# Appendices

## A. Proofs of Main Theorems

**Proof of Theorem 1.**

A brief roadmap of the proof is as follows. We first show that there exist polytopes in the $0-1$ hypercube, parameterized by $\gamma \in \mathbf{R}^n$, that correspond to worst-case topologies (see (12)); the remaining of the proof deals with identifying the worst-case polytope within this class, *i.e.*, the worst-case value of the parameter $\gamma$, utilizing symmetry and optimization theory arguments.

Geometrically, the $\alpha$-fair allocation of any convex utility set in the $0-1$ hypercube lies on its boundary. Consider now the supporting hyperplane at the $\alpha$-fair allocation, defined by the gradient of $W_\alpha$. Intuitively, any set that is contained in the polytope defined by that supporting hyperplane (and the $0-1$ hypercube) would have the same $\alpha$-fair allocation. However, that does not hold true for the utilitarian or max-min allocations. In fact, by considering convex supersets of the original utility set, contained in the described polytope, one could obtain higher values for the utilitarian and/or max-min objectives, while the $\alpha$-fair allocation remains constant. As such, one need only consider polytopes of the described form for worst-cases. Note that such an approach can be generalized in a straightforward manner for any similar settings where one considers multiple competing objective functions.

Without loss of generality, we assume that $U$ is monotone[5]. This is because both schemes we consider, namely utilitarian and $\alpha$-fairness yield Pareto optimal allocations. In particular, suppose there exist allocations $a \in U$ and $b \notin U$, with allocation $a$ dominating allocation $b$, *i.e.*, $0 \leq b \leq a$. Note that allocation $b$ can thus not be Pareto optimal. Then, we can equivalently assume that $b \in U$, since $b$ cannot be selected by any of the schemes.

We also assume that the maximum achievable utilities of the players are equal to 1; the proof can be trivially modified otherwise.

By combining the above two assumptions, we get

$$e_j \in U, \quad \forall j = 1, \ldots, n, \tag{5}$$

where $e_j$ is the unit vector in $\mathbf{R}^n$, with the $j$th component equal to 1.

Fix $\alpha > 0$ and let $z = z(\alpha) \in U$ be the unique allocation under the $\alpha$-fairness criterion (since $W_\alpha$ is strictly concave for $\alpha > 0$), and assume, without loss of generality, that

$$z_1 \geq z_2 \geq \ldots \geq z_n. \tag{6}$$

[5]A set $A \subset \mathbf{R}^n_+$ is called monotone if $\{b \in \mathbf{R}^n \mid 0 \leq b \leq a\} \subset A$, $\forall a \in A$, where the inequality sign notation for vectors is used for componentwise inequality.

The necessary first order condition for the optimality of $z$ can be expressed as

$$\nabla W_\alpha(z)^T(u-z) \le 0 \;\Rightarrow\; \sum_{j=1}^{n} z_j^{-\alpha}(u_j - z_j) \le 0, \quad \forall u \in U,$$

or equivalently

$$\gamma^T u \le 1, \quad \forall u \in U, \tag{7}$$

where

$$\gamma_j = \frac{z_j^{-\alpha}}{\sum_i z_i^{1-\alpha}}, \quad j = 1, \ldots, n. \tag{8}$$

Note that (6) implies

$$\gamma_1 \le \gamma_2 \le \ldots \le \gamma_n. \tag{9}$$

Using (5) and (7) we also get

$$\gamma_j = \gamma^T e_j \le 1, \quad j = 1, \ldots, n. \tag{10}$$

We now use (7), and the fact that each player has a maximum achievable utility of 1 to bound the sum of utilities under the utilitarian principle as follows:

$$\begin{aligned} \text{SYSTEM}\,(U) &= \max\left\{\mathbf{1}^T u \,\middle|\, u \in U\right\} \\ &\le \max\left\{\mathbf{1}^T u \,\middle|\, 0 \le u \le \mathbf{1}, \gamma^T u \le 1\right\}. \end{aligned} \tag{11}$$

