A Dual-Channel Vendor-Buyer System with Minimum Purchase Commitment

Yexin Wang [1], Rohit Bhatnagar [1,2,*], Stephen C. Graves [1,3]

[1]Singapore-MIT Alliance, Nanyang Technological University, Singapore

[2] Nanyang Business School, Nanyang Technological University, Singapore

[3] Sloan School of Management, Massachusetts Institute of Technology, USA

Abstract: In this paper, we consider a dual-channel vendor-buyer system consisting of a buyer and two vendor-owned facilities: a central distribution center (DC) and a regional DC. Orders for a single item are delivered through two distinct channels: an *indirect channel*, comprising the central DC, the regional DC and the buyer; and a *direct channel*, comprising the central DC and the buyer, bypassing the intermediate regional DC. Each facility periodically replenishes its inventory at a common time interval and safety stock is carried at each facility to maintain the desired service level. The vendor and buyer make a *minimum purchase commitment* (MPC), under which the buyer commits to purchase a predetermined and fixed quantity through the direct channel in each time period, and has the option to purchase a flexible quantity through the indirect channel in each time period, We study the impact of the MPC agreement on the inventory and safety stock at the vendor and buyer for this dual-channel vendor-buyer system, and introduce a simulation- based method to estimate this impact for *iid* normally distributed demand. We also study an integrated coordination problem in which the vendor and buyer cooperate to implement the optimal MPC agreement that minimizes total system cost.

Keywords: Inventory; Supply chain management; Minimum purchase commitment; Coordination

*Corresponding author. Tel: (65) 6790-6235.
Email: arbhatnagar@ntu.edu.sg

1. Introduction

Spurred by the increasing globalization of economic activities in the last two decades, firms are confronted with several new challenges on the logistics front. As production activities shift to locations with low labor costs and long-distance deliveries are used for distributing products to worldwide markets, a crucial imperative for logistics managers is to strive for efficient and responsive distribution systems by implementing innovative practices such as cross-docking and merge-in-transit among others. These emerging practices have also provided impetus to academic researchers to model and optimize the logistics systems.

The research presented in this paper is motivated by the multi-stage distribution system of a computer printer peripherals manufacturer (see Figure 1). The company operates a manufacturing factory in Europe and many distribution facilities. Raw materials and components are manufactured and assembled into bulk products at the factory. These bulk products are then delivered by sea freight to central DCs (e.g., the central DC for Asia-Pacific market is located in Singapore), and are customized and packaged into finished products according to specific regional requirements. From these central DCs, the finished products are delivered by sea freight to regional DCs (e.g., the regional DC for China market is located in Shanghai). The finished products are then delivered by trucks to geographically scattered customers (e.g., retailers and wholesalers).

Figure 1: A multi-stage distribution system

Our discussions with the logistics managers of the firm revealed their keen interest in implementing new logistics strategies to improve the distribution system performance. The company has some big customers who regularly place large orders and these orders exhibit relatively low variability. In the existing distribution system as presented in Figure 1, these large orders are delivered successively through the central DC and the regional DC as are all other orders. This distribution strategy ignores the impact of order quantity and variability and may lead to system inefficiency. To address this issue, we proposed a logistics strategy to design a more flexible distribution system by treating these large orders distinctly from the smaller orders.

The company's regional DCs are located close to the customers in order to provide short delivery lead times. At these regional DCs, the demands of different customers can be pooled to achieve economies of scale in inbound transportation to the regional DC from the central DC. However, our discussions indicated that the firm could reduce its transportation costs by delivering large orders directly from the central DCs to the customers since it could then use efficient transportation modes (e.g., full container load by sea freight) over shorter transportation distances. In addition, the variable operating costs at the regional DCs could also be reduced since the direct deliveries bypass these facilities.

For the deliveries to the aforementioned big customers, we propose a dual-channel distribution strategy as follows. A customer can receive orders through two distinct channels: an *indirect channel*, where orders are delivered from a central DC to a regional DC, and then to the customer; and a *direct channel*, where orders are delivered directly from the central DC to the customer, bypassing the intermediate regional DC. In addition, we propose a *minimum purchase commitment* (MPC) that allows a customer to purchase a fixed quantity through the direct channel and a flexible quantity through the indirect channel in each time period.

The proposed dual-channel distribution strategy aims to improve the system performance as follows. For the proportion of demand that is likely to be certain, regular quantities are delivered directly from the distant central DC to the customer to achieve low transportation and operating costs. For the remaining demand that is likely to be uncertain, flexible quantities are delivered from the nearby regional DC to achieve responsiveness.

The rest of this paper is organized as follows. In Section 2, we review the relevant literature on inventory models with a MPC agreement. In Section 3, we consider a traditional single-channel vendor-buyer system, and discuss the optimal replenishment policy in the standard settings. In Section 4, we introduce a dual-channel vendor-buyer system, investigate the impact of a MPC agreement on the inventory and safety stock in the system, and study an integrated coordination problem. In Section 5, we provide a simulation-based method to analyze the dual-channel vendor-buyer model. In Section 6, we present numerical cases to demonstrate our analysis and findings. Finally, conclusions, implications, and future research directions are outlined in Section 7.

2. Literature Review

In this section, we review the literature on inventory models with a MPC agreement, which restricts a buyer to periodically purchase a minimum quantity regardless of realized demand.

In one of the earliest works on MPC inventory models, Rosenshine and Obee [15] considered a situation where a buyer has to carry a high level of safety stock due to a long replenishment lead time. To eliminate the high level of safety stock, Rosenshine and Obee [15] studied an MPC inventory model, in which the buyer periodically places a fixed-quantity order with a long replenishment lead time and has the option to place an emergency order for a flexible quantity that is delivered instantaneously. They assumed that the buyer has a limited storage capacity such that any excess inventory has to be sold off at a salvage price. For discrete demand, the authors used a Markov chain model to determine the optimal committed quantity and order-up-to level. They showed that the MPC agreement reduces the buyer's cost by eliminating safety stock, despite incurring emergency order costs and salvage costs. Also see Chiang and Guttierrez [6] for related work.

Chiang [7] studied the same MPC inventory model as Rosenshine and Obee [15], but used dynamic programming to derive the optimal storage capacity and order-up-to level. He assumed average-cost and discounted-cost criteria, and studied backlogged and lost-sales problems in his paper. Chiang also showed that a convergence approach can be applied to determine the optimal system parameters with reasonable errors.

Anupindi and Akella [1] investigated the impact of an MPC agreement on order variance. They considered delivery lead time as a decision variable and assumed that a buyer needs to pay a price premium to adjust the order quantity above the committed quantity. The authors showed that the MPC agreement can reduce the variance in the order process to the supplier and proved that the optimal replenishment policy in a finite planning horizon is the periodic review order-up-to policy.

Moinzadeh and Nahmias [14] considered a problem similar to Anupindi and Akella [1], but in a continuous review and infinite planning horizon setting. They showed that the equations that need to be solved to find the optimal order-up-to level are intractable. The authors developed a diffusion approximation that is coupled with the solution to a deterministic version of the problem. They empirically derived a formula for computing the optimal committed quantity and established its accuracy with numerical tests.

Janssen and de Kok [9] studied an MPC inventory model by considering the fixed ordering costs, purchase cost and holding cost at the buyer. By showing the equivalence of the buyer's inventory level to the waiting time of a GI/G/1 queue, they used the moment-iteration method introduced in De Kok [8] to approximate the buyer's inventory level, and developed an algorithm to estimate the optimal order-up-to level and committed quantity that minimize the buyer's cost subject to a desired service level.

Urban [20] used mixed-integer linear programming and network optimization to formulate an MPC inventory model, where the buyer has limited flexibility to adjust the order quantity at an extra cost. The author provided a solution methodology for the general stochastic demand case and for several specific demand distributions. Urban [20] also gave a numerical analysis to extend the basic problem to a multiple-product, multiple-constraint problem.

Thomas and Hackman [17] studied an MPC inventory model in a finite horizon with price-sensitive demand. For *iid* normally distributed demand, they used a simulation-based method to approximate the expected inventory level and order quantities at the buyer as quadratic functions of the committed quantity and the reselling price. They showed that the approximation method can yield closed-form solutions to decide the optimal policy that maximizes the buyer's revenue.

Cheung and Yuan [5] considered an infinite horizon inventory model of a buyer with a periodic order commitment. The authors considered general discrete demand distributions and assumed that the buyer could order more than the minimum commitment without incurring any extra adjustment cost. They formulated a Markov chain to represent the buyer's inventory level and used the solution approach to the classical GI/M/1 queue to derive the steady-state results and obtain the exact closed-form cost function.

Beside the MPC inventory models mentioned above, *total minimum purchase commitment* (TMPC) inventory models have also received attention in literature. Bassok and Anupindi [2] considered a TMPC inventory model that requires a buyer to purchase a minimum cumulative quantity over a finite horizon to satisfy stochastic demand. They proved the optimality of the dual order-up-to policy given the committed quantity and showed that the optimal policy can be computed by solving two standard newsboy problems. Chen and Krass [4] extended the model of Bassok and Anupindi [2] to a more general setting of non-stationary demand, and different unit prices for the committed quantity and the remaining quantity.

Tibben-Lemke [18] studied another TMPC inventory model with order quantity restrictions in each time period. He showed that the computation of the optimal order-up-to levels is time-consuming. The author provided a heuristic method to derive near-optimal policies for a range of system parameters. For the relevant literature on inventory models with other types of supply contracts, we refer the reader to the review papers of Tsay et. al. [19], Cachon [3] and Kamrad and Siddique [10].

The research outlined in this paper makes several distinct contributions to the existing literature. First, our MPC inventory model addresses an *integrated* vendor-buyer coordination problem, in which the vendor and buyer can cooperate to decide the optimal committed quantity that minimizes total system cost. All previous studies reported in literature considered MPC inventory models from the individual perspective of the buyer. In contrast, our research investigates the impact of the MPC agreement on the inventory and safety stock at both the vendor and the buyer. Second, we introduce a simulation-based method to quantitatively estimate the inventory and safety stock levels at the vendor and buyer. Our method can be easily implemented as compared to models discussed in literature, e.g. diffusion approximation in Moinzadeh and Nahmias [14] and moment-iteration method in Janssen and de Kok [9]. While these methods provide reasonable accuracy in analyzing the MPC inventory model, they need complex modeling techniques and are computationally demanding. Finally, since our research is motivated by the real life distribution system of a computer printer peripherals manufacturer, it will provide useful insights to practitioners from an implementation perspective.

