Appendices

A. Proofs of Main Theorems

Proof of Theorem 1. A brief roadmap of the proof is as follows. We first show that there exist polytopes in the 0 – 1 hypercube, parameterized by $\gamma \in \mathbb{R}^n$, that correspond to worst-case topologies (see [12]); the remaining of the proof deals with identifying the worst-case polytope within this class, i.e., the worst-case value of the parameter $\gamma$, utilizing symmetry and optimization theory arguments.

Geometrically, the $\alpha$-fair allocation of any convex utility set in the 0 – 1 hypercube lies on its boundary. Consider now the supporting hyperplane at the $\alpha$-fair allocation, defined by the gradient of $W_\alpha$. Intuitively, any set that is contained in the polytope defined by that supporting hyperplane (and the 0 – 1 hypercube) would have the same $\alpha$-fair allocation. However, that does not hold true for the utilitarian or max-min allocations. In fact, by considering convex supersets of the original utility set, contained in the described polytope, one could obtain higher values for the utilitarian and/or max-min objectives, while the $\alpha$-fair allocation remains constant. As such, one need only consider polytopes of the described form for worst-cases. Note that such an approach can be generalized in a straightforward manner for any similar settings where one considers multiple competing objective functions.

Without loss of generality, we assume that $U$ is monotone\(^5\). This is because both schemes we consider, namely utilitarian and $\alpha$-fairness yield Pareto optimal allocations. In particular, suppose there exist allocations $a \in U$ and $b \notin U$, with allocation $a$ dominating allocation $b$, i.e., $0 \leq b \leq a$. Note that allocation $b$ can thus not be Pareto optimal. Then, we can equivalently assume that $b \in U$, since $b$ cannot be selected by any of the schemes.

We also assume that the maximum achievable utilities of the players are equal to 1; the proof can be trivially modified otherwise.

By combining the above two assumptions, we get

$$e_j \in U, \quad \forall j = 1, \ldots, n,$$

where $e_j$ is the unit vector in $\mathbb{R}^n$, with the $j$th component equal to 1.

Fix $\alpha > 0$ and let $z = z(\alpha) \in U$ be the unique allocation under the $\alpha$-fairness criterion (since $W_\alpha$ is strictly concave for $\alpha > 0$), and assume, without loss of generality, that

$$z_1 \geq z_2 \geq \ldots \geq z_n.$$  

\(^5\)A set $A \subset \mathbb{R}_+^n$ is called monotone if $\{b \in \mathbb{R}^n \mid 0 \leq b \leq a \} \subset A, \forall a \in A$, where the inequality sign notation for vectors is used for componentwise inequality.
The necessary first order condition for the optimality of $z$ can be expressed as
\[ \nabla W_\alpha(z)^T(u - z) \leq 0 \Rightarrow \sum_{j=1}^{n} z_j^{-\alpha}(u_j - z_j) \leq 0, \quad \forall u \in U, \]
or equivalently
\[ \gamma^T u \leq 1, \quad \forall u \in U, \quad (7) \]
where
\[ \gamma_j = \frac{z_j^{-\alpha}}{\sum_i z_i^{1-\alpha}}, \quad j = 1, \ldots, n. \quad (8) \]
Note that (8) implies
\[ \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n. \quad (9) \]
Using (5) and (7) we also get
\[ \gamma_j = \gamma^T e_j \leq 1, \quad j = 1, \ldots, n. \quad (10) \]

We now use (7), and the fact that each player has a maximum achievable utility of 1 to bound the sum of utilities under the utilitarian principle as follows:
\[
\text{SYSTEM} (U) = \max \left\{ 1^T u \bigg| u \in U \right\} \\
\leq \max \left\{ 1^T u \bigg| 0 \leq u \leq 1, \gamma^T u \leq 1 \right\}. \quad (11)
\]

Using the above inequality,
\[
\text{POF} (U; \alpha) = \frac{\text{SYSTEM} (U) - \text{FAIR} (U; \alpha)}{\text{SYSTEM} (U)} \\
= 1 - \frac{\text{FAIR} (U; \alpha)}{\text{SYSTEM} (U)} \\
= 1 - \frac{\sum_{j=1}^{n} z_j}{\text{SYSTEM} (U)} \\
\leq 1 - \frac{\sum_{j=1}^{n} z_j}{\max \left\{ 1^T u \bigg| 0 \leq u \leq 1, \gamma^T u \leq 1 \right\}}. \quad (12)
\]

The optimization problem in (12) is the linear relaxation of the well-studied knapsack problem, a version of which we review next. Let $w \in \mathbb{R}_+^n$ be such that $0 < w_1 \leq \ldots \leq w_n < 1$ (in particular, $\gamma$ satisfies those conditions). Then, one can show (see Bertsimas and Tsitsiklis (1997)) that the linear optimization problem
\[
\begin{align*}
\text{maximize} & \quad 1^T y \\
\text{subject to} & \quad w^T y \leq 1 \\
& \quad 0 \leq y \leq 1,
\end{align*} \quad (13)
\]
has an optimal value equal to $\ell(w) + \delta(w)$, where

$$\ell(w) = \max \left\{ i \left| \sum_{j=1}^{i} w_j \leq 1, \ i \leq n - 1 \right. \right\} \in \{1, \ldots, n - 1\}$$

(14)

$$\delta(w) = \frac{1 - \sum_{j=1}^{\ell(w)} w_j}{w_{\ell(w)+1}} \in [0, 1].$$

(15)

We can apply the above result to compute the optimal value of the problem in (12),

$$\max \left\{ 1^T u \left| 0 \leq u \leq 1, \gamma^T u \leq 1 \right. \right\} = \ell(\gamma) + \delta(\gamma).$$

(16)

The bound from (12) can now be rewritten,

$$\text{POF}(U; \alpha) \leq 1 - \frac{\sum_{j=1}^{n} z_j}{\ell(\gamma) + \delta(\gamma)}.$$ 

(17)

Consider the set $S$ in the $(n + 3)$-dimensional space, defined by the following constraints with variables $d \in \mathbb{R}$, $\lambda \in \mathbb{N}$ and $x_1, \ldots, x_\lambda, \varpi_{\lambda+1}, \varpi_{\lambda+1}, x_{\lambda+2}, \ldots, x_n \in \mathbb{R}$. The variables $d$ and $\lambda$ correspond to $\delta$ and $\lambda$ accordingly, whereas $x$ corresponds to $z$. Note also that we associate two variables, $\varpi_{\lambda+1}$ and $\varpi_{\lambda+1}$, with $z_{\lambda+1}$.

$$0 \leq d \leq 1$$

(18a)

$$1 \leq \lambda \leq n - 1$$

(18b)

$$0 \leq x_n \leq \ldots \leq x_{\lambda+2} \leq \varpi_{\lambda+1} \leq x_\lambda \leq \ldots \leq x_1 \leq 1$$

(18c)

$$x_n^{-\alpha} \leq x_1^{-1-\alpha} + \ldots + x_\lambda^{-1-\alpha} + d \varpi_{\lambda+1}^{-\alpha} + (1 - d) x_{\lambda+1}^{-1-\alpha} + x_{\lambda+2}^{-1-\alpha} + \ldots + x_n^{-1-\alpha}$$

(18d)

$$x_1^{-\alpha} + \ldots + x_\lambda^{-\alpha} + d \varpi_{\lambda+1}^{-\alpha} \leq x_1^{-1-\alpha} + \ldots + x_\lambda^{-1-\alpha} + d \varpi_{\lambda+1}^{-1-\alpha} + (1 - d) x_{\lambda+1}^{-1-\alpha} + x_{\lambda+2}^{-1-\alpha} + \ldots + x_n^{-1-\alpha}.$$ 

(18e)

The introduction of those new variables will allow us to further simplify (17). In particular, we show that

$$\frac{\sum_{j=1}^{n} z_j}{\ell(\gamma) + \delta(\gamma)} \geq \min_{(d, \lambda, x) \in S} \frac{x_1 + \ldots + x_\lambda + d \varpi_{\lambda+1} + (1 - d) x_{\lambda+1} + x_{\lambda+2} + \ldots + x_n}{\lambda + d}.$$ 