Using the above inequality,

$$\begin{aligned} \text{POF}\,(U;\alpha) &= \frac{\text{SYSTEM}\,(U) - \text{FAIR}\,(U;\alpha)}{\text{SYSTEM}\,(U)} \\ &= 1 - \frac{\text{FAIR}\,(U;\alpha)}{\text{SYSTEM}\,(U)} \\ &= 1 - \frac{\sum_{j=1}^n z_j}{\text{SYSTEM}\,(U)} \\ &\le 1 - \frac{\sum_{j=1}^n z_j}{\max\left\{\mathbf{1}^T u \,|\, 0 \le u \le \mathbf{1}, \gamma^T u \le 1\right\}}. \end{aligned} \tag{12}$$

The optimization problem in (12) is the linear relaxation of the well-studied knapsack problem, a version of which we review next. Let $w \in \mathbf{R}_+^n$ be such that $0 < w_1 \le \ldots \le w_n \le 1$ (in particular, $\gamma$ satisfies those conditions). Then, one can show (see Bertsimas and Tsitsiklis (1997)) that the linear optimization problem

$$\begin{aligned} \text{maximize} \quad & \mathbf{1}^T y \\ \text{subject to} \quad & w^T y \le 1 \\ & 0 \le y \le \mathbf{1}, \end{aligned} \tag{13}$$

has an optimal value equal to $\ell(w)+\delta(w)$, where

$$\ell(w) = \max\left\{ i \,\middle|\, \sum_{j=1}^{i} w_j \le 1,\, i \le n-1 \right\} \in \{1,\ldots,n-1\} \tag{14}$$

$$\delta(w) = \frac{1-\sum_{j=1}^{\ell(w)} w_j}{w_{\ell(w)+1}} \in [0,1]. \tag{15}$$

We can apply the above result to compute the optimal value of the problem in (12),

$$\max\left\{ \mathbf{1}^T u \,\middle|\, 0 \le u \le \mathbf{1}, \gamma^T u \le 1 \right\} = \ell(\gamma)+\delta(\gamma). \tag{16}$$

The bound from (12) can now be rewritten,

$$\mathrm{POF}\,(U;\alpha) \le 1 - \frac{\sum_{j=1}^{n} z_j}{\ell(\gamma)+\delta(\gamma)}. \tag{17}$$

Consider the set $S$ in the $(n+3)$-dimensional space, defined by the following constraints with variables $d \in \mathbf{R}$, $\lambda \in \mathbf{N}$ and $x_1,\ldots,x_\lambda,\overline{x}_{\lambda+1},\underline{x}_{\lambda+1},x_{\lambda+2},\ldots,x_n \in \mathbf{R}$. The variables $d$ and $\lambda$ correspond to $\delta$ and $\lambda$ accordingly, whereas $x$ corresponds to $z$. Note also that we associate two variables, $\overline{x}_{\lambda+1}$ and $\underline{x}_{\lambda+1}$, with $z_{\lambda+1}$.

$$0 \le d \le 1 \tag{18a}$$

$$1 \le \lambda \le n-1 \tag{18b}$$

$$0 \le x_n \le \ldots \le x_{\lambda+2} \le \underline{x}_{\lambda+1} \le \overline{x}_{\lambda+1} \le x_\lambda \le \ldots \le x_1 \le 1 \tag{18c}$$

$$x_n^{-\alpha} \le x_1^{1-\alpha} + \ldots + x_\lambda^{1-\alpha} + d\,\overline{x}_{\lambda+1}^{1-\alpha} + (1-d)\,\underline{x}_{\lambda+1}^{1-\alpha} + x_{\lambda+2}^{1-\alpha} + \ldots + x_n^{1-\alpha} \tag{18d}$$

$$\begin{aligned} & x_1^{-\alpha} + \ldots + x_\lambda^{-\alpha} + d\,\overline{x}_{\lambda+1}^{-\alpha} \le \\ & \qquad x_1^{1-\alpha} + \ldots + x_\lambda^{1-\alpha} + d\,\overline{x}_{\lambda+1}^{1-\alpha} + (1-d)\,\underline{x}_{\lambda+1}^{1-\alpha} + x_{\lambda+2}^{1-\alpha} + \ldots + x_n^{1-\alpha}. \end{aligned} \tag{18e}$$

