$

$$\bar{X}_2(s) = \frac{\xi_2}{s\left(1-\frac{\underline{s}+\bar{s}}{2}\right)} \text{ for } s \ge \hat{s}_2.$$

Finally, the system of equations (A10)-(A11) simplifies to:

$$x = \frac{\xi_1}{s_1(1-\frac{\underline{s}+s_2}{2})}$$

$$x = \frac{\xi_2}{s_2(1-\frac{\underline{s}+s_1}{2})}$$

Substituting $s_2 = \frac{\xi_2}{x\left(1-\frac{\underline{s}+s_1}{2}\right)}$ into the first equation and simplifying, we obtain:

$$xs_1\left(x\left(2-\underline{s}-s_1\right)-\underline{s}x\left(1-\frac{\underline{s}+s_1}{2}\right)-\xi_2\right) = \xi_1 x\left(2-\underline{s}-s_1\right).$$

This yields a quadratic equation for s_1, one of which roots (the smaller one) belongs to interval $[\underline{s}, \hat{s}_1]$. Hence, the case of uniform distribution of signals has a closed form solution for inverse initiation functions. Direct initiation thresholds are also recovered from them by inversion.

Details of the equilibrium construction for the model with entry deterrence (Section 5.1).

Step 1: "no deterrence" region, $s \in [\underline{s}, \hat{s}_1)$. Consider a bidder with signal $s \in [\underline{s}, \hat{s}_1)$ and time t satisfying $\max_{u \le t} X(u) \ge \tilde{X}_{pd}(\hat{s}_1)$. If such bidder initiates the auction at threshold $\hat{X}$, its expected value at time t is

$$\Pi_B \frac{X_t}{r-\mu} + \left(\frac{X_t}{\hat{X}}\right)^{\beta} \int_{\underline{s}}^{\tilde{X}^{-1}(\hat{X})} \left(s(1-z)\psi(v_h,v_l)\frac{\hat{X}}{r-\mu} - I\right) \frac{dF(z)}{F\left(\tilde{X}^{-1}(\max_{u\le t} X_u)\right)}$$
$$+ \int_{\tilde{X}^{-1}(\hat{X})}^{\tilde{X}^{-1}\left(\max_{u\le t} X_u\right)} \left(\frac{X_t}{\tilde{X}(z)}\right)^{\beta} \left(s(1-z)\psi(v_h,v_l)\frac{\tilde{X}(z)}{r-\mu} - I\right) \frac{dF(z)}{F\left(\tilde{X}^{-1}(\max_{u\le t} X_u)\right)},$$

which coincides with (8) in the basic model. Therefore, by the same argument, the equilibrium initiation threshold in this range is given by

$$\tilde{X}(s) = \bar{X}(s) = \frac{\beta}{\beta-1}\frac{(r-\mu)I}{sm(s)\psi(v_h,v_l)}. \tag{A14}$$

Step 2: "partial deterrence" region, $s \in (\hat{s}_1, \hat{s}_2)$. Consider a bidder with signal $s \in (\hat{s}_1, \hat{s}_2)$ and any time t satisfying $\max_{u\le t} X(u) \ge \tilde{X}_{fd}(\hat{s}_2)$ and $\max_{u\le t} X(u) < \tilde{X}_{pd}(\hat{s}_1)$. If such bidder initiates the auction at threshold $\hat{X}$, its expected value at time t is