(19)

We pick values for $d$, $\lambda$ and $x$ that are such that (a) they are feasible for $S$, and (b) the function argument of the minimum, if evaluated at $(d, \lambda, x)$, is equal to the left-hand side of (19). In
particular, let
\[ d = \delta(\gamma), \quad \lambda = \ell(\gamma), \]
\[ x_j = z_j, \quad j \neq \lambda + 1, \quad x_{\lambda+1} = x_{\lambda+1} = z_{\lambda+1}. \]

Then, (18a), (18b) and (18c) are satisfied because of (15), (14) and (6) respectively. By the definition of \( \gamma \) and the selected value of \( x \), (18d) can be equivalently expressed as
\[ \gamma_n \leq 1, \]
which is implied by (10). Similarly, (18e) is equivalent to
\[ \gamma_1 + \ldots + \gamma_{\ell(\gamma)} + \delta(\gamma)\gamma_{\ell(\gamma)+1} \leq 1, \]
which again holds true (by (15)). The function argument of the minimum, evaluated at the selected point, is clearly equal to the left-hand side of (19). Finally, the minimum is attained by the Weierstrass Theorem, since the function argument is continuous, and \( S \) is compact. Note that (18d) in conjunction with (18e) bound \( x_n \) away from 0. In particular, if \( \alpha \geq 1 \), we get
\[ x_n^{-\alpha} \leq x_1^{1-\alpha} + \ldots + x_n^{1-\alpha} \leq nx_n^{1-\alpha} \Rightarrow x_n \geq \frac{1}{n}. \]
Similarly, for \( \alpha < 1 \) we get
\[ x_n \geq \left( \frac{1}{n} \right)^{\frac{1}{\alpha}}. \]

To evaluate the minimum in (19), one can assume without loss of generality that for a point \((d', \lambda', x') \in S\) that attains the minimum, we have
\[ x'_1 = \ldots = x'_{\lambda} = x'_{\lambda+1}, \quad x'_{\lambda+1} = x'_{\lambda+2} = \ldots = x'_n. \tag{20} \]

Technical details are included in Section C. Using this observation, we can further simplify (19). In particular, consider the set \( T \subset \mathbb{R}^3 \), defined by the following constraints, with variables \( x_1, x_2 \) and \( y \) (since \( x'_1 = \ldots = x'_{\lambda} = x'_{\lambda+1} \), we associate \( x_1 \) with them, and similarly we associate \( x_2 \) with the remaining variables of \( x' \); variable \( y \) is associated with \( \lambda + d \)):
\begin{align*}
0 \leq x_2 & \leq x_1 \leq 1 \tag{21a} \\
1 \leq y & \leq n \tag{21b} \\
x_2^{-\alpha} & \leq yx_1^{1-\alpha} + (n - y)x_2^{1-\alpha} \tag{21c} \\
yx_1^{-\alpha} & \leq yx_1^{1-\alpha} + (n - y)x_2^{1-\alpha}. \tag{21d}
\end{align*}
Using similar arguments as in showing (19), one can then show that
\[
\min_{(d, \lambda, x) \in S} x_1 + \ldots + x_\lambda + d x_{\lambda+1} + (1 - d) x_{\lambda+2} + \ldots + x_n \geq \min_{(x_1, x_2, y) \in T} y x_1 + (n - y) x_2.
\] (22)

If we combine (17), (19), (22) we get
\[
\text{POF}(U; \alpha) \leq 1 - \min_{(x_1, x_2, y) \in T} y x_1 + (n - y) x_2.
\] (23)

The final step is the evaluation of the minimum above. Let \((x_1^*, x_2^*, y^*) \in T\) be a point that attains the minimum. Then, we have
\[
y^* < n, \quad x_2^* < x_1^*.
\] (24)

To see this, suppose that \(x_2^* = x_1^*\). Then, the minimum is equal to \(\frac{nx_1^*}{y^*}\). But, constraint (21d) yields that \(nx_1^* \geq y^*\), in which case the minimum is greater than or equal to 1. Then, (23) yields that the price of fairness is always 0, a contradiction. If \(y^* = n\), (21d) suggests that \(x_1^* = 1\). Also, the minimum is equal to \(1\), a contradiction.

We now show that (21c, 21d) are active at \((x_1^*, x_2^*, y^*)\). We argue for \(\alpha \geq 1\) and \(\alpha < 1\) separately.

\(\alpha \geq 1\) : Suppose that (21c) is inactive. Then, a small enough reduction in the value of \(x_2^*\) preserves feasibility (with respect to \(T\)), and also yields a strictly lower value for the minimum (since \(y^* < n\), by (21)), thus contradicting that the point attains the minimum. Similarly, if (21d) is inactive, a small enough reduction in the value of \(x_1^*\) leads to a contradiction.

\(\alpha < 1\) : Suppose that (21d) is inactive at \((x_1^*, x_2^*, y^*)\). Then, we increase \(y^*\) by a small positive value, such that (21d) and (21b) are still satisfied. Constraint (21c) is then relaxed, since \((x_1^*)^{1-\alpha} > (x_2^*)^{1-\alpha}\). The minimum then has a strictly lower value, a contradiction. Hence, (21d) is active at any point that attains the minimum. If we solve for \(y\) and substitute back, the objective of the minimum becomes
\[
x_1 + x_2^\alpha (x_1^{1-\alpha} - x_2^{1-\alpha}),
\] (25)

and the constraints defining the set \(T\) simplify to
\[
0 \leq x_2 \leq x_1 \leq 1
\] (26a)
\[
x_1^{-\alpha} - x_1^{1-\alpha} + x_2^{1-\alpha} \leq nx_1^{-\alpha} x_2.
\] (26b)

In particular, constraint (26b) corresponds to constraint (21c). In case (21c) is not active at a minimum, so is (26b). But then, a small enough reduction in the value of \(x_2^*\) leads to a
Since for any point that attains the minimum constraints (21c, 21d) are active, we can use the corresponding equations to solve for $x_1$ and $x_2$. We get

$$x_1 = \frac{\frac{1}{\alpha} + n - y + y^{\frac{1}{\alpha}}}{n - y + y^{\frac{1}{\alpha}}},$$  \hfill (27)  

$$x_2 = \frac{1}{n - y + y^{\frac{1}{\alpha}}}. \hfill (28)$$

If we substitute back to (23), we get

$$\text{POF} (U; \alpha) \leq 1 - \min_{x \in [1, n]} \frac{x^{1 + \frac{1}{\alpha}} + n - x}{x^{1 + \frac{1}{\alpha}} + (n - x)x}.$$ 

The asymptotic analysis is included in Section C.

**Proof of Theorem 2.** We follow similar steps to the ones in the proof of Theorem 1. Thus, assume that $U$ is monotone, the maximum achievable utilities of the players are equal to 1 and that $z_1 \geq z_2 \geq \ldots \geq z_n$ (where $z = z(\alpha) \in U$ is the unique $\alpha$-fair allocation). Then, for the variable $\gamma$ (defined as in (8)), we similarly have

$$\gamma^T u \leq 1, \quad \forall u \in U,$$

and

$$\gamma_1 \leq \gamma_2 \leq \ldots \gamma_n \leq 1.$$

We use the above to bound the maximum value of the fairness metric

$$\max \left\{ \min_{j=1, \ldots, n} u_j \bigg| u \in U \right\} \leq \max \left\{ \min_{j=1, \ldots, n} u_j \bigg| 0 \leq u \leq 1, \gamma^T u \leq 1 \right\} = \frac{1}{\gamma^T \gamma},$$

where the equality follows from $z \leq 1$ and $1^T \gamma \geq 1$. 
We bound the price of efficiency using $z_1 \geq \ldots \geq z_n$, $\gamma_n \leq 1$ and the inequality above as follows:

$$
\text{POE} (U; \alpha) = \frac{\max_{u \in U} \min_{j=1, \ldots, n} u_j - \min_{j=1, \ldots, n} z_j(\alpha)}{\max_{u \in U} \min_{j=1, \ldots, n} u_j}
$$

$$
= 1 - \frac{z_n}{\max_{u \in U} \min_{j=1, \ldots, n} u_j}
$$

$$
\leq 1 - z_n \sum_{j=1}^T \gamma_j
$$

$$
= 1 - \frac{z_n \left( z_1^{-\alpha} + z_2^{-\alpha} + \ldots + z_n^{-\alpha} \right)}{z_1^{1-\alpha} + z_2^{1-\alpha} + \ldots + z_n^{1-\alpha}}
$$

$$
= 1 - f^*,
$$

where $f^*$ is the optimal value of the problem

$$
\begin{align*}
\text{minimize} & \quad \frac{z_n \left( z_1^{-\alpha} + z_2^{-\alpha} + \ldots + z_n^{-\alpha} \right)}{z_1^{1-\alpha} + z_2^{1-\alpha} + \ldots + z_n^{1-\alpha}} \\ \text{subject to} & \quad 0 \leq z_n \leq z_{n-1} \leq \ldots \leq z_1 \leq 1 \\ & \quad z_n^{-\alpha} \leq z_1^{1-\alpha} + z_2^{1-\alpha} + \ldots + z_n^{1-\alpha}.
\end{align*}
$$