3. A Single-Channel Vendor-Buyer Model

Consider a single-channel vendor-buyer system comprising a buyer and two vendor-owned facilities: a central DC and a regional DC (see Figure 2). The three facilities periodically replenish their inventories for a single item at a common review interval. These replenishment orders are delivered from the central DC to the regional DC, and then to the buyer.

Figure 2: A single-channel vendor-buyer system

At the buyer, the exogenous customer demand D is *iid* normally distributed with mean μ and standard deviation (STD) σ in each time period. We use n (n=1, 2, 3,…, N) to denote the index of time period, and use D_n to denote the realized customer demand in time period n. We also assume that the customer demand D does not depend on the selling price.

Safety stock is carried at each facility to maintain the desired service level α at the buyer, which is defined as the probability that no stockout occurs in any given time period. We assume that other mechanisms such as the spot market or expediting can be used to fulfill the demand beyond the service level α.

We will use the base-stock modeling framework introduced by Kimball, whose 1955 manuscript was later reprinted in 1988. Kimball [11] studied a single stage inventory model with a periodic order-up-to replenishment policy and an assumption of bounded demand. Kimball showed that the safety stock should be used to satisfy the maximum demand over the net replenishment lead time, which is defined as the incoming service time plus the production time at the stage minus the outgoing service time. Using Kimball's work as a building block, Simpson [16] considered a serial supply chain and studied the problem of determining the safety stock at each stage by setting the service times. Simpson also provided an alternate interpretation of the demand bound as the maximum demand the firm wants to satisfy from safety stocks.

We use SS_b, SS_{rdc}, and SS_{cdc} to denote the safety stock levels and L_b, L_{rdc} and L_{cdc} to denote the replenishment lead times at the buyer, the regional DC and the central DC, respectively. All replenishment lead times are assumed to be deterministic. At each facility in the single-channel vendor-buyer system, we model the safety stock level as follows:

$$SS = \eta_\alpha \cdot \sigma \cdot \sqrt{L}\,, \tag{3.1}$$

where η_α represents the safety factor that is uniquely associated with service level α in the instance of *iid* normally distributed demand. Using the safety stock level, we can determine the order-up-to level as follows:

$$S = \mu \cdot L + SS\,, \tag{3.2}$$

where the term $\mu \cdot L$ represents the average demand during the replenishment time L. At the beginning of each time period, a facility needs to place an order on its upstream facility to raise the inventory position to the order-up-to level, S. Thus, the order quantity is always equal to the realized demand in the previous time period.

4. A Dual-Channel Vendor-Buyer Model with Minimum Purchase Commitment

Now consider a dual-channel vendor-buyer system (see Figure 3), in which orders are delivered through two distinct channels: an *indirect channel*, in which orders are delivered from the central DC to the regional DC, and then to the buyer; and a *direct channel*, in which orders are delivered directly from the central DC to the buyer, bypassing the intermediate regional DC.

The buyer replenishes its inventory through the two channels with an MPC agreement as follows.

Figure 3: A dual-channel vendor-buyer system

At the beginning of time period n, the buyer places a *regular order* for a predetermined and fixed quantity Q through the direct channel. The regular order is purchased with a percentage purchase discount λ, which is offered by the vendor to encourage regular orders. Note that the MPC (or regular order) quantity Q should be smaller than the mean demand μ; otherwise the buyer's inventory level will rise without bound in an infinite horizon. After placing the regular order, the buyer places no further order if the resulting inventory position is at or above the order-up-to level S; otherwise the buyer can place a *supplementary order* for quantity q_n through the indirect channel to raise the inventory position up to the order-up-to level S. The supplementary order is purchased at the unit price p, with no purchase discount.

We consider the channel supply cost, and the inventory holding costs at the buyer, and at the vendor. At each facility, inventory holding cost is incurred in proportion to the average inventory level and the holding cost rate. We use h_b , h_{cdc}, and h_{rdc} to denote the holding cost rates at the buyer, the central DC and the regional DC, respectively. For the sake of simplicity, we assume the same holding cost rate h_b for the inventories of both regular orders and supplementary orders at the buyer, even though the unit purchase costs are different. We also assume that the holding cost rate is increasing as material moves down the supply chain: $h_{cdc} < h_{rdc} < h_b$.

4.1. Channel Supply Cost

The channel supply cost includes the transportation costs, and the operating cost incurred at the regional DC (e.g. loading/ unloading cost and other handling costs). We assume that the channel supply cost is incurred in proportion to the quantity delivered through each channel. We use c_1 and c_2 to denote the channel supply cost rate for the direct channel and for the indirect channel. We use c_3 to denote the channel supply cost rate for that part of the demand that is satisfied by a "backup" mechanism like a spot market or expediting. We use β to denote the fill rate that is defined as the proportion of demand that is satisfied from inventory on hand. Thus, we can express the expected channel supply cost per time period C_{supply} as follows.

$$C_{supply} = c_1 Q + c_2\left(\mu\beta - Q\right) + c_3(1-\beta)\mu \tag{4.1}$$

In (4.1), the first term c_1Q represents the cost of supplying the MPC quantity Q through the direct channel; the second term $c_2(\mu\beta-Q)$ represents the cost of supplying the average supplementary order quantity $\mu\beta-Q$ through the indirect channel; the third term $c_3(1-\beta)\mu$ represents the cost of supplying the quantity $(1-\beta)\mu$ that is satisfied by the "backup" mode. We can rewrite the expected channel supply cost C_{supply} as a linear function of the MPC quantity Q.

$$C_{supply} = \left[c_2\beta + c_3\left(1-\beta\right)\right]\mu - \left(c_2 - c_1\right)Q \tag{4.2}$$

In (4.2), we interpret the first term $[c_2\beta+c_3(1-\beta)]\mu$ as the expected channel supply cost in the corresponding single-channel system, and the second term $(c_2-c_1)Q$ as the channel supply cost savings by delivering the MPC quantity Q through the direct channel. We assume that $c_1<c_2$, which implies that utilizing the direct channel reduces the channel supply cost. Utilizing the direct channel reduces the channel supply cost for three reasons: the transportation distance is shorter, we can use a more efficient transportation mode, and we can avoid the operating or handling cost incurred at the regional DC.

4.2. Inventory at the Buyer

Under the MPC agreement, the buyer is committed to purchase a minimum quantity Q in each time period. The buyer's inventory position IP may thus exceed the order-up-to level S by an overshoot when the quantity Q is larger than the realized demand during the previous time period. We refer to this inventory overshoot as *surplus inventory* and use SI_n to denote the surplus inventory level at the buyer in the time period n. We have

$$O_n = Max\left(D_{n-1} - SI_{n-1}, Q\right) \tag{4.3}$$

$$IP_n = Max\left\{S, IP_{n-1} + Q - D_{n-1}\right\} = SS_b + L_b\mu + SI_n \tag{4.4}$$

where $S = SS_b + L_b\mu$

As shown above, the buyer's order quantity O_n in time period n is the maximum of the realized demand D_{n-1} in time period n-1, net of any surplus inventory in the prior period, and the MPC quantity Q. If this order quantity exceeds the MPC quantity Q, then the difference is obtained by placing an order on the regional DC. At the beginning of time period n, the inventory position IP_n includes three components: the surplus inventory SI_n, the safety stock at the buyer, SS_b, and the expected demand during the replenishment lead time, $L_b \cdot \mu$. We can approximate the buyer's average on-hand inventory level I_b as follows.

$$I_b = SS_b + \tfrac{1}{2}\mu + SI \,, \tag{4.5}$$

where we use SI to denote the expectation of SI_n.

From (4.4), we can derive a Lindley type iterative equation (see Lindley [13]) for the surplus inventory level SI_n.

$$SI_n = Max\{0, SI_{n-1} + Q - D_{n-1}\} \tag{4.6}$$

Equation (4.6) shows that the surplus inventory SI depends on the MPC quantity Q and the stochastic demand D, but is independent of the order-up-to level S. The Lindley type equation also shows that the surplus inventory level SI is equivalent to the customer waiting time in a single-stage GI/D/1 queue, where the customers arrive with general independent inter-arrival times D and are served by a single *first-come-first-served* (FCFS) server with deterministic service time Q. Unfortunately, there is no closed-form analytical solution for the average customer waiting time in a single-stage GI/D/1 queue.

Despite the lack of a closed-form solution, we can establish some useful properties for the surplus inventory level SI_n when we assume that the demand is normally distributed.

<u>PROPOSITION 1.</u> *Given that the stochastic demand D is iid normally distributed in each time period with parameters μ, σ and the starting surplus inventory SI_0, is zero, we can express the expected surplus inventory SI_n in period n as a product of the demand standard deviation σ and a function of the time horizon n and the standardized MPC quantity z=(μ-Q)/σ.* That is

$$SI_n = \sigma \cdot k(n, z) \tag{4.7}$$

Proof. To prove (4.7), we show that the cumulative distribution function for the variable (SI_n/σ) depends only on the time period n and on the standardized MPC quantity z. We define the complementary cumulative distribution function $G_n(x) = \Pr\left[\frac{SI_n}{\sigma} > x\right]$ and will show that this function depends only on n and z by induction:

For $n = 2$, we have

$$G_2[x] = \Pr\left[\left(\frac{SI_2}{\sigma}\right) > x\right] = \Pr[D_1 \le Q - \sigma x] = \Phi(-x - z), \qquad \forall x > 0 \tag{4.8}$$

where $\Pr[D_n \le x]$ denotes the CDF for the demand in period n, which is assumed to be normal with parameters μ, σ, and where $\Phi(\cdot)$ is the CDF of the standard normal distribution $N(0, 1)$.

Now, suppose the induction hypothesis is true for $n = 2, \dots m$. That is, the variable $\left(SI_n/\sigma\right)$ has a unique CDF $G_n(\cdot)$ that depends only on n and the standardized MPC quantity $z=(\mu-Q)/\sigma$ for $n = 2, \dots m$.