$$\Pi_B \frac{X_t}{r-\mu} + \left(\frac{X_t}{\hat{X}}\right)^{\beta} \int_{\underline{s}}^{\tilde{X}^{-1}(\hat{X})} \left(s(1-z)\psi(v_h,v_l)\frac{\hat{X}}{r-\mu} - I\right) \frac{dF(z)}{F\left(\tilde{X}^{-1}(\max_{u\le t} X_u)\right)}$$
$$+ \left(\frac{X_t}{\hat{X}}\right)^{\beta} s\psi(v_h,v_l)\frac{\hat{X}}{r-\mu} \int_{\underline{s}}^{\frac{(r-\mu)I}{\psi(v_h,v_l)(1-\tilde{X}^{-1}(\hat{X}))\hat{X}}} z \frac{dF(z)}{F\left(\tilde{X}^{-1}(\max_{u\le t} X_u)\right)} \tag{A15}$$
$$+ \int_{\tilde{X}^{-1}(\hat{X})}^{\tilde{X}^{-1}\left(\max_{u\le t} X_u\right)} \left(\frac{X_t}{\tilde{X}(z)}\right)^{\beta} \max\left\{s(1-z)\psi(v_h,v_l)\frac{\tilde{X}(z)}{r-\mu} - I, 0\right\} \frac{dF(z)}{F\left(\tilde{X}^{-1}(\max_{u\le t} X_u)\right)}.$$

In (A15), the first term reflects the stand-alone value of the bidder; the second term reflects the expected value from the auction initiated by the bidder, assuming that the rival participates; the third term reflects additional value from the auction given that the bidder does not participate if its signal is low enough; and the fourth term reflects the expected value from the auction initiated by the rival. Differentiating (A15) with respect to $\hat{X}$ and applying the equilibrium condition that the maximum must be reached at

$\hat{X} = \tilde{X}(s)$, we obtain the following differential equation on $\tilde{X}(s)$ in this region, denoted $\tilde{X}_{pd}(s)$:

$$(\beta-1)\,s\psi(v_h,v_l)\frac{\tilde{X}_{pd}(s)}{r-\mu}\left(1-\int_{\frac{(r-\mu)I}{\psi(v_h,v_l)(1-s)\tilde{X}_{pd}(s)}}^{s} z\frac{dF(z)}{F(s)}\right)-\beta I$$

$$= -\frac{f\left(\frac{(r-\mu)I}{\psi(v_h,v_l)(1-s)\tilde{X}_{pd}(s)}\right)}{F(s)}\frac{s(r-\mu)I^2}{\psi(v_h,v_l)}\frac{1-s-\frac{\tilde{X}_{pd}(s)}{\tilde{X}'_{pd}(s)}}{(1-s)^3\tilde{X}_{pd}(s)}. \tag{A16}$$

Step 3: "full deterrence" region, $s \in (\hat{s}_2, \bar{s}]$. Consider a bidder with signal $s \in (\hat{s}_2, \bar{s}]$ and any time t satisfying $\max_{u\le t} X(u) < \tilde{X}_{fd}(\hat{s}_2)$. If such bidder follows the strategy of initiating the auction at threshold $\hat{X}$, provided that the rival has not initiated the auction yet, and not participating in the auction initiated by the bidder, its expected value at time t is

$$\Pi_B\frac{X_t}{r-\mu}+\left(\frac{X_t}{\hat{X}}\right)^{\beta}\int_{\underline{s}}^{\tilde{X}^{-1}(\hat{X})}\left(s\psi(v_h,v_l)\frac{\hat{X}}{r-\mu}-I\right)\frac{dF(z)}{F\left(\tilde{X}^{-1}(\max_{u\le t}X_u)\right)}. \tag{A17}$$

In (A17), the first and second terms reflect the stand-alone value of the bidder and the expected value from the auction, respectively. If the signal of the rival exceeds $\tilde{X}^{-1}\left(\hat{X}\right)$, the rival initiates the auction before $X(t)$ reaches threshold $\hat{X}$, and the bidder gets zero payoff, because it does not enter the auction. If the signal of the rival is below $\tilde{X}^{-1}\left(\hat{X}\right)$, the bidder initiates the auction first at threshold $\hat{X}$ and acquires the target for $V(v_l) - V_o$, because the rival does not enter the auction. Thus, the payoff to the bidder from the auction in this case is $s\psi(v_h,v_l)\frac{\hat{X}}{r-\mu} - I$. Differentiating (A17) with respect to $\hat{X}$ and applying the equilibrium condition that the maximum must be reached at $\hat{X} = \tilde{X}(s)$, we obtain the following differential equation on $\tilde{X}(s)$ in this region, denoted $\tilde{X}_{fd}(s)$:

$$(\beta-1)\,s\psi(v_h,v_l)\frac{\tilde{X}_{fd}(s)}{r-\mu}-\beta I=\left(s\psi(v_h,v_l)\frac{\tilde{X}_{fd}(s)}{r-\mu}-I\right)\frac{f(s)}{F(s)}\frac{\tilde{X}_{fd}(s)}{\tilde{X}'_{fd}(s)}. \tag{A18}$$