(29)

Let $z^*$ be an optimal solution of (29) (guaranteed to exist by the Weierstrass Theorem). Then, without loss of generality we can assume that (a) $z^*_1 = z^*_2 = \ldots = z^*_{n-1}$ and (b) $z^*_1 = 1$. Technical details are included in the Section C. Using those two assumptions, $f^*$ is then equal to

$$
\text{minimize} & \quad \frac{(n-1)x + x^{1-\alpha}}{n-1 + x^{1-\alpha}} \\ \text{subject to} & \quad 0 \leq x \leq 1 \\ & \quad x^{1-\alpha} \leq n-1 + x^{1-\alpha}.
$$

(30)

Finally, note that for $x \in [0, 1]$ the function $x^{-\alpha} - x^{1-\alpha} - n - 1$ is strictly decreasing, is positive for $x$ small and negative for $x = 1$. Hence, for $x \in [0, 1]$ the constraint $x^{-\alpha} \leq n-1 + x^{1-\alpha}$ is equivalent to $x \geq \rho$. As a result,

$$
f^* = \min_{\rho \leq x \leq 1} \frac{(n-1)x + x^{1-\alpha}}{n-1 + x^{1-\alpha}}.
$$

The asymptotic analysis is similar to the analysis in Theorem 1 and is omitted.

**B. More on Near Worst-case Examples for the Price of Fairness**

We demonstrate how one can construct near worst-case examples, for which the price of fairness is very close to the bounds implied by Theorem 1 for any values of the problem parameters; the
number of players $n$ and the value of the inequality aversion parameter $\alpha$. We then provide details about the bandwidth allocation problem in Section 3.1.1.

For any $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha > 0$, we create a utility set using Procedure 1.

**Procedure 1** Creation of near worst-case utility set

**Input:** $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha > 0$

**Output:** utility set $U$

1: compute $y := \arg\min_{x \in [1, n]} \frac{x^{1+\frac{1}{\alpha}} + n - x}{x^{1+\frac{1}{\alpha}} + (n - x)x}$

2: $x_1 \leftarrow \frac{y}{n - y + y^n} \quad \text{(as in (27))}$

3: $x_2 \leftarrow \frac{1}{n - y + y^n} \quad \text{(as in (28))}$

4: $\ell \leftarrow \min \{\text{round}(y), n - 1\}$

5: $\gamma_i \leftarrow \frac{x_i^{\frac{1}{\alpha}}}{y^{\frac{1}{\alpha}} + (n - y)x_i^{\frac{1}{\alpha}}}$ for $i = 1, 2$

6: $U \leftarrow \{u \in \mathbb{R}_+^n \mid \gamma_1 u_1 + \ldots + \gamma_1 u_\ell + \gamma_2 u_{\ell+1} + \ldots + \gamma_2 u_n \leq 1, \ u \leq 1 \ \forall j\}$

The following proposition demonstrates why Procedure 1 creates utility sets that achieve a price of fairness very close to the bounds implied by Theorem 1.

**Proposition 1.** For any $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha > 0$, the output utility set $U$ of Procedure 1 satisfies the conditions of Theorem 1. If $y \in \mathbb{N}$, the output utility set $U$ satisfies the bound of Theorem 1 with equality.

**Proof.** The output utility set $U$ is a bounded polyhedron, hence convex and compact. Boundedness follows from positivity of $\gamma_1$ and $\gamma_2$.

Note that the selection of $x_1$, $x_2$ and $y$ in Procedure 1 corresponds to a point that attains the minimum of (23), hence all properties quoted in the proof of Theorem 1 apply. In particular, by (18d) we have $\gamma_2 \leq 1$ and (21d) is tight, $y\gamma_1 = 1$. Moreover, the bound from Theorem 1 can be expressed as

$$\text{POF} (U; \alpha) \leq 1 - \frac{yx_1 + (n - y)x_2}{y}.$$ 

The maximum achievable utility of the $j$th player is equal to 1. To see this, note that the definition of $U$ includes the constraint $u_j \leq 1$, so it suffices to show that $e_j \in U$. For $j \leq \ell$, we have $\gamma_1 \leq \gamma_1 y = 1$. For $j > \ell$, we have $\gamma_2 \leq 1$. It follows that $U$ satisfies the conditions of Theorem 1.

Suppose that $y \in \mathbb{N}$. By (24) and the choice of $\ell$ in Procedure 1 we get $\ell = y$. Consider the vector $z \in \mathbb{R}^n$ with $z_1 = \ldots = z_\ell = x_1$ and $z_{\ell+1} = \ldots = z_n = x_2$. Then, the sufficient first order optimality condition for $z$ to be the $\alpha$-fair allocation of $U$ is satisfied, as for any $u \in U$

$$\sum_{j=1}^n z_j^{-\alpha} (u_j - z_j) = x_1^{-\alpha} (u_1 + \ldots + u_\ell) + x_2^{-\alpha} (u_{\ell+1} + \ldots + u_n) - yx_1^{1-\alpha} - (n - y)x_2^{1-\alpha} \leq 0,$$
since $\gamma_1(u_1 + \ldots + u_\ell) + \gamma_2(u_{\ell+1} + \ldots + u_n) \leq 1$. Hence,

$$\text{FAIR}(U; \alpha) = 1^T z = y x_1 + (n - y) x_2.$$  

For the efficiency-maximizing solution, since $y \gamma_1 = 1$, we get

$$\text{SYSTEM}(U) = y.$$  

Then,

$$\text{POF}(U; \alpha) = 1 - \frac{y x_1 + (n - y) x_2}{y},$$

which is exactly the bound from Theorem 1.

The above result demonstrates why one should expect Procedure 1 to generate examples that have a price of fairness very close to the established bounds. In particular, Proposition 1 shows that the source of error between the price of fairness for the utility sets generated by Procedure 1 and the bound is the (potential) non-integrality of $y$. In case that error is “large”, one can search in the neighborhood of parameters $\gamma_1$ and $\gamma_2$ for an example that achieves a price closer to the bound, for instance by using finite-differencing derivatives and a gradient descent method (respecting feasibility).

Near worst-case bandwidth allocation

We utilize Proposition 1 and Procedure 1 to construct near worst-case network topologies. In particular, one can show that the line-graph discussed in Section 3.1.1 actually corresponds to a worst-case topology in this setup.

Suppose that we fix the number of players $n \geq 2$, the desired inequality aversion parameter $\alpha > 0$, and follow Procedure 1. Further suppose that $y \in \mathbb{N}$, as in Proposition 1. Consider then a network with $y$ links of unit capacity, in a line-graph topology: the routes of the first $y$ flows are disjoint and they all occupy a single (distinct) link. The remaining $n - y$ flows have routes that utilize all $y$ links. Each flow derives a utility equal to its assigned nonnegative rate, which we denote $u_1, \ldots, u_n$. We next show that the price of fairness for this network is equal to the bound of Theorem 1.