For $n = m+1$, we have

$$G_{m+1}\left[x\right]=\Pr\left[\left(\frac{SI_{m+1}}{\sigma}\right)>x\right]=\Pr\left[D_m\le SI_m+Q-\sigma x\right]=\Pr\left[\frac{D_m-\mu}{\sigma}\le\frac{SI_m}{\sigma}+\frac{Q-\mu}{\sigma}-x\right]$$
$$=\Pr\left[\frac{D_m-\mu}{\sigma}\le\frac{SI_m}{\sigma}-z-x\right]=-\int_{y=0}^{\infty}\Phi\left(y-z-x\right)dG_m\left(y\right)$$

(4.9)

Thus, we see that we can express the complementary cumulative distribution function for (SI_{m+1}/σ) in terms of the normal CDF and the complementary cumulative distribution function for (SI_m/σ). By the induction hypothesis, we now see that the CDF of (SI_n/σ) depends only on the period n and the standardized MPC quantity z. Therefore, Proposition 1 is true. □

PROPOSITION 2. *The expected surplus inventory level in period n increases with n, where we assume that the starting surplus inventory SI_0, is zero. When n goes to infinity, the expected surplus inventory level converges to a constant level that is independent of the starting surplus inventory level SI_0.*

Proof. The proof of the convergence of surplus inventory level is the same as the proof of the convergence of average customer waiting time in a single-stage GI/G/1 queue. We refer the interested reader to the discussion in Kingman [12].

Based on Propositions 1 and 2, we can write the expected surplus inventory SI in a simple form as follows.

$$SI=\sigma\cdot k(z) \tag{4.10}$$

To determine the expected surplus inventory SI, it is sufficient to know the value of the coefficient function $k(z)$. As we cannot determine an analytical expression for this function, we will determine it numerically.

4.3. Safety Stock at the Buyer

As mentioned earlier, at the beginning of time period n the buyer's inventory position IP_n equals its order-up-to level S plus the surplus inventory SI_n. The inventory position represents what inventory is available to meet demand over the next L_b time periods. The buyer desires to set its order-up-to level S to achieve some desired service level α; thus, we will set S so that the probability that the inventory position covers the total demand during the next L_b time periods is at least the desired service level α. Compared to the traditional single-channel system, the buyer will need less safety stock SS_b (or lower order-up-to level S_b) to maintain the same service level α due to the surplus inventory in the dual-channel system. In the following proposition we find that the required safety stock level SS_b at the buyer has a similar property to the surplus inventory level SI.

PROPOSITION 3. *Given that the stochastic demand D is iid normally distributed in each time period with parameters μ, σ, we can determine the safety stock level at the buyer SS_b by*

$$SS_b = \sigma \cdot \sqrt{L_b} \cdot \psi_{\alpha, L_b}(z), \tag{4.11}$$

where the function ψ depends on the service level α and replenishment lead time L_b, and its argument is the standardized MPC $z = (\mu - Q)/\sigma$.

Proof: We set the order-up-to level $S_b = SS_b + \mu L_b$ so that the following condition holds:

$$\Pr\left[D_n + \ldots + D_{n+L_b-1} \le S_b + SI_n\right] = \alpha \tag{4.12}$$

We will show that we can express the left-hand-side as a function of $SS_b/(\sigma\sqrt{L_b})$, the replenishment lead time L_b and the standardized MPC z.

$$\begin{aligned} \Pr\left[D_n + \ldots + D_{n+L_b-1} \le S_b + SI_n\right] &= \Pr\left[\frac{D_n + \ldots + D_{n+L_b-1} - \mu L_b}{\sigma\sqrt{L_b}} \le \frac{SI_n + SS_b}{\sigma\sqrt{L_b}}\right] \\ &= -\int_{y=0}^{\infty} \Phi\left(\frac{y}{\sqrt{L_b}} + \frac{SS_b}{\sigma\sqrt{L_b}}\right) dG_n(y) \end{aligned} \tag{4.13}$$

From the above, we see that we can express the service level as a function of $SS_b/(\sigma\sqrt{L_b})$ and of the lead time L_b. From proposition 1, we see that the service level depends on the standardized MPC. We can also argue that the service level is a monotonic function of the safety stock, from which we can conclude Proposition 3.□

4.4. Safety Stock at the Vendor

We have two vendor-owned facilities: the central DC and the regional DC. At the beginning of each time period each facility places an order to raise its inventory position to an order-up-to level. We note that these two facilities observe similar order processes.

The order process at the regional DC is the supplementary order placed by the buyer in each period. The order process at the central DC is the constant MPC quantity Q, plus the order placed by the regional DC, which is equal to the supplementary order from the buyer. Thus, each facility needs to carry a safety stock due to the variability from the buyer's supplementary orders, which depends on the MPC. In the following, we focus on investigating how the MPC agreement affects the safety stock at the regional DC.

At the beginning of time period n, the regional DC receives a supplementary order q_n from the buyer if the regular order quantity Q does not raise the buyer's inventory position up to the order-up-to level S. We have

$$q_n = D_n + SI_{n+1} - Q - SI_n . \tag{4.14}$$

The regional DC should set its order-up-to level S_{rdc} to satisfy the total order during the next L_{rdc} time periods for some desired service level α. We have

$$\left.\begin{aligned} q_n &= D_n + SI_{n+1} - Q - SI_n \\ q_{n+1} &= D_{n+1} + SI_{n+2} - Q - SI_{n+1} \\ \dots\dots &= \dots\dots \\ q_{n+L_{rdc}-1} &= D_{n+L_{rdc}-1} + SI_{n+L_{rdc}} - Q - SI_{n+L_{rdc}-1} \end{aligned}\right\} \Rightarrow \sum_{m=n}^{m=n+L_{rdc}-1} q_m = \sum_{m=n}^{m=n+L_{rdc}-1} D_m + SI_{n+L_{rdc}} - QL_{rdc} - SI_n$$

(4.15)

Wang [21] has developed an approximation to show that the required safety stock level SS_{rdc} at the regional DC has a similar property to the safety stock level SS_b at the buyer.

Approximation 1. *Given that the stochastic demand D is iid normally distributed in each time period with parameters μ, σ, we can determine the safety stock level at the regional DC SS_{rdc} by*

$$SS_{rdc} = \sigma \cdot \sqrt{L_{rdc}} \cdot \varphi_{\alpha, L_{rdc}}(z) , \tag{4.16}$$

where the function φ depends on the service level α and replenishment lead time $L_{rdc.}$ and its argument is the standardized MPC $z = (\mu - Q)/\sigma$.

Argument: We set the order-up-to level $S_{rdc} = SS_{rdc} + (\mu - Q)L_{rdc}$ so that the following condition holds:

$$\Pr\left[q_n + \ldots + q_{n+L_{rdc}-1} \le S_{rdc} \right] = \alpha \tag{4.17}$$

Based on (4.15) and (4.17), we will argue that we can express the left-hand-side as a function of $SS_{rdc} \big/ (\sigma\sqrt{L_{rdc}})$, the replenishment lead time L_{rdc} and the standardized MPC z.

$$\Pr\left[q_n + \ldots + q_{n+L_{rdc}-1} \le S_{rdc}\right]$$
$$= \Pr\left[\frac{D_n + \ldots + D_{n+L_{rdc}-1} + SI_{n+L_{rdc}} - QL_{rdc} - SI_n - \mu L_{rdc}}{\sigma\sqrt{L_{rdc}}} \le \frac{SS_{rdc} + (\mu - Q)L_{rdc} - \mu L_{rdc}}{\sigma\sqrt{L_{rdc}}}\right] \quad (4.18)$$
$$= \Pr\left[\frac{D_n + \ldots + D_{n+L_{rdc}-1} - \mu L_{rdc}}{\sigma\sqrt{L_{rdc}}} \le \frac{SI_n - SI_{n+L_{rdc}} + SS_{rdc}}{\sigma\sqrt{L_{rdc}}}\right]$$

To develop the approximation, suppose that we assume that $(SI_n - SI_{n+L_{rdc}})/\sigma$ is independent of the demands $D_n, \ldots, D_{n+L_{rdc}-1}$. From proposition 1, we know the cumulative distribution function for the variable SI_n depends on the time period n and on the standardized MPC quantity z. Thus, the cumulative distribution function for the variable $(SI_n - SI_{n+L_{rdc}})/\sigma$ depends only on the time period n, the lead time L_{rdc} and the standardized MPC quantity z. We define the complementary cumulative distribution function $H_{n,L_{rdc}}(x) = \Pr\left[\frac{SI_n - SI_{n+L_{rdc}}}{\sigma} > x\right]$. With the assumption of independence we can rewrite equation (4.18) by the following approximation

$$\Pr\left[q_n + \ldots + q_{n+L_{rdc}-1} \le S_{rdc}\right] = -\int_{y=0}^{\infty} \Phi\left(\frac{y}{\sqrt{L_{rdc}}} + \frac{SS_{rdc}}{\sigma\sqrt{L_{rdc}}}\right) dH_{n,L_{rdc}}(y) \quad (4.19)$$

From the similar argument as the proof of Proposition 3, we can conclude Approximation 1.□

The central DC's order process is the same as that for the regional DC, plus the constant MPC quantity Q. Thus we can also use this approximation for the required safety stock level SS_{cdc}. We use the index v to denote either vendor-owned facility, and we have

$$SS_v = \sigma \cdot \sqrt{L_v} \cdot \varphi_{\alpha,L_v}(z), \quad (4.20)$$

Comparing equations (4.13) and (4.18), we can observe that for the same lead time the vendor requires higher safety stock to maintain a given service level than the buyer does. That is

$$\varphi_{\alpha,L_v}(z) \ge \psi_{\alpha,L_b}(z) \quad (4.21)$$

From the above analysis, however, it is still not clear whether the MPC agreement can reduce the safety stock level at the vendor, since the demand process D and surplus inventory level SI_n depend on each other in (4.18). We can get some intuitive insights from the following scenarios:

(I). If the total demand realized during the time periods n to $n+L_v$-1, is "*large*" and greater than expected, the surplus inventory SI_{n+L_v} is more likely to be zero. Thus, the total order quantity $\sum D_m - SI_n$ is smaller than the total demand quantity $\sum D_m$ if $SI_n > 0$.