Step 4: equations for $\hat{s}_1$ *and* $\tilde{X}_{pd}(\hat{s}_1)$. Cutoff type $\hat{s}_1$ and the initial value condition in differential equation (A16), $\tilde{X}_{pd}(\hat{s}_1)$, must satisfy:

$$\left(\frac{\tilde{X}_{pd}(\hat{s}_1)}{\bar{X}(\hat{s}_1)}\right)^{\beta}\int_{\underline{s}}^{\hat{s}_1}\left(\frac{\hat{s}_1(1-z)\psi(v_h,v_l)\bar{X}(\hat{s}_1)}{r-\mu}-I\right)dF(z) \tag{A19}$$
$$=\left(\int_{\underline{s}}^{\hat{s}_1}\left(\frac{\hat{s}_1\left(1-z1\left\{z>\frac{(r-\mu)I}{\psi(v_h,v_l)(1-\hat{s}_1)\tilde{X}_{pd}(\hat{s}_1)}\right\}\right)\psi(v_h,v_l)\tilde{X}_{pd}(\hat{s}_1)}{r-\mu}-I\right)dF(z)\right);$$

$$(\beta-1)\hat{s}_1\psi(v_h,v_l)\frac{\tilde{X}_{pd}(\hat{s}_1)}{r-\mu}\left(1-\int_{\frac{(r-\mu)I}{\psi(v_h,v_l)(1-\hat{s}_1)\tilde{X}_{pd}(\hat{s}_1)}}^{\hat{s}_1}z\frac{dF(z)}{F(\hat{s}_1)}\right)-\beta I \tag{A20}$$
$$=-\frac{f\left(\frac{(r-\mu)I}{\psi(v_h,v_l)(1-\hat{s}_1)\tilde{X}_{pd}(\hat{s}_1)}\right)}{F(\hat{s}_1)}\frac{\hat{s}_1(r-\mu)I^2}{\psi(v_h,v_l)(1-\hat{s}_1)^2\tilde{X}_{pd}(\hat{s}_1)}.$$

where $1\{\cdot\}$ is an indicator function. (A19) is the indifference condition stating that type $\hat{s}_1$ must be indifferent between initiating the auction at threshold $\bar{X}(\hat{s}_1)$ and facing entry of the rival with probability one and initiating the auction at threshold $\tilde{X}_{pd}(\hat{s}_1)<\bar{X}(\hat{s}_1)$ and facing entry of the rival only if its signal z is sufficiently high. If (A19) did not hold, then either type s just above $\hat{s}_1$ would be better off deviating from initiating the auction at threshold $\tilde{X}_{pd}(s)$ to threshold $\bar{X}(\hat{s}_1)$ (if the left-hand side of (A19) exceeded the right-hand side) or type s just below $\hat{s}_1$ would be better off deviating from initiating the auction at threshold $\bar{X}(s)$ to threshold $\tilde{X}_{pd}(\hat{s}_1)$. Hence, (A19) must hold in equilibrium. (A20) states that the action of the lowest type in the signaling region, $\hat{s}_1$, coincides with its action in the game without signaling incentives, that is, in the modified game in which signal $\hat{s}_1$ is truthfully revealed to the rival bidder when type $\hat{s}_1$ initiates the auction. This is the threshold that maximizes