The output utility set of Procedure 1 achieves the bound, by Proposition 1 since $y \in \mathbb{N}$. Moreover, we also get that $y \gamma_1 = 1$ and $\gamma_2 = 1$. Hence, the output utility set that achieves the bound can be formulated as

$$U = \{u \geq 0 \mid u_1 + \ldots + u_y + y (u_{y+1} + \ldots + u_n) \leq y, u \leq 1\}.$$  

The utility set corresponding to the line-graph example above can be expressed using the non-negativity constraints of the flow rates, and the capacity constraints on each of the $y$ links as
follows,

$$\mathcal{U} = \{ u \geq 0 \mid u_j + u_{y+1} + \ldots + u_n \leq 1, \; j = 1, \ldots, y \}.$$ 

Clearly, the maximum sum of utilities under both sets is equal to $y$, simply by setting the first $y$ components of $u$ to 1. It suffices then to show that the two sets also share the same $\alpha$-fair allocation. In particular, by symmetry of $U$ and strict concavity of $W_\alpha$, if $u^F$ is its $\alpha$ fair allocation, then $u_1^F = \ldots = u_y^F$, and $u_{y+1}^F = \ldots = u_n^F$. As a result, it follows that $u^F \in \mathcal{U}$. Finally, noting that all inequalities in the definition of $U$ are also valid for $\mathcal{U}$, it follows that $\mathcal{U} \subset U$ and that $u^F$ is also the $\alpha$-fair allocation of $\mathcal{U}$.

C. Auxiliary Results

**Proposition 2.** For a point $(d, \lambda, x) \in S$ that attains the minimum of (17),

(a) if $\lambda + 1 < n$, then without loss of generality

$$x_{\lambda+1} = x_{\lambda+2} = \ldots = x_n,$$ and,

(b) without loss of generality

$$x_1 = \ldots = x_\lambda = x_{\lambda+1}.$$ 

**Proof.** (a) We drop the underline notation for $x_{\lambda+1}$ to simplify notation. Suppose that $x_j > x_{j+1}$, for some index $j \in \{\lambda + 1, \ldots, n - 1\}$. We will show that there always exists a new point, $(d, \lambda, x') \in S$, for which $x'_i = x_i$, for all $i \in \{1, \ldots, n\} \setminus \{j, j + 1\}$, and which either achieves the same objective with $x'_j = x'_{j+1}$, or it achieves a strictly lower objective.

If $j = \lambda + 1$ and $d = 1$, we set $x'_j = x'_{j+1} = x_{j+1}$. The new point is feasible, and the objective attains the same value.

Otherwise, let $x'_j = x_j - \epsilon$, for some $\epsilon > 0$. We have two cases.

$\alpha \geq 1$: Let $x'_{j+1} = x_{j+1}$ and pick $\epsilon$ small enough, such that $x'_j \geq x'_{j+1}$. Moreover, for the new point (compared to the feasible starting point) the left-hand sides of (18d) and (18e) are unaltered, whereas the right-hand sides are either unaltered (for $\alpha = 1$) or greater, since $x_j^{1-\alpha} < (x_j - \epsilon)^{1-\alpha}$ for $\alpha > 1$. Hence, the new point is feasible. It also achieves a strictly lower objective value.
\( \alpha < 1 \): Let \( x'_{j+1} = x_{j+1} + \rho \epsilon \), where

\[
\begin{align*}
  b &= \begin{cases} 
    1 - d, & \text{if } j = \lambda + 1, \\
    1, & \text{otherwise}, 
  \end{cases} \\
  \rho &\in \left( \frac{x_j^{-\alpha}}{x_{j+1}^{-\alpha}}, 1 \right).
\end{align*}
\]

For \( \epsilon \) small enough, we have \( x'_j \geq x'_{j+1} \). For the new point, the left-hand side of (18d) either decreases (if \( j + 1 = n \)), or remains unaltered. The left-hand side of (18e) remains also unaltered. For the right-hand sides, since the only terms that change are those involving \( x_j \) and \( x_{j+1} \), we use a first order Taylor series expansion to get

\[
\begin{align*}
  b \left(x'_j \right)^{1-\alpha} + (x'_{j+1})^{1-\alpha} &= b(x_j - \epsilon)^{1-\alpha} + (x_{j+1} + \rho \epsilon)^{1-\alpha} \\
  &= bx_j^{1-\alpha} - b\epsilon(1 - \alpha)x_j^{-\alpha} + x_{j+1}^{1-\alpha} + \rho \epsilon(1 - \alpha)x_{j+1}^{-\alpha} + O(\epsilon^2) \\
  &= (bx_j^{1-\alpha} + x_{j+1}^{1-\alpha}) + b(1 - \alpha) \left( \rho x_{j+1}^{-\alpha} - x_j^{-\alpha} \right) \epsilon + O(\epsilon^2).
\end{align*}
\]

By the selection of \( \rho \), the coefficient of the first order term (with respect to \( \epsilon \)) above is positive, and hence, for small enough \( \epsilon \) we get

\[
\begin{align*}
  b \left(x'_j \right)^{1-\alpha} + (x'_{j+1})^{1-\alpha} > bx_j^{1-\alpha} + x_{j+1}^{1-\alpha}.
\end{align*}
\]

That shows that the right hand side increases, and the new point is feasible. Finally, the difference in the objective value is \(-b\epsilon + \rho \epsilon\), and thus negative.

(b) We drop the overline notation for \( \overline{x}_{\lambda+1} \) to simplify notation. Suppose that \( x_j > x_{j+1} \), for some index \( j \in \{1, \ldots, \lambda\} \).

We will show that there always exists a new point, \((d, \lambda, x') \in S\), for which \( x'_i = x_i \), for all \( i \in \{1, \ldots, n\} \setminus \{j, j+1\} \), and which either achieves the same objective with \( x'_j = x'_{j+1} \), or it achieves a strictly lower objective.

If \( j + 1 = \lambda + 1 \) and \( d = 0 \), we set \( x'_j = x'_{j+1} = x_j \). The new point is feasible, and the objective attains the same value.

Otherwise, let

\[
\begin{align*}
  x'_j &= x_j - \epsilon \\
  x'_{j+1} &= x_{j+1} + \rho \epsilon,
\end{align*}
\]
for some $\epsilon > 0$, where

$$
\rho \in \left( \frac{x_{j+1}}{x_j} \cdot \frac{x_{j+1}^{-\alpha}}{x_j^{-\alpha}} \right)
$$

$$
c = \frac{x_j^{-\alpha}}{bx_{j+1}^{-\alpha}}
$$

$$
b = \begin{cases} 
  d, & \text{if } j + 1 = \lambda + 1, \\
  1, & \text{otherwise.}
\end{cases}
$$

For $\epsilon$ small enough, we have $x'_j \geq x'_{j+1}$. For the new point, the left-hand side of (18d) remains unaltered. For the left-hand side of (18e) we use a first order Taylor series expansion (similarly as above) to get

$$
\left( x_j' \right)^{-\alpha} + b \left( x'_{j+1} \right)^{-\alpha} = (x_j - \epsilon)^{-\alpha} + b(x_{j+1} + \rho \epsilon)^{-\alpha}
$$

$$
= x_j^{-\alpha} + \epsilon \alpha x_j^{-\alpha-1} + b x_{j+1}^{-\alpha} - b \rho \epsilon \alpha x_j^{-\alpha} x_{j+1}^{-1} + O(\epsilon^2)
$$

$$
= \left( x_j^{-\alpha} + bx_{j+1}^{-\alpha} \right) + \epsilon \alpha x_j^{-\alpha-1} - \rho \epsilon \alpha x_j^{-\alpha} x_{j+1}^{-1} + O(\epsilon^2)
$$

$$
= \left( x_j^{-\alpha} + bx_{j+1}^{-\alpha} \right) + \alpha x_j^{-\alpha-1} \left( 1 - \rho \frac{x_j}{x_{j+1}} \right) \epsilon + O(\epsilon^2).
$$

By the selection of $\rho$, the coefficient of the first order term (with respect to $\epsilon$) above is negative, and hence, for small enough $\epsilon$ we get that the left-hand side decreases.