(II). If total demand realized during the time periods n to $n+L_v-1$, is "*small*" and less than expected, the surplus inventory SI_{n+L_v} is more likely to be positive. Thus, the total order quantity $\sum D_m - SI_n + SI_{n+L_v}$ is larger than the total demand quantity $\sum D_m$ if $SI_{n+L_v} - SI_n > 0$.

Thus, in the dual-channel vendor-buyer system, the order process observed at the vendor is less variable or smoother than that in the single-channel vendor-buyer system and, consequently less safety stock is required at each vendor facility to maintain the same service level.

4.5. Integrated Vendor-Buyer Coordination Problem

We now consider an *integrated vendor-buyer coordination problem*, in which the buyer and vendor can fully cooperate with each other to decide the MPC quantity Q that minimizes total system cost. We express the expected system cost C_{sys} as follows.

$$C_{sys} = \left\{ \left[c_2\beta + c_3(1-\beta) \right]\mu - (c_2 - c_1)Q \right\} + (\frac{1}{2}\mu + SI + SS_b)h_b + SS_{rdc}h_{rdc} + SS_{cdc}h_{cdc} \quad (4.22)$$

The first term in (4.22), represents the system supply cost. The second term represents the buyer's inventory holding cost, which includes the cycle inventory cost $0.5\mu h_b$, the surplus inventory cost $SI \cdot h_b$, and the safety stock cost $SS_b \cdot h_b$. The last two terms represent just the safety stock costs at the regional DC and central DC. Furthermore, we can rewrite the expected system cost C_{sys} as a function of the standardized MPC quantity z by substituting $Q = \mu - z\sigma$ and replacing the safety stock and surplus inventory terms as follows.

$$C_{sys}(z) = \mu\left[(c_3 - c_2)(1-\beta) + c_1 + \frac{1}{2}h_b \right] + \sigma\left\{ (c_2 - c_1)z + \left[k(z) + \sqrt{L_b}\psi_{\alpha,L_b}(z) \right] h_b \right.$$
$$\left. + \sqrt{L_{rdc}}\varphi_{a,L_{rdc}}(z)h_{rdc} + \sqrt{L_{cdc}}\varphi_{a,L_{cdc}}(z)h_{cdc} \right\}$$

(4.23)

Then, the objective of an integrated coordination problem is to find the optimal standardized MPC z^* that minimizes the expected system cost $C_{sys}(z)$.

$$z^* = \arg\min_{0<z\le 1} C_{sys}(z) \quad (4.24)$$

Below, we outline the effects of the system parameters on the optimal solution z^*.

I. Effect of demand parameters μ and σ

From the expression of the expected system cost C_{sys} (4.23), we observe that the first term is proportional to the demand mean μ but remains constant when the standardized MPC quantity z varies, and the second term is a product of the demand standard deviation σ and a value that does not depend on the demand parameters μ and σ. Thus, the optimal solution z^* is independent of the demand parameters μ and σ.

II. Effect of channel supply cost rates c_1 and *c_2*

The channel supply cost rates c_1 and c_2 contribute to the expected system cost C_{sys} in that the cost savings per unit (c_2-c_1) can be obtained by supplying the product through the direct channel. When the value of (c_2-c_1) increases, the optimal MPC quantity Q^* should increase and, consequently, the optimal standardized MPC quantity z^* decreases.

III. Effect of holding cost rate h

From the previous analysis, we know that the MPC agreement increases the inventory level at the buyer and decreases the inventory levels at the regional DC and central DC. Thus, the optimal solution z^* increases when the holding cost rate h_b increases, and decreases when the holding cost rate h_{rdc} or h_{cdc} increases.

One important issue in the integrated coordination problem is to allocate the benefit between the vendor and buyer so that each party is willing to participate in implementing the integrated coordination. By setting $z = \mu/\sigma$ we can use (4.23) to compute the expected system cost without a MPC, i.e., when Q = 0. We use Π to denote the amount of the savings that needs to be shifted from the vendor to the buyer under an *equal allocation scheme*. We have

$$\begin{aligned}\Pi = &\frac{1}{2}\sigma k(z)h_b - \frac{1}{2}\sqrt{L_b}\sigma\left[\psi_{\alpha,L_b}(\mu/\sigma) - \psi_{\alpha,L_b}(z)\right]h_b + \frac{1}{2}(c_2 - c_1)(\mu - \sigma z) \\ &+ \frac{1}{2}\sqrt{L_{RDC}}\sigma\left[\varphi_{a,L_{rdc}}(\mu/\sigma) - \varphi_{a,L_{rdc}}(z)\right]h_{rdc} + \frac{1}{2}\sqrt{L_{cdc}}\sigma\left[\varphi_{a,L_{cdc}}(\mu/\sigma) - \varphi_{a,L_{cdc}}(z)\right]h_{cdc}\end{aligned} \tag{4.25}$$

In (4.25), the first two terms represent half of the increase in the buyer's cost, and the last three terms represent half of the decrease in the vendor's cost. When the profit allocation is in the form of purchase discount, the vendor should offer a purchase discount for the MPC quantity Q^* at the percentile λ defined as

$$\lambda = \Pi / Q^* \tag{4.26}$$

5. Simulation-Based Approximation Method

Based on our analysis in Section 4, the surplus inventory coefficient function $k(z)$ and the safety stock coefficient functions $\psi_{\alpha,L}(z)$ and $\varphi_{\alpha,L}(z)$ are invariant for any case where the demand D is *iid* normally distributed. We now introduce a method to use simulation to estimate these coefficient functions, which can then be used to quantitatively analyze the dual-channel vendor-buyer system.

5.1. Surplus Inventory Coefficient Function *k*(*z*)

We conducted simulations using VBA programming in Microsoft Excel™ to estimate the long-term surplus inventory coefficient function $k(z)$ with the following parameters:

- In each time period, the demand D is *iid* normally distributed with a mean μ of 400 and a standard deviation σ of 100.
- The MPC quantity Q varies between [300, 400] with an increment of 1; that is, the standard MPC quantity z varies between [0, 1] with an increment of 0.01.
- The simulation horizon N is 20,000. Each simulation trial includes 1,000 random runs, and these trial results have a 99% confidence interval that is at most within 1% of their mean. The simulation results of the surplus inventory coefficient function $k(z)$ are presented in Table 1.

$k(z)$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.00	79.45	40.19	23.33	15.83	11.98	9.478	7.794	6.591	5.687	5.005
.10	4.443	3.968	3.608	3.305	3.028	2.785	2.571	2.398	2.246	2.099
.20	1.964	1.848	1.743	1.651	1.561	1.475	1.399	1.333	1.269	1.211
.30	1.152	1.103	1.055	1.010	0.969	0.928	0.891	0.856	0.822	0.791
.40	0.761	0.733	0.706	0.681	0.656	0.632	0.610	0.590	0.571	0.550
.50	0.531	0.514	0.498	0.483	0.466	0.451	0.437	0.424	0.411	0.398
.60	0.386	0.375	0.363	0.353	0.342	0.332	0.322	0.312	0.303	0.295
.70	0.287	0.278	0.270	0.263	0.256	0.249	0.242	0.235	0.228	0.223
.80	0.216	0.211	0.205	0.199	0.194	0.189	0.184	0.179	0.174	0.169
.90	0.165	0.161	0.156	0.152	0.148	0.144	0.141	0.137	0.133	0.130

Table 1: Simulation results for the surplus inventory coefficient function $k(z)$

Given the simulation results presented in Table 1, we can use a *linear interpolation* *method* to estimate the expected surplus inventory level *SI* by assuming a linear function of *k*(*z*) between any two neighboring *z* values given in Table 1. We have

$$k(z) \approx k(z_i) + \frac{z - z_i}{z_j - z_i}\left[k(z_j) - k(z_i)\right], \tag{5.1}$$

where z_i and z_j are the two neighboring values for *z* in Table 1.

The simulation results for surplus inventory function *k*(*z*) are shown in Figure 4

Figure 4: Surplus inventory coefficient function *k*(*z*)

From the above figure, we can make the following observations about the surplus inventory coefficient function *k*(*z*).

(I). The surplus inventory coefficient function *k*(*z*) is exponentially increasing as the standardized MPC quantity *z* decreases to zero. This exponential trend can be interpreted as follows. Decreasing MPC quantity *z* increases the probability of the MPC quantity *Q* being larger than demand *D* and consequently increases the probability of the surplus inventory being built up over consecutive time periods. The accumulation of surplus inventory results in the exponential trend of the surplus inventory coefficient function *k*(*z*).

(II). The surplus inventory coefficient function *k*(*z*) is equal to 4.43 and 0.130 when the standardized MPC quantity *z* is 0.1 and 0.99. We view this as a reasonable range for our choice of *z*. For smaller values of z, the function grows dramatically; for larger values of z, there is very little reduction possible. Thus, we assert that a reasonable MPC quantity *Q* should fall in the range of [μ-σ, μ-0.1σ].

5.2. Safety Stock Coefficient Functions $\psi(z)$ and $\varphi(z)$

We also conducted simulations to estimate the long-term safety stock coefficient functions ψ(z) and φ(z) using the following parameters:

- In each time period, the demand *D* is *iid* normally distributed with the mean μ of 400 and the standard deviation σ of 100.
- The MPC quantity *Q* varies between [300, 390] with an increment of 1; that is, the standard MPC quantity *z* varies between [0.1, 1] with an increment of 0.01.
- For each safety stock coefficient function, we consider the service levels α of [98%, 95%, 90%] and the lead times *L* of [1, 3, 5, 7, 15, 25].

- The simulation horizon N is 20,000. Each simulation trial includes 1,000 random runs, and these trial results have a 99.5% confidence interval that is at most within 1% of their mean.

In Table 2 and 3, we give the simulation results of the safety stock coefficient functions $\psi_{98\%,1}(z)$ and $\varphi_{98\%,1}(z)$.