$$\hat{X}^{-\beta}\int_{\underline{s}}^{\hat{s}_1}\left(s(1-z)\psi(v_h,v_l)\frac{\hat{X}}{r-\mu}-I\right)\frac{dF(z)}{F(\hat{s}_1)}+s\psi(v_h,v_l)\frac{\hat{X}^{1-\beta}}{r-\mu}\int_{\underline{s}}^{\frac{(r-\mu)I}{\psi(v_h,v_l)(1-\hat{s}_1)\hat{X}}}z\frac{dF(z)}{F(\hat{s}_1)}, \tag{A21}$$

which is analogous to (A15) but sets $\tilde{X}^{-1}\left(\hat{X}\right)=\hat{s}_1$. We refer the reader to Mailath (1987) for the formal argument,[1] but the intuition is as follows. Suppose a bidder is the lowest type in region $[\hat{s}_1,\hat{s}_2]$, and suppose that its initiation threshold $\tilde{X}_{pd}(\hat{s}_1)$ violates (A20). If it deviates to threshold that maximizes (A21), its expected payoff (A15) increases for two reasons: first, the payoff, assuming same entry of the rival $\left(z>\frac{(r-\mu)I}{\psi(v_h,v_l)(1-\hat{s}_1)\hat{X}}\right)$, increases; second, the entry of the rival cannot increase, because the current

[1] See Grenadier and Malenko (2011) for an adaptation of this argument to signaling games in the real options context.

equilibrium belief of the rival ($\hat{s}_1$) is already the lowest. Therefore, $\hat{s}_1$ and $\tilde{X}_{pd}(\hat{s}_1)$ satisfy (A19)-(A20).

Step 5: equations for $\hat{s}_2$ and $\tilde{X}_{fd}(\hat{s}_2)$. Cutoff type $\hat{s}_2$ and $\tilde{X}_{fd}(\hat{s}_2)$ must satisfy:

$$\int_{\underline{s}}^{\hat{s}_2} \left(\frac{\hat{s}_2 \left(1 - z1\left\{z > \frac{(r-\mu)I}{\psi(v_h,v_l)(1-\hat{s}_2)\tilde{X}_{pd}(\hat{s}_2)}\right\}\right) \psi(v_h,v_l)\tilde{X}_{pd}(\hat{s}_2)}{r-\mu} - I \right) dF(z)$$

$$= \left(\frac{\tilde{X}_{pd}(\hat{s}_2)}{\tilde{X}_{fd}(\hat{s}_2)} \right)^{\beta} \left(\hat{s}_2 \psi(v_h,v_l) \frac{\tilde{X}_{fd}(\hat{s}_2)}{r-\mu} - I \right), \tag{A22}$$

$$\tilde{X}_{fd}(\hat{s}_2) = \frac{(r-\mu) I}{\psi(v_h,v_l)(1-\hat{s}_2)\hat{s}_2}. \tag{A23}$$

(A22) is the indifference condition saying that type $\hat{s}_2$ must be indifferent between initiating the auction at threshold $\tilde{X}(\hat{s}_2)$ and facing entry of the rival if its signal is high enough and initiating the auction at threshold $\tilde{X}_{fd}(\hat{s}_2) < \tilde{X}_{pd}(\hat{s}_2)$ and not facing entry of the rival. The proof of (A22) is similar to the proof of (A19). (A23) must hold for the following reason. First, $\tilde{X}_{fd}(\hat{s}_2)$ cannot exceed (A23), because otherwise entry of types of the rival that are close enough to $\hat{s}_2$ is not deterred, which contradicts the assertion that types just above $\hat{s}_2$ deter entry of the rival with probability one. Second, if $\tilde{X}_{fd}(\hat{s}_2)$ were below (A23), type $\hat{s}_2$ would be better off deviating from threshold $\tilde{X}_{fd}(\hat{s}_2)$ to threshold (A23), because the entry of the rival is deterred in both cases and the expected payoff of the bidder is strictly increasing in the initiation threshold in this range. Therefore, $\hat{s}_2$ and $\tilde{X}_{fd}(\hat{s}_2)$ satisfy (A22)-(A23).

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