For the right-hand side of (18d) and (18e), we similarly get that

$$
\left( x_j' \right)^{1-\alpha} + b \left( x'_{j+1} \right)^{1-\alpha} = (x_j - \epsilon)^{1-\alpha} + b(x_{j+1} + \rho \epsilon)^{1-\alpha}
$$

$$
= x_j^{1-\alpha} - \epsilon(1 - \alpha)x_j^{-\alpha} + bx_{j+1}^{1-\alpha} + b \rho \epsilon \alpha x_j^{1-\alpha} + O(\epsilon^2)
$$

$$
= \left( x_j^{1-\alpha} + bx_{j+1}^{1-\alpha} \right) + (1 - \alpha)x_j^{-\alpha} (\rho - 1) \epsilon + O(\epsilon^2).
$$

If for $\alpha > 1$ we pick $\rho < 1$, and for $\alpha < 1$ we pick $\rho > 1$, the first order term (with respect to $\epsilon$) above is positive, and hence, for small enough $\epsilon$ we get that the right-hand side increases for $\alpha \neq 1$. For $\alpha = 1$, the right-hand side remains unaltered.

In all cases, the new point is feasible, and the difference in the objective value is

$$
- \epsilon + \rho \epsilon \alpha = (\rho \epsilon - 1) \epsilon = \left( \frac{x_j^{-\alpha}}{x_{j+1}^{-\alpha}} - 1 \right) \epsilon,
$$

and thus negative (by the selection of $\rho$).
Proposition 3. Let \( n \in \mathbb{N} \setminus \{0, 1\} \) and \( f : [1, n] \to \mathbb{R} \) be defined as

\[
f(x; \alpha, n) = \frac{x^{1+\frac{1}{\alpha}} + n - x}{x^{1+\frac{1}{\alpha}} + (n-x)x}.
\]

For any \( \alpha > 0 \),

(a) \(-f\) is unimodal over \([1, n]\), and thus has a unique minimizer \( \xi^* \in [1, n] \).

(b) \( \min_{x \in [1, n]} f(x; \alpha, n) = f(\xi^*; \alpha, n) = \Theta \left( n^{-\frac{\alpha}{\alpha+1}} \right) \).

Proof. (a) The derivative of \( f \) is

\[
f'(x; \alpha, n) = \frac{g(x)}{(x^{1+\frac{1}{\alpha}} + (n-x)x)^2},
\]

where

\[
g(x) = \left(1 - \frac{1}{\alpha}\right) x^{2+\frac{4}{\alpha}} + \frac{n+1}{\alpha} x^{1+\frac{4}{\alpha}} - n \left(1 + \frac{1}{\alpha}\right) x^{\frac{1}{\alpha}} - (x-n)^2.
\]

Note that the sign of the derivative is determined by \( g(x) \), since the denominator is positive for \( 1 \leq x \leq n \), that is,

\[
\text{sgn } f'(x; \alpha, n) = \text{sgn } g(x).
\] (31)

We will show that \( g \) is strictly increasing over \([1, n]\). To this end, we have

\[
g'(x) = x^{\frac{1}{\alpha}-1} q(x) + 2(n-x),
\]

where

\[
q(x) = \left(2 + \frac{1}{\alpha}\right) \left(1 - \frac{1}{\alpha}\right) x^2 + \left(1 + \frac{1}{\alpha}\right) \left(\frac{n+1}{\alpha}\right) x - n \left(1 + \frac{1}{\alpha}\right).
\]

Since we are interested in the domain \([1, n]\), it suffices to show that \( q(x) > 0 \) over it. For \( \alpha > 1 \), \( q \) is a convex quadratic, with its minimizer being equal to

\[
-\frac{\left(1 + \frac{1}{\alpha}\right) \left(\frac{n+1}{\alpha}\right)}{2 \left(2 + \frac{1}{\alpha}\right) \left(1 - \frac{1}{\alpha}\right)} < 0.
\]

Hence, \( q(x) \geq q(1) \) for \( x \in [1, n] \). Similarly, for \( \alpha < 1 \), \( q \) is a concave quadratic, and as such, for \( x \in [1, n] \) we have \( q(x) \geq \min\{q(1), q(n)\} \). For \( \alpha = 1 \), \( q(x) = 2(n+1)x - 2n \), which is positive for \( x \geq 1 \). Then, \( q(x) > 0 \) in \([1, n]\) for all \( \alpha > 0 \), if and only if \( q(1) > 0 \) and \( q(n) > 0 \).

Note that for \( r = 1 \), we get \( q(1) = 2 \) and \( q(n) = 2n^2 \), and

\[
\frac{dq(1)}{dr} = 2 > 0, \quad \frac{dq(n)}{dr} = 2n^2 > 0,
\]

13
which demonstrates that \( q(1) \) and \( q(n) \) are positive. Furthermore,

\[
g(n) = n^{1 + \frac{1}{\alpha}}(n - 1) > 0.
\]

Using the above, the fact that \( g \) is continuous and strictly increasing over \([1, n]\) and (31), we deduce that if \( g(1) < 0 \), there exists a unique \( m \in (1, n) \) such that

\[
\text{sgn} f'(x; \alpha, n) \begin{cases} < 0, & \text{if } 1 \leq x < m, \\ > 0, & \text{if } m < x \leq n. \end{cases}
\]

Similarly, if \( g(1) \geq 0 \), \( f \) is strictly increasing for \( 1 \leq x \leq n \). It follows that \(-f\) is unimodal.

(b) Let \( \theta_n = n^{\frac{\alpha}{\alpha+1}} \). Using the mean value Theorem, for every \( n \geq 2 \), there exists a \( \psi_n \in [\theta_n, \xi^*] \) (or \([\xi^*, \theta_n]\), depending on if \( \theta_n \leq \xi^* \)), such that

\[
f(\theta_n; \alpha, n) = f(\xi^*; \alpha, n) + f'(\psi_n; \alpha, n)(\theta_n - \xi^*),
\]

or, equivalently,

\[
\frac{f(\xi^*; \alpha, n)}{f(\theta_n; \alpha, n)} = 1 - \frac{f'(\psi_n; \alpha, n)(\theta_n - \xi^*)}{f(\theta_n; \alpha, n)}.
\]

We will show that, for a sufficiently small \( \epsilon > 0 \)

(I.) \( f'(\psi_n; \alpha, n) = O \left( n^{-\min\{1, \alpha\alpha+1\} + 2\epsilon} \right) \),

(II.) \( \theta_n - \xi^* = O \left( n^{-\frac{\alpha}{\alpha+1} + \epsilon} \right) \),

(III.) \( f(\theta_n; \alpha, n) = \Theta \left( n^{-\frac{\alpha}{\alpha+1}} \right) \).

Using the above facts, it is easy to see that

\[
\frac{f(\xi^*; \alpha, n)}{f(\theta_n; \alpha, n)} = 1 - \frac{f'(\psi_n; \alpha, n)(\theta_n - \xi^*)}{f(\theta_n; \alpha, n)} = 1 - O \left( n^{-\min\{1, \alpha\alpha+1\} + 3\epsilon} \right) \to 1,
\]

and thus \( f(\xi^*; \alpha, n) = \Theta \left( n^{-\frac{\alpha}{\alpha+1}} \right) \).