$\Psi(z)$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.10	1.273	1.317	1.358	1.394	1.426	1.455	1.482	1.507	1.531	1.552
.20	1.572	1.591	1.608	1.624	1.639	1.654	1.668	1.681	1.693	1.705
.30	1.716	1.726	1.737	1.746	1.756	1.765	1.773	1.782	1.789	1.797
.40	1.804	1.811	1.818	1.824	1.831	1.837	1.842	1.848	1.854	1.859
.50	1.864	1.869	1.874	1.878	1.883	1.887	1.891	1.895	1.899	1.903
.60	1.907	1.910	1.914	1.918	1.921	1.924	1.927	1.930	1.933	1.936
.70	1.939	1.942	1.944	1.947	1.950	1.952	1.954	1.957	1.959	1.961
.80	1.963	1.965	1.967	1.969	1.971	1.973	1.975	1.977	1.979	1.981
.90	1.982	1.984	1.986	1.987	1.989	1.990	1.992	1.993	1.994	1.996

Table 2: Simulation results for the s safety stock coefficient functions $\psi_{98\%,1}(z)$

$\varphi(z)$	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.10	1.210	1.257	1.301	1.340	1.374	1.406	1.435	1.462	1.486	1.509
.20	1.530	1.550	1.569	1.586	1.603	1.619	1.633	1.647	1.659	1.672
.30	1.684	1.695	1.706	1.716	1.725	1.735	1.744	1.753	1.761	1.769
.40	1.777	1.784	1.792	1.799	1.805	1.812	1.818	1.825	1.831	1.837
.50	1.843	1.848	1.853	1.858	1.863	1.868	1.872	1.877	1.881	1.885
.60	1.889	1.893	1.897	1.901	1.904	1.908	1.911	1.914	1.917	1.921
.70	1.924	1.927	1.930	1.932	1.935	1.938	1.940	1.943	1.945	1.948
.80	1.950	1.953	1.955	1.957	1.959	1.962	1.964	1.966	1.968	1.970
.90	1.972	1.974	1.976	1.977	1.979	1.981	1.982	1.984	1.985	1.987

Table 3: Simulation results for the s safety stock coefficient functions $\varphi_{98\%,1}(z)$

Using the same *linear interpolation method*, we can estimate the expected safety stock coefficient functions. The simulations results in Table 2 and 3 are also show in Figure 5.

Figure 5: Safety stock coefficient functions $\psi_{98\%,1}(z)$ and $\varphi_{98\%,1}(z)$

From the above figure, we observe that the required safety stock levels are increasing when the standardized MPC quantity z increases. In addition the required safety stock coefficient converges to 2.055, which is the safety stock coefficient to maintain a service level of 98% in the traditional single-channel vendor-buyer system. Further details of the simulation can be found in Wang [21].

5.3 Quadratic Approximation

Using the linear interpolation method discussed in Section 5.1 and 5.2, we can quantitatively estimate the impacts of the MPC agreement on the dual-channel vendor-buyer system. In some circumstances, however, an analytical method might still be desirable or necessary. We introduce a quadratic approximation method to estimate the three coefficient functions $k(z)$, $\psi_{\alpha,L}(z)$ and $\varphi_{\alpha,L}(z)$ as piece-wise quadratic functions, which have the following structure:

$$k(z) \approx A_k^r z^2 + B_k^r z + C_k^r, \quad \forall z \in (Z_r, Z_{r+1}] \tag{5.2}$$

$$\psi_{\alpha,L_b}(z) \approx A_{\psi,\alpha,L_b}^r z^2 + B_{\psi,\alpha,L_b}^r z + C_{\psi,\alpha,L_b}^r, \quad \forall z \in (Z_r, Z_{r+1}] \tag{5.3}$$

$$\varphi_{\alpha,L_v}(z) \approx A_{\varphi,\alpha,L_v}^r z^2 + B_{\varphi,\alpha,L_v}^r z + C_{\varphi,\alpha,L_v}^r, \quad \forall z \in (Z_r, Z_{r+1}] \tag{5.4}$$

where all the values of A, B, C are constant in the three ranges $(Z_r, Z_{r+1}]$: (0.1, 0.2], (0.2, 0.3] and (0.3, 1]. We conducted regression on the simulation results of the coefficient functions $k(z)$, $\psi_{98\%,1}(z)$, $\varphi_{98\%,3}(z)$ and $\varphi_{98\%,5}(z)$. The regression parameters are given in Table 4, and the quadratic approximation functions are presented in Figures 6 and 7.

		$k(z)$	$\psi_{98\%,1}(z)$	$\varphi_{98\%,3}(z)$	$\varphi_{98\%,5}(z)$
	A^1	165.45	-14.07	-10.763	-10.089
$0.1 < z \le 0.2$	B^1	-73.48	7.138	5.602	5.172
	C^1	10.08	0.704	0.989	1.107
	A^2	32.50	-3.976	-3.053	-3.098
$0.2 < z \le 0.3$	B^2	-24.28	3.410	2.662	2.552
	C^2	5.52	1.051	1.271	1.353
	A^3	3.63	-0.533	-0.419	-0.359
$0.3 < z \le 1.0$	B^3	-5.69	1.058	0.825	0.696
	C^3	2.48	1.463	1.598	1.676

Table 4: Simulation results for the s safety stock coefficient functions $\varphi_{98\%,1}$(z)

Figure 6: Quadratic approximation functions of $k(z)$

Figure 7: Quadratic approximation functions of $\psi_{98\%,1}(z)$, $\varphi_{98\%,3}(z)$ and $\varphi_{98\%,5}(z)$

Given these approximation functions, we can estimate the total system cost $C_{sys}(z)$ as a piece-wise quadratic function of the standardized MPC quantity z as follows.

$$
\begin{aligned}
C_{sys}(z) \approx{}& \sigma\left(A_k^r h_b + \sqrt{L_b+1}A^r_{\psi,\alpha,L_b+1}h_b + \sqrt{L_{rdc}}A^r_{\varphi,\alpha,L_{rdc}}h_{rdc} + \sqrt{L_{cdc}}A^r_{\varphi,\alpha,L_{cdc}}h_{cdc}\right)z^2 \\
&+\sigma\left((c_2-c_1) + B_k^r h_b + B^r_{\psi,\alpha,L_b+1}h_b + B^r_{\varphi,\alpha,L_{rdc}}h_{rdc} + B^r_{\varphi,\alpha,L_{cdc}}h_{cdc}\right)z \\
&+\left[(c_3-c_2)(1-\beta)\mu + c_1\mu + 0.5\mu h_b + C_k^r h_b + C^r_{\psi,\alpha,L_b+1}h_b + C^r_{\varphi,\alpha,L_{rdc}}h_{rdc} + C^r_{\varphi,\alpha,L_{cdc}}h_{cdc}\right]
\end{aligned}
$$

(5.5)

which has a quadratic structure as follows:

$$C_{sys}(z) \approx A^r_{sys}z^2 + B^r_{sys}z + C^r_{sys},\ \forall z \in (Z_r, Z_{r+1}] \tag{5.6}$$

According to our discussion in Section 5.1, it is not likely that the optimal standardized MPC quantity z^* falls outside of the range of [0.1, 1.0]. Thus, we only need to compare the limits and the value of $-B^r_{sys}/2A^r_{sys}$ in each range of $(Z_r, Z_{r+1}]$; we have

$$-B^r_{sys}/2A^r_{sys} = \frac{-\left[(c_2-c_1) + B_k^r h_b + B^r_{\psi,\alpha,L_b+1}h_b + B^r_{\varphi,\alpha,L_{rdc}}h_{rdc} + B^r_{\varphi,\alpha,L_{cdc}}h_{cdc}\right]}{2\left(A_k^r h_b + \sqrt{L_b+1}A^r_{\psi,\alpha,L_b+1}h_b + \sqrt{L_{rdc}}A^r_{\varphi,\alpha,L_{rdc}}h_{rdc} + \sqrt{L_{cdc}}A^r_{\varphi,\alpha,L_{cdc}}h_{cdc}\right)},\ z^* \in (Z_r, Z_{r+1}]$$

(5.7)

The equation (5.7) supports the observations stated in Section 4.4 that the optimal solution z^* is independent of the demand parameters *μ and σ*, and the optimal solution z^* decreases when the channel supply cost rates difference (c_2-c_1) increases

6. Numerical Cases

In this section, we present a numerical study for a dual-channel vendor-buyer system with the following parameters as a base case.

- At the buyer, the product has an *iid* normally distributed demand D (1000, 400) per week. The demand D is independent of the selling price, which is \$30.

- Through the indirect channel, the buyer purchases the product at the price P of \$27. For the orders delivered through the direct channel, a discount λ is offered.

- The product has cumulated product costs of \$27, \$23, and \$22 at the buyer, the regional DC, and the central DC, respectively.

- Holding cost is incurred based on an annual interest rate of 25% and 50 weeks in a year.

- Channel supply cost rates are c_1=\$0.8 for the direct channel, and c_2=\$1.2 for the indirect channel. For the purpose of simplicity, we also assume that c_3=\$1.2 for the "backup" channel.

- Each location replenishes its inventory every week.

- At the buyer, the target service level α is 98%.

- The net replenishment lead time at the central DC L_{cdc} is 5 weeks, at the regional DC L_{rdc} is 3 weeks and at the buyer L_b is 0 week.

For the system described above, we investigated the effects of different system parameters on the optimal solution z^*.

I. Effect of demand parameters μ and σ

We have four numerical cases with demand parameters of [(μ=1000, σ=400), (μ=1000, σ=450), (μ=800, σ=500), (μ=800, σ=550)]. The total system cost C_{sys} for each case is shown in Figure 8.

Figure 8: Effect of demand mean μ and STD σ

From the above figure, we observe that the total system cost C_{sys} is a convex function of the standardized MPC quantity z and is exponentially increasing when the value z gets close to zero. This trend is mainly due the exponentially increasing surplus inventory cost. For all four cases, the optimal standardized MPC quantity z^* is 0.248. This confirms our previous analysis that the optimal standardized MPC quantity z^* is independent of the demand parameters μ and σ.

For the case of demand parameters (μ = 1000, σ = 400), we have

- The optimal value z^* of 0.248 and the optimal MPC quantity Q^* of 900.
- The minimum system cost C^*_{sys} is $1392 per week. The total system cost for the case of no MPC agreement is $1744 per week; thus direct delivery of 900 units results in a system cost savings of $352 per week.
- The holding cost for the surplus inventory is $80.5 per week.
- The cycle inventory holding cost at the buyer is $67.5 per week.
- The channel supply cost is $840 per week and the supply cost savings is $360 per week.
- The safety stock holding costs at central DC, regional DC, and buyer are $176, $139, and $89 per week.
- To equally allocate the benefit between the buyer and vendor, a purchase discount of 0.26 $/unit should be offered for the regular orders.