(I.) We first show that for any sufficiently large \( n \),

\[
n^{\frac{\alpha}{\alpha+1} - \epsilon} \leq \xi^* \leq n^{\frac{\alpha}{\alpha+1} + \epsilon}.
\]

By part (a), \( \xi^* \) is the unique root of \( g \) in the interval \([1, n]\). Moreover, \( g \) is strictly increasing.
The dominant term of
\[ g \left( n \frac{\alpha}{\alpha + 1 - \epsilon} \right) = \left( 1 - \frac{1}{\alpha} \right) n^{2 + \frac{1}{\alpha}}(\frac{\alpha}{\alpha + 1 - \epsilon}) + \frac{1}{\alpha} n^{1 - \frac{\alpha + 1}{\alpha}} + \frac{1}{\alpha} n^{2 - \frac{\alpha + 1}{\alpha}} \]
\[ - \left( 1 + \frac{1}{\alpha} \right) n^{1 + \frac{1}{\alpha} - \frac{1}{\alpha}} - n - 2 n^{2 - \frac{2\alpha}{\alpha + 1} - 2\epsilon} + 2 n^{1 + \frac{\alpha}{\alpha + 1} - \epsilon}, \]
is \(-n^2\), and hence, for sufficiently large \( n \) we have \( g \left( n \frac{\alpha}{\alpha + 1 - \epsilon} \right) < 0 \). Similarly, the dominant term of \( g \left( n \frac{\alpha}{\alpha + 1 + \epsilon} \right) \) is \( \frac{1}{\alpha} n^{2 + \frac{\alpha + 1}{\alpha}} \), and for sufficiently large \( n \) we have \( g \left( n \frac{\alpha}{\alpha + 1 + \epsilon} \right) > 0 \). The claim then follows. Using the above bound, for sufficiently large \( n \), we also get that \( \psi_n \geq n \frac{\alpha}{\alpha + 1 - \epsilon} \). We now provide a bound for the denominator of \( f'(\psi_n; \alpha, n) \). In particular, for sufficiently large \( n \), we get that for \( x \leq n \frac{\alpha}{\alpha + 1 - \epsilon} \),
\[ \frac{d}{dx} \left( x^{\frac{1}{\alpha}} + n x - x^2 \right) = \left( 1 + \frac{1}{\alpha} \right) x^{\frac{1}{\alpha}} + n - 2 x > 0, \]
which shows that the denominator is strictly increasing. Hence, using the lower bound on \( \psi_n \),
\[ \frac{1}{\left( \psi_n^{1 + \frac{1}{\alpha}} + n \psi_n - \psi_n^2 \right)^2} \leq \frac{1}{\left( n \frac{\alpha}{\alpha + 1 - \epsilon} \right)^{1 + \frac{1}{\alpha}} + n^{1 + \frac{\alpha + 1}{\alpha} - \epsilon} - n^{2 - \frac{2\alpha}{\alpha + 1} - 2\epsilon} \}^2}
\[ \leq \frac{n^{-2 - \frac{2\alpha}{\alpha + 1} + 2\epsilon}}{\left( n - \frac{\alpha + 1}{\alpha + 1} - 1 + n \frac{1}{\alpha + 1} \right)^2} = O \left( n^{-2 - \frac{2\alpha}{\alpha + 1} + 2\epsilon} \right). \]
We now provide a bound for the numerator. Since \( g \) is strictly increasing and \( \xi^* \) is a root, we get
\[ |g(\psi_n)| \leq |g(\theta_n)| \]
\[ = \left| \left( 1 - \frac{1}{\alpha} \right) \alpha + n - \frac{1}{\alpha} + 2 + n \left( 1 + \frac{1}{\alpha} \right) \alpha + \frac{1}{\alpha + 1} n^{-\alpha + 1 + 2} - \alpha + \frac{1}{\alpha + 1} n^{-\alpha + 1 + 2} + 2 \alpha + \frac{1}{\alpha + 1} n^{-\alpha + 1 + 2} \right| \]
\[ = O \left( n^{-\frac{\min(1, \alpha)}{\alpha + 1} + 2} \right). \]
If we combine the above results, we get \( f'(\psi_n; \alpha, n) = O \left( n^{-\frac{\min(1, \alpha)}{\alpha + 1} + 2} \right). \)
(II.) Follows from 322.
(III.) We have
\[ f(\theta_n; \alpha, n) = \frac{n + n - \frac{\alpha}{\alpha + 1}}{n + 1 \frac{\alpha}{\alpha + 1} - n \frac{\alpha}{\alpha + 1}} \]
\[ = \frac{n \left( 2 - n^{\frac{1}{\alpha}} \right)}{n \left( \frac{\alpha}{\alpha + 1} + 1 - n^{\frac{1}{\alpha + 1}} \right)} = \Theta \left( n^{-\frac{\alpha}{\alpha + 1}} \right). \]
15
Proposition 4. There exists a point \( z \in \mathbb{R}^n \) that attains the minimum of (29), for which
\[
z_1 = \ldots = z_{n-1} = 1.
\]

Proof. For \( \alpha = 1 \), problem (29) is written as
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{n} \left( \frac{z_n}{z_1} + \frac{z_n}{z_2} + \ldots + \frac{z_n}{z_{n-1}} + 1 \right) \\
\text{subject to} & \quad \frac{1}{n} \leq z_n \leq z_{n-1} \leq \ldots \leq z_1 \leq 1.
\end{align*}
\]
If \( z \) is an optimal solution of the above, then clearly \( z_1 = \ldots = z_{n-1} = 1 \).

We now deal with the case of \( \alpha \neq 1 \). We first show that if \( z \) is an optimal solution of (29), then \( z_1 = \ldots = z_{n-1} \). We analyze the cases \( 0 < \alpha < 1 \) and \( \alpha > 1 \) separately.

For \( 0 < \alpha < 1 \), the function \( z_1^{1-\alpha} + \ldots + z_{n-1}^{1-\alpha} \) is strictly concave, and the function \( z_1^{-\alpha} + \ldots + z_{n-1}^{-\alpha} \) is strictly convex. If \( z \) is an optimal solution of (29) for which \( z_1 = \ldots = z_{n-1} \) is violated, we construct a point \( \tilde{z} \in \mathbb{R}^n \), such that its first \( n-1 \) components are all equal to the mean of \( z_1, \ldots, z_{n-1} \) and \( \tilde{z}_n = z_n \). We show that \( \tilde{z} \) is feasible for (29) and it achieves a strictly lower objective value compared to \( z \), a contradiction. Note that by strict concavity/convexity we get
\[
\tilde{z}_1^{1-\alpha} + \ldots + \tilde{z}_{n-1}^{1-\alpha} > z_1^{1-\alpha} + \ldots + z_{n-1}^{1-\alpha},
\]
and
\[
\tilde{z}_1^{-\alpha} + \ldots + \tilde{z}_{n-1}^{-\alpha} < z_1^{-\alpha} + \ldots + z_{n-1}^{-\alpha},
\]
respectively. For feasibility, \( 0 \leq \tilde{z}_n \leq \ldots \leq \tilde{z}_1 \leq 1 \) is immediate and
\[
\tilde{z}_n^{-\alpha} = z_n^{-\alpha} \leq z_1^{-\alpha} + \ldots + z_{n-1}^{-\alpha} + z_{n-1}^{-\alpha} < z_1^{1-\alpha} + \ldots + z_{n-1}^{1-\alpha} + z_n^{1-\alpha}.
\]

Finally, compared to \( z \), if we evaluate the objective of (29) at \( \tilde{z} \), the numerator strictly decreases and the denominator strictly increases, hence the objective value strictly decreases.

For \( \alpha > 1 \), let \( z \) be an optimal solution of (29) for which \( z_{j+1} < z_j \) for some \( j = 1, \ldots, n-2 \). We similarly construct a feasible point \( \tilde{z} \) for (29) that achieves a strictly lower objective value than \( z \). Let \( \tilde{z}_i = z_i \) for all \( i \neq j, j+1 \), \( \tilde{z}_j = z_j - \epsilon \) and \( \tilde{z}_{j+1} = z_{j+1} + \delta \epsilon \), where \( \epsilon > 0 \) and
\[
\delta = \frac{z_j^{-\alpha} - \mu}{z_{j+1}^{-\alpha}}, \quad \mu \in \left( 0, z_j^{-\alpha} \left( \frac{z_j - z_{j+1}}{z_j} \right) \right).
\]

16
For small enough \( \epsilon \), \( 0 \leq \tilde{z}_n \leq \ldots \leq \tilde{z}_1 \leq 1 \) is immediate. Using a first order Taylor series expansion,

\[
\tilde{z}_j^{-\alpha} + \frac{1}{\tilde{z}_j + 1} = z_j^{-\alpha} + \frac{1}{z_j + 1} + (z_j^{-\alpha} - \delta z_j^{-\alpha})(\alpha - 1)\epsilon + O(\epsilon^2)
\]

for small enough \( \epsilon \), since \( z_j^{-\alpha} > \delta z_j^{-\alpha} \Leftrightarrow \mu > 0 \). As a result,

\[
z_1^{-\alpha} + \ldots + \tilde{z}_{n-1}^{-\alpha} + \tilde{z}_n^{-\alpha} > z_1^{-\alpha} + \ldots + \tilde{z}_{n-1}^{-\alpha} + \tilde{z}_n^{-\alpha},
\]

and \( \tilde{z} \) is feasible. Moreover, the denominator of the objective strictly increases. Thus it suffices to show that the numerator decreases. To this end, we have

\[
\tilde{z}_j^{-\alpha} + \frac{1}{\tilde{z}_j + 1} = z_j^{-\alpha} + \frac{1}{z_j + 1} + (z_j^{-\alpha} - \delta z_j^{-\alpha})(\alpha - 1)\epsilon + O(\epsilon^2)
\]

for small enough \( \epsilon \), since \( z_j^{-\alpha-1} < \delta z_j^{-\alpha-1} \Leftrightarrow \mu < z_j^{-\alpha} \left( \frac{z_j^{-\alpha+1}}{1} \right) \).