II. Effect of channel supply cost rates

We studied four numerical cases with channel supply cost rates parameters of [(c_1=0.8, c_2=1), (c_1=0.8, c_2=1.2), (c_1=0.8, c_2=1.4), (c_1=0.8, c_2=1.6)]. The total system cost C_{sys} for each case is shown in Figure 9.

Figure 9: Effect of channel supply cost rates c_1 and c_2

We observe that the optimal standardized MPC quantity z^* decreases when the channel supply cost difference (c_2-c_1) increases, and consequently, the optimal MPC quantity Q^* increases. When the standardized MPC quantity z increases, the system cost difference is increasing mainly due to the increasing cost difference in supplying product in cost c_2.

III. Effect of holding cost rates h_b and h_{cdc}

In Figure 10, we demonstrate the effect of the holding cost rate h_b on the optimal solution z^* in the integrated coordination model. The four sets of parameters are [(h_b=5), (h_b=6), (h_b=7), (h_b=8)].

From the numerical results, we observe that the optimal solution z^* increases when holding cost rate h_b increases, which confirms our analysis in Section 4.5.

Figure 10: Effect of holding cost rate h_b

In Figure 11, we demonstrate the effect of holding cost rate h_{cdc} on the optimal solution z^* in the integrated coordination model. The four sets of parameters are [(h_{cdc} =5), (h_{cdc} =6), (h_{cdc} =7), (h_{cdc} =8)]. We observe that the optimal standardized MPC quantity z^* increases when the holding cost rate h_{cdc} decreases.

Figure 11: Effect of holding cost rate h_{cdc}

In addition, we use the quadratic approximation parameters to derive the expression of the total system cost for the case of demand (1000, 400). The quadratic parameters are given as follows.

	A	B	C
$0.1 < z \le 0.2$	6324.4	-2467.2	1637.6
$0.2 < z \le 0.3$	992.27	-503.77	1456.7
$0.3 < z \le 1.0$	98.57	44.083	1372.5

Table 5: Quadratic parameters for total system cost

The approximation and simulation results are shown in Figure 12.

Figure 12: Quadratic approximation functions of total system cost $C_{sys}(z)$

From the above figure, we see that the quadratic approximation function fits the simulation results well within the range of [0.1, 0.8] and the error grows when the standardized MPC quantity z approaches 1. The minimum total system cost occurs in the interval $z \in (0.2, 0.3)$, and the optimal standardized MPC quantity z^* should take the value of $\left(-B_{sys}^2 / 2A_{sys}^2\right) = 0.254$ and the estimated optimal system cost C_{sys} is \$1392 per week. Compared to the optimal simulation result of [z^*=0.248, C_{sys}=1392], the quadratic approximation method results in a solution that is very close to the optimal solution.

7. Conclusions

In this paper, we consider a dual-channel vendor-buyer system in which the buyer can replenish its inventory through two distinct channels: an indirect channel, which is characterized by short lead time and high channel supply cost; and a direct channel, which is characterized by long lead time and low channel supply cost. We propose a minimum purchase commitment (MPC) agreement; that is, the buyer commits to purchase a predetermined and fixed quantity through the direct channel in each time period, and has the option to purchase a flexible quantity through the indirect channel in each time period. We study the impacts of the MPC agreement on the inventory, safety stock and cost of each facility in the dual-channel vendor-buyer system, and develop a simulation-based method to estimate these impacts.

This paper contributes to the literature by incorporating the vendor-buyer coordination issue into the traditional dual-channel inventory model. The analysis presented in this research can serve as a building block and a decision support tool for several aspects of vendor-buyer coordination, supply chain network design, supply strategy development, and supply contracts negotiation. It can also provide insights for a vendor to compete for a single sourcing agreement; that is, the vendor could design a dual-channel MPC supply contract to provide both economies of scale and substantial flexibility that make it unfavorable for the buyer to consider an alternate vendor. This research can also be applied to global supply chain management issues in which global suppliers offer a cheaper price, but require long lead times because of the long shipping distances. Therefore, these suppliers are contracted with a stable replenishment quantity; in situations where the magnitude of demand is larger than expected, a more expensive domestic supplier is used. The models assist in allocating the purchase volume between these supply options.

In this paper, we assumed that safety stock is carried to maintain a desired service level. A reasonable future research direction could be to consider fill rate as the performance measure in these models. Our assumption of stationary and price-insensitive demand could be inapplicable for cases of seasonal products or short planning contexts. When the demand is price sensitive, the vendor and buyer can cooperate in determining both the MPC quantity and the selling price that maximize the total system revenue. Thus, incorporating non-stationary and/or price-sensitive demand in the dual-channel vendor-buyer coordination problem represents an important future research direction.

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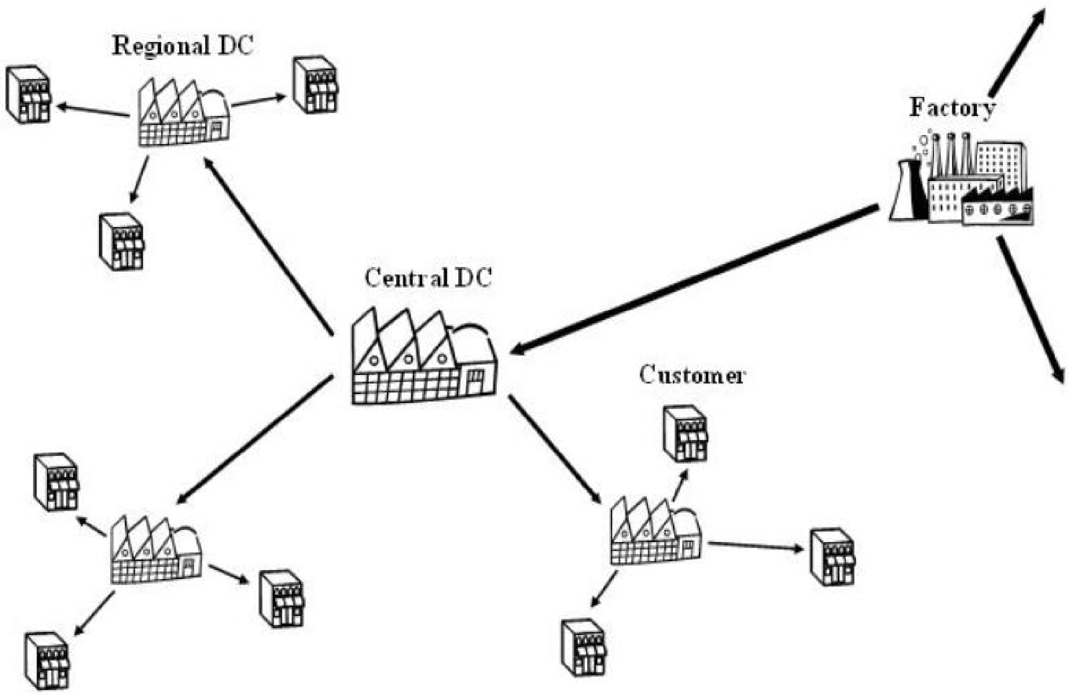

Figure 1: A multi-stage distribution system

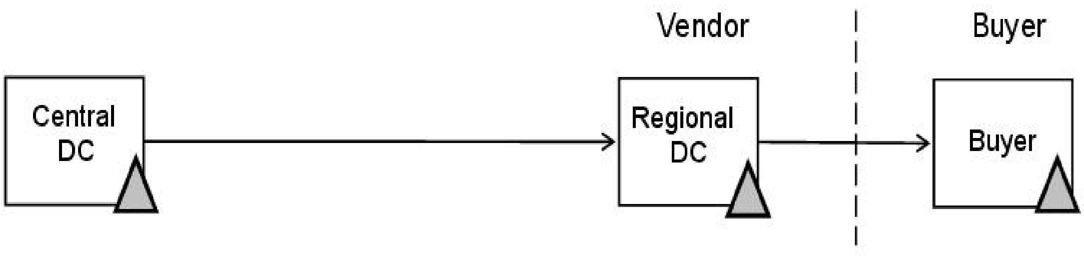

Figure 2: A single-channel vendor-buyer system

Figure 3: A dual-channel vendor-buyer system

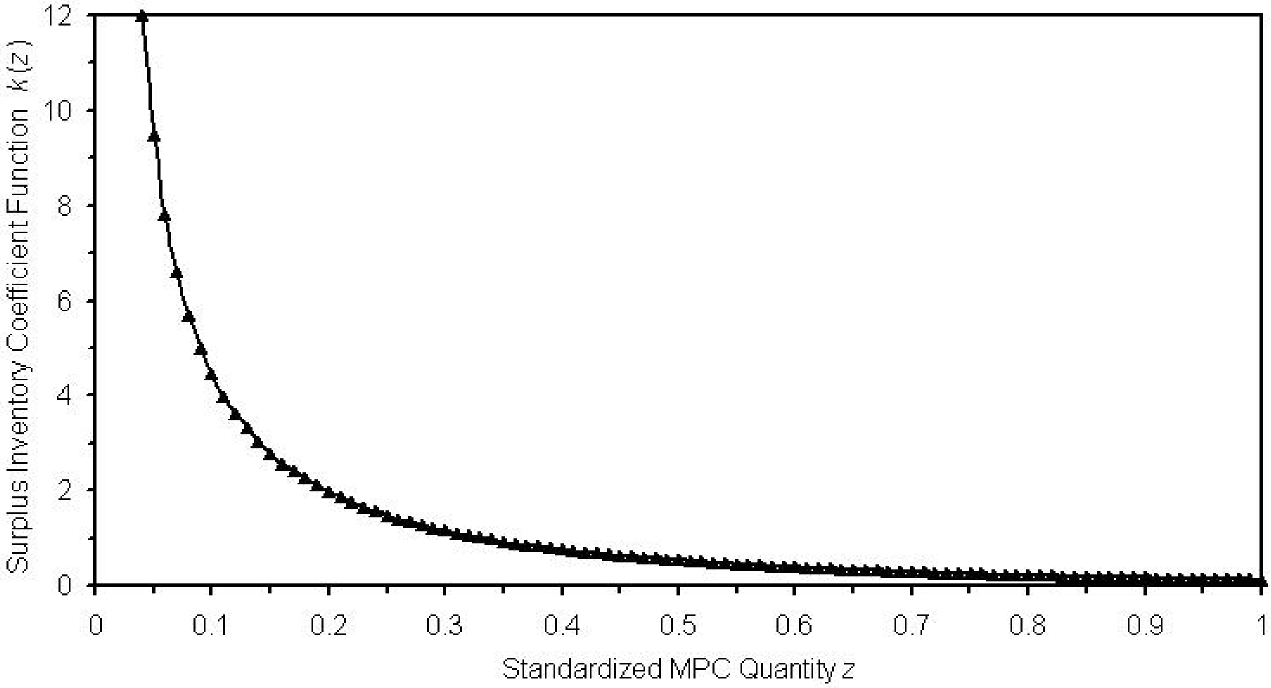

Figure 4: Surplus inventory coefficient function $k(z)$

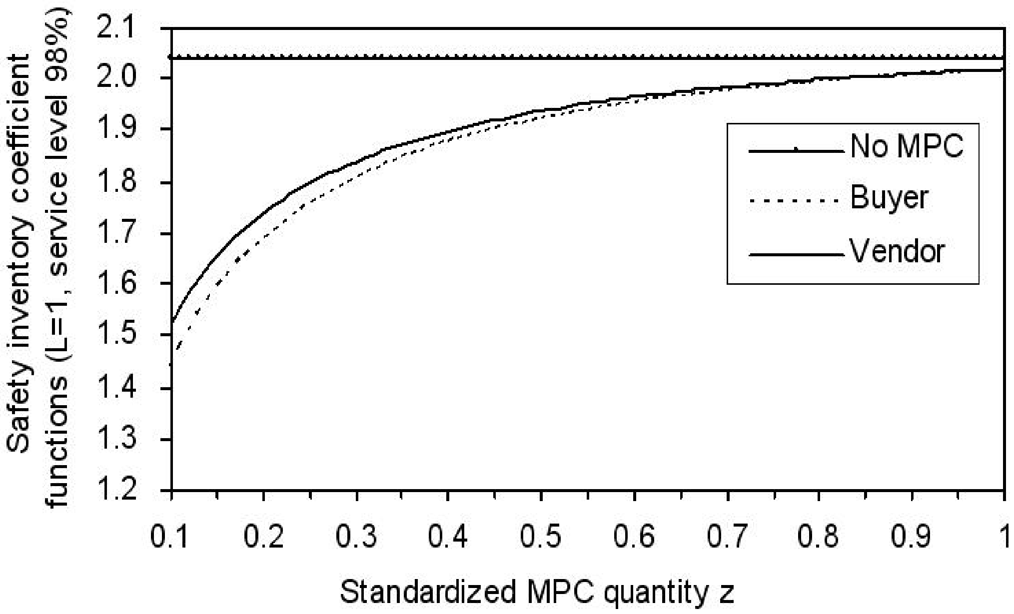

Figure 5: Safety stock coefficient functions $\psi_{98\%,1}(z)$ and $\varphi_{98\%,1}(z)$

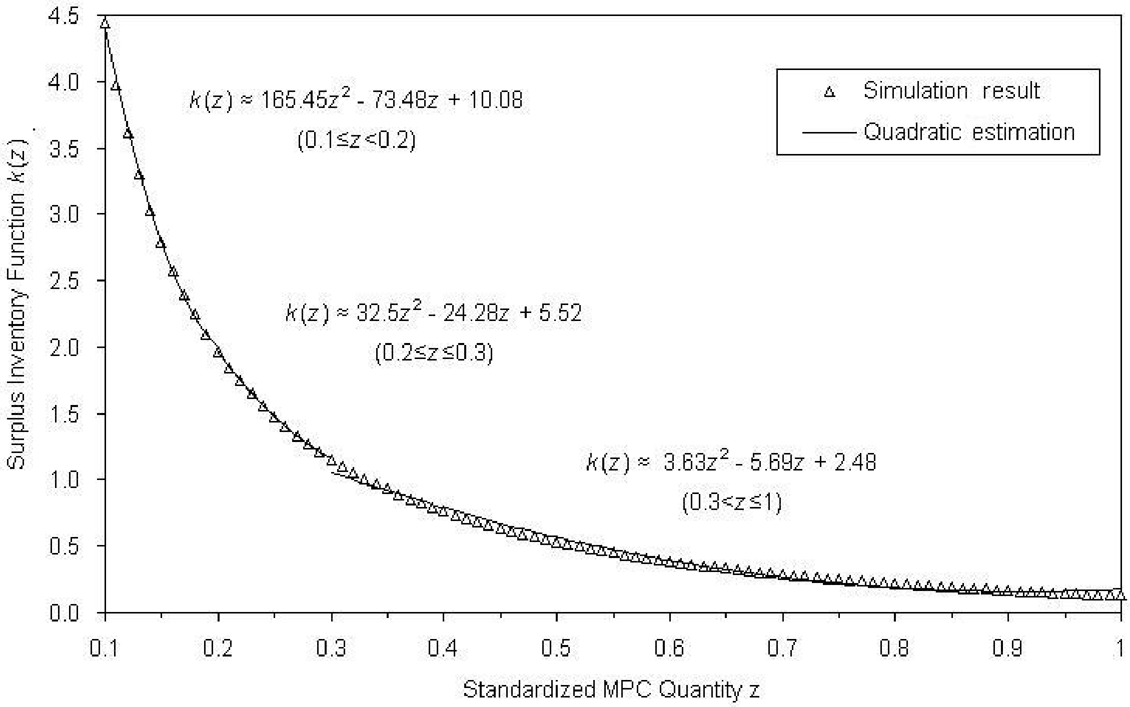

Figure 6: Quadratic approximation functions of $k(z)$

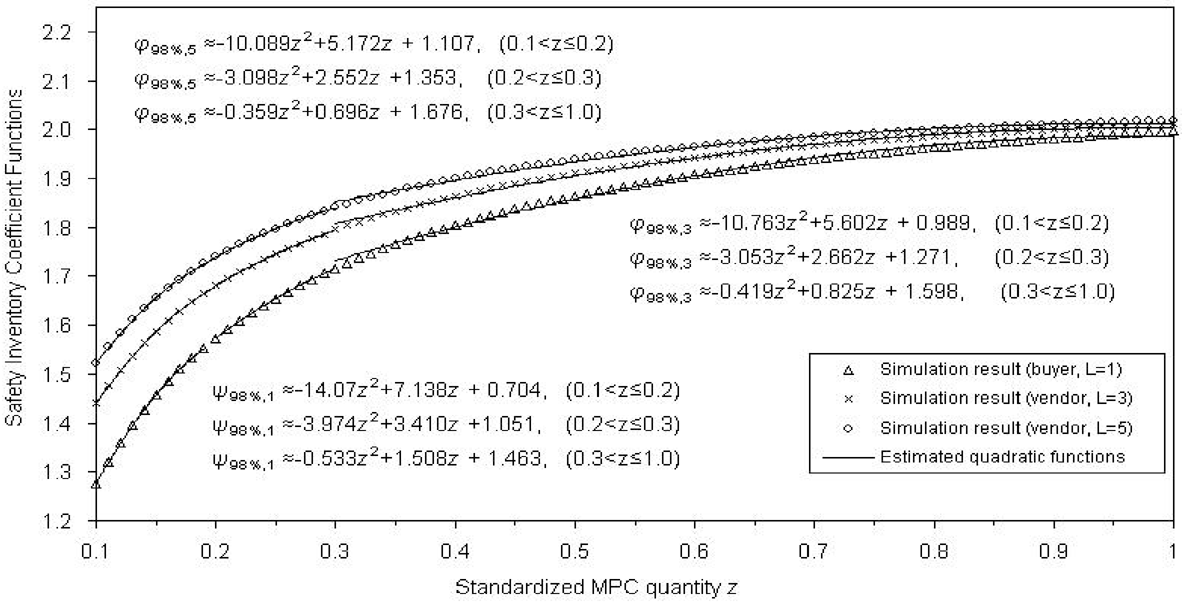

Figure 7: Quadratic approximation functions of $\psi_{98\%,1}(z)$, $\varphi_{98\%,3}(z)$ and $\varphi_{98\%,5}(z)$