Since for every optimal solution of (29), we have \( z_1 = \ldots = z_{n-1} \), problem (29) can be written equivalently as

\[
\text{minimize } g(z_1, z_2) = \frac{(n-1)z_1^{-\alpha}z_2 + z_2^{-\alpha}}{(n-1)z_1^{-\alpha} + z_2^{-\alpha}} \\
\text{subject to } 0 \leq z_2 \leq z_1 \leq 1 \\
z_2^{-\alpha} \leq (n-1)z_1^{-\alpha} + z_2^{-\alpha}.
\]

(33)

It suffices to show that there exists an optimal solution \( z \) of (33) for which \( z_1 = 1 \).

Let \( z \) be an optimal solution of (33).

If \( 0 < \alpha < 1 \), assume that \( z_1 < 1 \). Then, increase \( z_1 \) by a small enough amount such that it remains less than 1. The quantity \( z_1^{-\alpha} \) increases, so the new point we get is feasible. Also, the quantity \( z_1^{-\alpha} \) decreases. Hence, the new point is feasible and achieves a strictly lower objective value, a contradiction.

If \( \alpha > 1 \), the point \( z \) lies on the boundary of the feasible set or is a stationary point of the objective. Suppose that \( z \) is not a stationary point, \( i.e., \nabla g(z_1, z_2) \neq 0 \). If \( z_1 = z_2 \), the objective evaluates to 1 for any such \( z \), so we can assume \( z_1 = 1 \). We next rule out the possibility of \( z \) lying on the \( z_2^{-\alpha} = (n-1)z_1^{-\alpha} + z_2^{-\alpha} \) boundary with \( z_1 < 1 \). Suppose that it does. We will demonstrate that we can always find a feasible direction along which the objective decreases. We have

\[
\frac{\partial g}{\partial z_1} = \frac{(n-1)z_1^{-\alpha}z_2}{(n-1)z_1^{-\alpha} + z_2^{-\alpha}} \left( -(n-1)z_1^{-\alpha} - \alpha z_1^{-1}z_2^{-\alpha} + (\alpha - 1)z_2^{-\alpha} \right),
\]

\[
\frac{\partial g}{\partial z_2} = -\frac{z_1}{z_2} \frac{\partial g}{\partial z_1}.
\]
Note that we assumed that $\nabla g(z) \neq 0$, hence $\frac{\partial g}{\partial z_1}(z) \neq 0$. Suppose that $\frac{\partial g}{\partial z_2}(z) > 0$. Then, $(1, \delta)$ is a direction along which the objective decreases, for large enough $\delta > 0$, since

$$
\frac{\partial g}{\partial z_1}(z) + \delta \frac{\partial g}{\partial z_2}(z) = \frac{\partial g}{\partial z_1}(z) \left(1 - \delta \frac{z_1}{z_2}\right) < 0.
$$

It is also a feasible direction, since for $\epsilon > 0$ small enough, $0 \leq z_2 + \delta \epsilon \leq z_1 + \epsilon \leq 1$, and is also a direction along which $(n - 1)z_1^{1-\alpha} + z_2^{1-\alpha} + z_2^{-\alpha}$ increases, since

$$
(n - 1)z_1^{-\alpha} + \delta \left((1 - \alpha)z_2^{-\alpha} + \alpha z_2^{-\alpha-1}\right) = (1 - \alpha)(z_2^{-\alpha} - z_2^{1-\alpha}) + \delta \left((1 - \alpha)z_2^{-\alpha} + \alpha z_2^{-\alpha-1}\right) = z_2^{-\alpha}\left((1 - \alpha)(1 - z_2) + \delta \left(\frac{a}{z_2} - (\alpha - 1)\right)\right) > 0
$$

for large enough $\delta$. Similarly, if $\frac{\partial g}{\partial z_1}(z) < 0$, one can show that $(1, \delta)$ is again a feasible direction along which the objective decreases, for

$$
\frac{(\alpha - 1)(1 - z_2)z_2}{\alpha - (\alpha - 1)z_2} < \delta < \frac{z_2}{z_1},
$$

if one can select such $\delta$. Otherwise, one can show that $(-1, -\delta)$ is a feasible direction along which the objective decreases, for

$$
\frac{z_2}{z_1} < \delta < \frac{(\alpha - 1)(1 - z_2)z_2}{\alpha - (\alpha - 1)z_2}.
$$

We have thus established that if $z$ is not a stationary point, then there also exists an optimal solution for which $z_1 = 1$. We next show that the same holds true if $z$ is a stationary point.

Suppose that $z$ is a stationary point, i.e., $\nabla g(z_1, z_2) = 0$. Then, we have

$$(n - 1)z_1^{1-\alpha} + z_2^{1-\alpha} - (\alpha - 1)z_1z_2^{-\alpha} = 0.$$ 

Using the above, the objective evaluates to

$$g(z_1, z_2) = \frac{\alpha}{\alpha - 1} \frac{z_2}{z_1}.$$ 

Moreover, if $z_1 = \lambda z_2$ for some $\lambda \geq 1$, the stationarity condition yields

$$(n - 1)\lambda^{1-\alpha} - (\alpha - 1)\lambda + \alpha = 0,$$

an equation that has a unique solution in $[1, \infty)$. Let $\tilde{\lambda}$ be the solution. Then, the problem (33)
constrained on the stationary points of its objective can be expressed as

\[
\begin{align*}
\text{minimize} & \quad \frac{\alpha}{\alpha-1} z_2 \\
\text{subject to} & \quad z_1 = \bar{\lambda} z_2, \quad z_1 \leq 1 \\
& \quad z_2^{1-\alpha} \leq (n-1) z_1^{1-\alpha} + z_2^{1-\alpha},
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
\text{minimize} & \quad \frac{\alpha-1}{\alpha-1} \frac{1}{\bar{\lambda}} \\
\text{subject to} & \quad z_1 = \bar{\lambda} z_2 \\
& \quad \frac{1}{(\alpha-1)(\bar{\lambda}-1)} \leq z_2 \leq \frac{1}{\bar{\lambda}}.
\end{align*}
\]

In case the above problem is feasible, we pick \(z_2 = \frac{1}{\bar{\lambda}}\), and \(z_1 = 1\) and the proof is complete. \[\blacksquare\]

**Proposition 5.** Consider a resource allocation problem with \(n\) players, \(n \geq 2\). Let the utility set, denoted by \(U \subset \mathbb{R}^n\), be compact and convex. If the players have equal maximum achievable utilities (greater than zero),

\[
\text{POF} (U; 1) \leq 1 - \frac{2\sqrt{n} - 1}{n}. \quad \text{(price of proportional fairness)}
\]

Let \(\{\alpha_k \in \mathbb{R} \mid k \in \mathbb{N}\}\) be a sequence such that \(\alpha_k \to \infty\) and \(\alpha_k \geq 1, \forall k\). Then,

\[
\limsup_{k \to \infty} \text{POF} (U; \alpha_k) \leq 1 - \frac{4n}{(n+1)^2}. \quad \text{(price of max-min fairness)}
\]