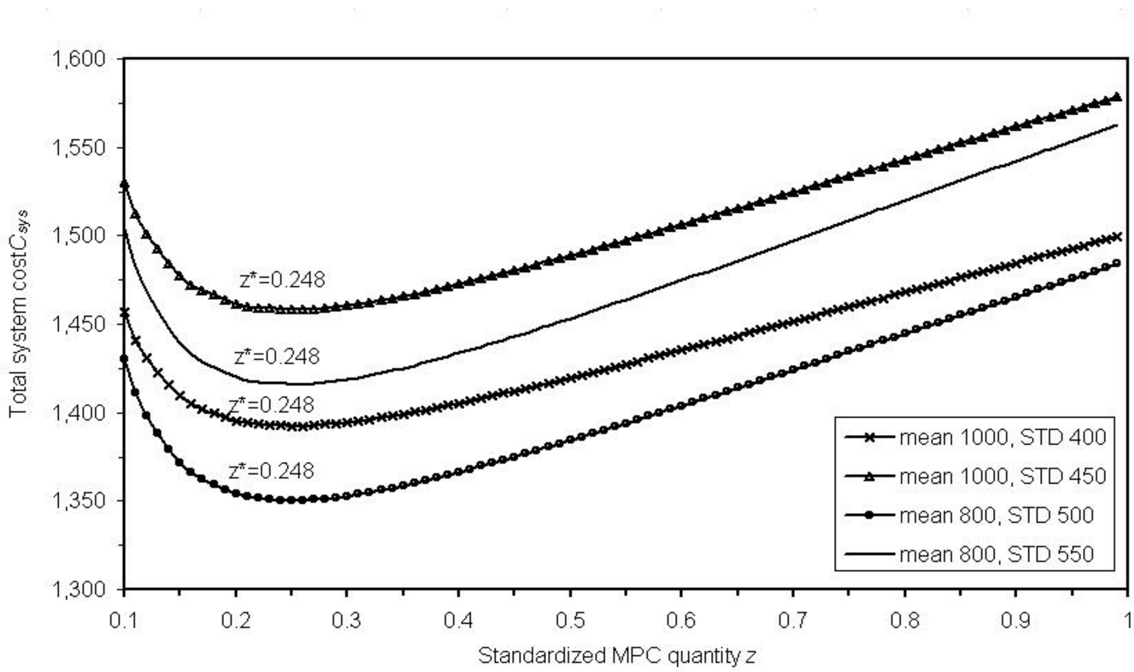

Figure 8: Effect of demand mean μ and STD σ

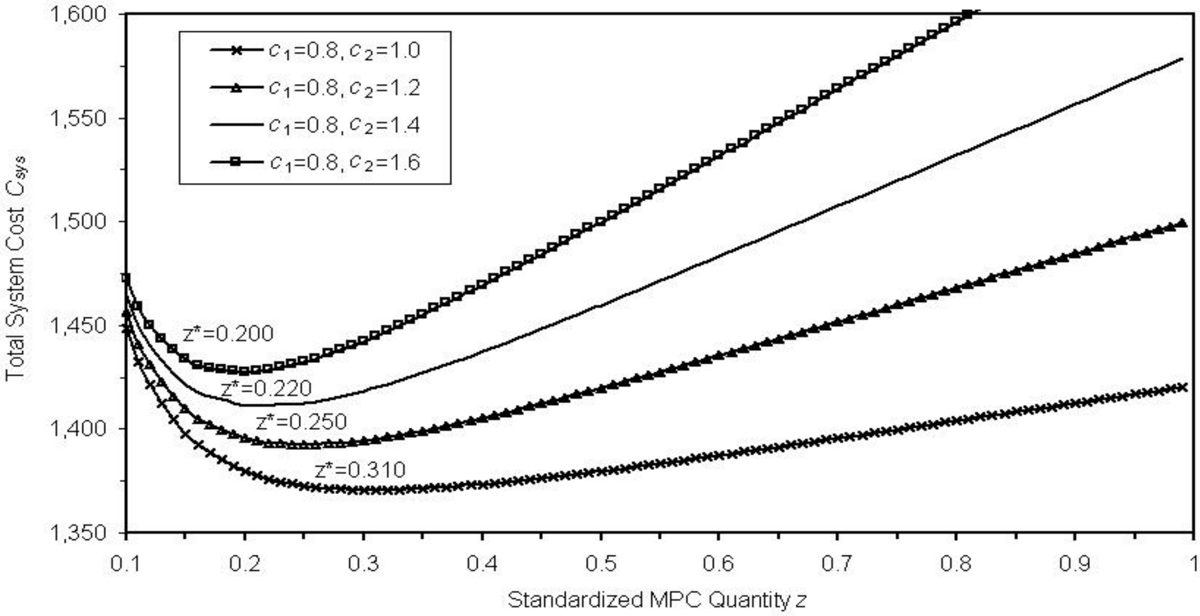

Figure 9: Effect of channel supply cost rates c_1 and c_2

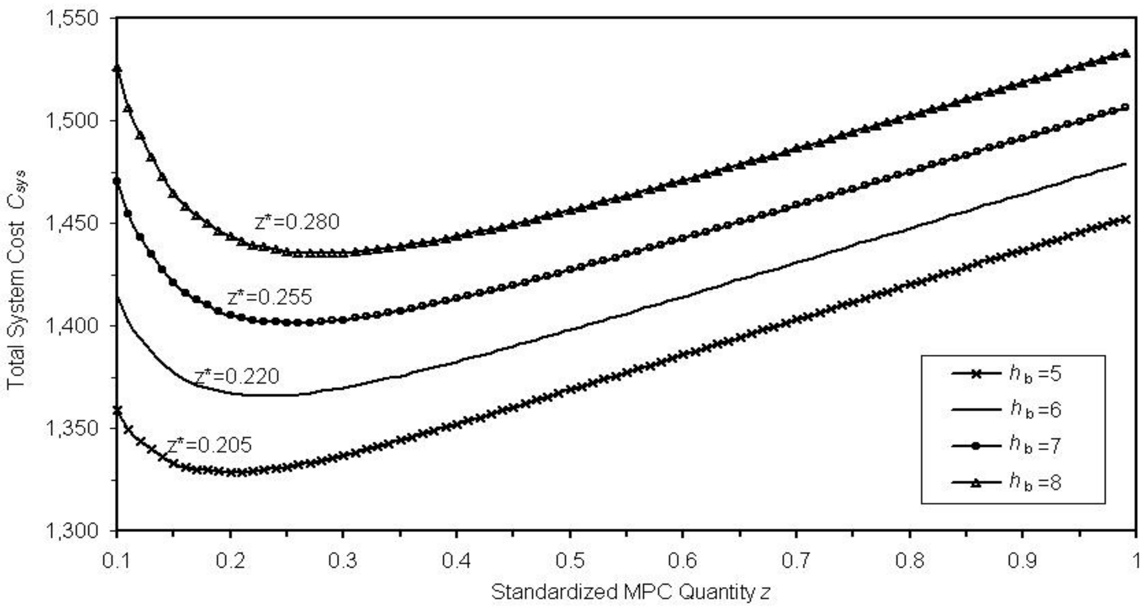

Figure 10: Effect of holding cost rate h_b

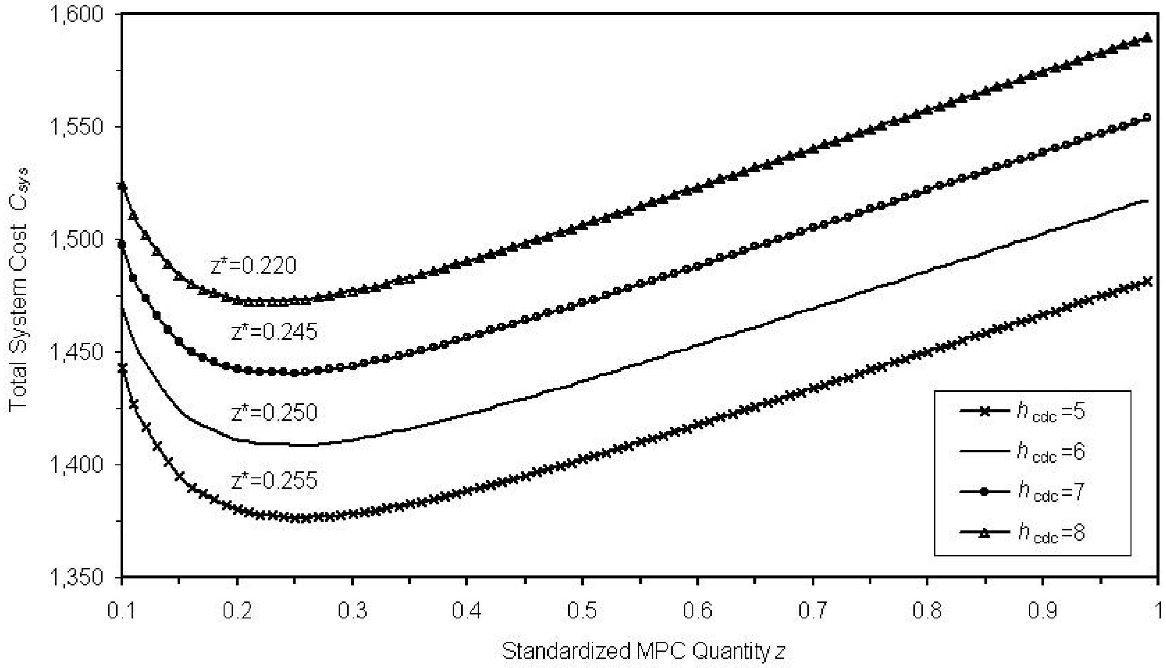

Figure 11: Effect of holding cost rate h_{cdc}

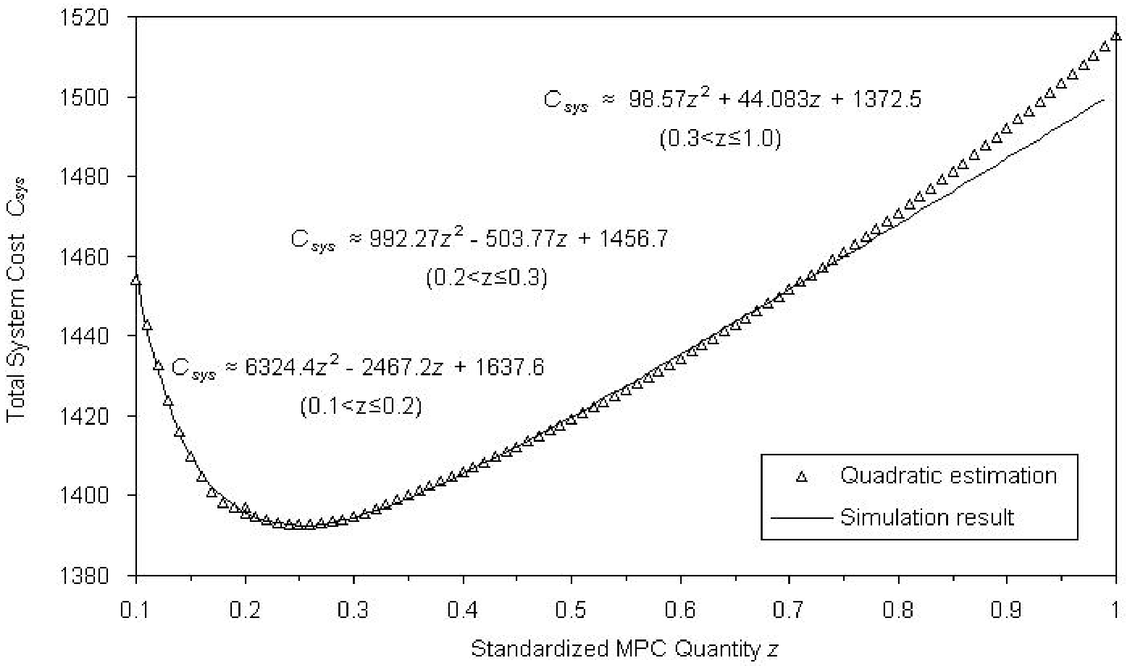

Figure 12: Quadratic approximation functions of total system cost $C_{sys}(z)$