**Proof.** Let \(f\) be defined as in Proposition 3. Using Theorem 1 for \(\alpha = 1\) we get

\[
\begin{align*}
\text{POF} (U; 1) & \leq 1 - \min_{x \in [1, n]} f(x; 1, n) \\
& = 1 - \min_{x \in [1, n]} \frac{x^2 + n - x}{nx} \\
& = 1 - \frac{2\sqrt{n} - 1}{n}.
\end{align*}
\]

Similarly, for any \(k \in \mathbb{N}\) and \(\alpha = \alpha_k\)

\[
\begin{align*}
\text{POF} (U; \alpha_k) & \leq 1 - \min_{x \in [1, n]} f(x; \alpha_k, n),
\end{align*}
\]

which implies that

\[
\limsup_{k \to \infty} \text{POF} (U; \alpha_k) \leq \limsup_{k \to \infty} \left(1 - \min_{x \in [1, n]} f(x; \alpha_k, n)\right) \\
& \leq 1 - \liminf_{k \to \infty} \min_{x \in [1, n]} f(x; \alpha_k, n). \quad (34)
\]
Consider the set of (real-valued) functions \( \{ f(\cdot; \alpha_k, n) \mid k \in \mathbb{N} \} \) defined over the compact set \([1, n]\). We show that the set is equicontinuous, and that the closure of the set \( \{ f(x; \alpha_k, n) \mid k \in \mathbb{N} \} \) is bounded for any \( x \in [1, n] \). Boundedness follows since \( 0 \leq f(x; \alpha, n) \leq 1 \) for any \( \alpha > 0 \) and \( x \in [1, n] \). The set of functions \( \{ f(\cdot; \alpha_k, n) \mid k \in \mathbb{N} \} \) shares the same Lipschitz constant, as for any \( k \in \mathbb{N}, \alpha_k \geq 1 \) and \( x \in [1, n] \) we have

\[
|f'(x; \alpha_k, n)| = \left| \frac{(1 - \frac{1}{\alpha_k}) x^{2 + \frac{1}{\alpha_k}} + n + 1 - n (1 + \frac{1}{\alpha_k}) x^{\frac{1}{\alpha_k}} - (x - n)^2}{x^{1 + \frac{1}{\alpha_k} + (n - x)x}^2} \right| \\
\leq \left| \frac{(1 - \frac{1}{\alpha_k}) x^{2 + \frac{1}{\alpha_k}} + n + 1 - n (1 + \frac{1}{\alpha_k}) x^{\frac{1}{\alpha_k}} - (x - n)^2}{x^{1 + \frac{1}{\alpha_k} + (n - x)x}^2} \right| \\
\leq \left(1 - \frac{1}{\alpha_k} \right) x^{2 + \frac{1}{\alpha_k}} + n + \frac{1}{\alpha_k} x^{1 + \frac{1}{\alpha_k}} - n \left(1 + \frac{1}{\alpha_k}\right) x^{\frac{1}{\alpha_k}} + (x - n)^2 \\
\leq n^3 + (n + 1)n^2 + 2n^2 + n^2 = 2(n^3 + 2n^2).
\]

As a result, the set of functions \( \{ f(\cdot; \alpha_k, n) \mid k \in \mathbb{N} \} \) is equicontinuous.

Using the above result,

\[
\lim_{k \to \infty} \min_{x \in [1, n]} f(x; \alpha_k, n) = \min_{x \in [1, n]} \lim_{k \to \infty} f(x; \alpha_k, n).
\]

Thus, (34) yields

\[
\limsup_{k \to \infty} \text{POF}(U; \alpha_k) \leq 1 - \liminf_{k \to \infty} \min_{x \in [1, n]} f(x; \alpha_k, n) \\
= 1 - \min_{x \in [1, n]} \lim_{k \to \infty} f(x; \alpha_k, n) \\
= 1 - \min_{x \in [1, n]} \lim_{k \to \infty} \frac{x^{1 + \frac{1}{\alpha_k}} + n - x}{x^{1 + \frac{1}{\alpha_k}} + (n - x)x} \\
= 1 - \min_{x \in [1, n]} \frac{n}{x + (n - x)x} \\
= 1 - \frac{4n}{(n + 1)^2}.
\]

D. A Model for Air Traffic Flow Management

The following is a model for air traffic flow management due to [Bertsimas and Stock-Patterson (1998)]. Consider a set of flights, \( \mathcal{F} = \{1, \ldots, F\} \), that are operated by airlines over a (discretized) time period in a network of airports, utilizing a capacitated airspace that is divided into sectors. Let \( \mathcal{F}_a \subset \mathcal{F} \) be the set of flights operated by airline \( a \in \mathcal{A} \), where \( \mathcal{A} = \{1, \ldots, A\} \) is the set of airlines. Similarly, \( \mathcal{T} = \{1, \ldots, T\} \) is the set of time steps, \( \mathcal{K} = \{1, \ldots, K\} \) the set of airports,
and \( J = \{1, \ldots, J\} \) the set of sectors. Flights that are continued are included in a set of pairs, \( \mathcal{C} = \{(f', f) : f' \text{ is continued by flight } f\} \). The model input data, the main decision variables, and a description of the feasibility set are described below:

### Data.

- \( N_f \) = number of sectors in flight \( f \)'s path,
- \( P(f,i) \) =
  \[
  \begin{cases}
    \text{the departure airport, if } i = 1, \\
    \text{the } (i-1)\text{th sector in flight } f \text{'s path, if } 1 < i < N_f, \\
    \text{the arrival airport, if } i = N_f,
  \end{cases}
  \]
- \( P_f \) = \( (P(f,i) : 1 \leq i \leq N_f) \),
- \( D_k(t) \) = departure capacity of airport \( k \) at time \( t \),
- \( A_k(t) \) = arrival capacity of airport \( k \) at time \( t \),
- \( S_j(t) \) = capacity of sector \( j \) at time \( t \),
- \( d_f \) = scheduled departure time of flight \( f \),
- \( r_f \) = scheduled arrival time of flight \( f \),
- \( s_f \) = turnaround time of an airplane after flight \( f \),
- \( l_{fj} \) = number of time steps that flight \( f \) must spend in sector \( j \),
- \( T^j_f \) = set of feasible time steps for flight \( f \) to arrive to sector \( j = \{T^j_f, \ldots, T^{f}_{max}\} \),
- \( T^j_{first} \) = first time step in the set \( T^j_f \), and
- \( T^j_{last} \) = last time step in the set \( T^j_f \).

### Decision Variables.

\[
  w^j_{ft} = \begin{cases}
    1, & \text{if flight } f \text{ arrives at sector } j \text{ by time step } t, \\
    0, & \text{otherwise}.
  \end{cases}
\]

### Feasibility Set. The variable \( w \) is feasible if it satisfies the constraints:

- \( \sum_{f : P(f,1)=k}(w^k_{ft} - w^k_{f,t-1}) \leq D_k(t) \ \forall k \in \mathcal{K}, t \in \mathcal{T} \),
- \( \sum_{f : P(f,N_f)=k}(w^k_{ft} - w^k_{f,t-1}) \leq A_k(t) \ \forall k \in \mathcal{K}, t \in \mathcal{T} \),
- \( \sum_{f : P(f,i)=j, P(f,i+1)=j', i < N_f}(w^j_{f,t} - w^{j'}_{f,t}) \leq S_j(t) \ \forall j \in \mathcal{J}, t \in \mathcal{T} \),
- \( w^j_{f,t+l_{fj}} - w^j_{f,t} \leq 0 \ \forall f \in \mathcal{F}, t \in T^j_f, j = P(f,i), j' = P(f,i+1), i < N_f \),
- \( w^k_{f,t} - w^k_{f,t-s_f} \leq 0 \ \forall (f', f) \in \mathcal{C}, t \in T^k_f, k = P(f,i) = P(f', N_f) \),
- \( w^j_{f,t} - w^j_{f,t-1} \geq 0 \ \forall f \in \mathcal{F}, j \in P_f, t \in T^j_f \),
- \( w^j_{f,t} \in \{0, 1\} \ \forall f \in \mathcal{F}, j \in P_f, t \in T^j_f. \)

The constraints correspond to capacity constraints for airports and sectors, connectivity between sectors and airports, and connectivity in time (for more details, see Bertsimas and Stock-Patterson (1998)).