The introduction of those new variables will allow us to further simplify (17). In particular, we show that

$$\frac{\sum_{j=1}^{n} z_j}{\ell(\gamma)+\delta(\gamma)} \ge \min_{(d,\lambda,x)\in S} \frac{x_1 + \ldots + x_\lambda + d\,\overline{x}_{\lambda+1} + (1-d)\,\underline{x}_{\lambda+1} + x_{\lambda+2} + \ldots + x_n}{\lambda + d}. \tag{19}$$

We pick values for $d$, $\lambda$ and $x$ that are such that (a) they are feasible for $S$, and (b) the function argument of the minimum, if evaluated at $(d,\lambda,x)$, is equal to the left-hand side of (19). In particular, let

$$d = \delta(\gamma), \qquad \lambda = \ell(\gamma),$$
$$x_j = z_j, \quad j \neq \lambda + 1, \qquad \overline{x}_{\lambda+1} = \underline{x}_{\lambda+1} = z_{\lambda+1}.$$

Then, (18a), (18b) and (18c) are satisfied because of (15), (14) and (6) respectively. By the definition of $\gamma$ and the selected value of $x$, (18d) can be equivalently expressed as

$$\gamma_n \leq 1,$$

which is implied by (10). Similarly, (18e) is equivalent to

$$\gamma_1 + \ldots + \gamma_{\ell(\gamma)} + \delta(\gamma)\gamma_{\ell(\gamma)+1} \leq 1,$$

which again holds true (by (15)). The function argument of the minimum, evaluated at the selected point, is clearly equal to the left-hand side of (19). Finally, the minimum is attained by the Weierstrass Theorem, since the function argument is continuous, and $S$ is compact. Note that (18d) in conjunction with (18c) bound $x_n$ away from 0. In particular, if $\alpha \geq 1$, we get

$$x_n^{-\alpha} \leq x_1^{1-\alpha} + \ldots + x_n^{1-\alpha} \leq n x_n^{1-\alpha} \Rightarrow x_n \geq \frac{1}{n}.$$

Similarly, for $\alpha < 1$ we get

$$x_n \geq \left(\frac{1}{n}\right)^{\frac{1}{\alpha}}.$$

To evaluate the minimum in (19), one can assume without loss of generality that for a point $(d', \lambda', x') \in S$ that attains the minimum, we have

$$x'_1 = \ldots = x'_\lambda = \overline{x}'_{\lambda+1}, \quad \underline{x}'_{\lambda+1} = x'_{\lambda+2} = \ldots = x'_n. \tag{20}$$

Technical details are included in Section C. Using this observation, we can further simplify (19). In particular, consider the set $T \subset \mathbf{R}^3$, defined by the following constraints, with variables $x_1$, $x_2$ and $y$ (since $x'_1 = \ldots = x'_\lambda = \overline{x}'_{\lambda+1}$, we associate $x_1$ with them, and similarly we associate $x_2$ with the remaining variables of $x'$; variable $y$ is associated with $\lambda + d$ ):

$$0 \leq x_2 \leq x_1 \leq 1 \tag{21a}$$
$$1 \leq y \leq n \tag{21b}$$
$$x_2^{-\alpha} \leq y x_1^{1-\alpha} + (n-y) x_2^{1-\alpha} \tag{21c}$$
$$y x_1^{-\alpha} \leq y x_1^{1-\alpha} + (n-y) x_2^{1-\alpha}. \tag{21d}$$

Using similar arguments as in showing (19), one can then show that

$$\min_{(d,\lambda,x)\in S} \frac{x_1 + \ldots + x_\lambda + d\,\overline{x}_{\lambda+1} + (1-d)\,\underline{x}_{\lambda+1} + x_{\lambda+2} + \ldots + x_n}{\lambda + d} \geq \min_{(x_1,x_2,y)\in T} \frac{yx_1 + (n-y)x_2}{y}. \tag{22}$$

If we combine (17), (19), (22) we get

$$\mathrm{POF}\,(U;\alpha) \leq 1 - \min_{(x_1,x_2,y)\in T} \frac{yx_1 + (n-y)x_2}{y}. \tag{23}$$

The final step is the evaluation of the minimum above. Let $(x_1^\star, x_2^\star, y^\star) \in T$ be a point that attains the minimum. Then, we have

$$y^\star < n, \quad x_2^\star < x_1^\star. \tag{24}$$

To see this, suppose that $x_2^\star = x_1^\star$. Then, the minimum is equal to $\frac{nx_1^\star}{y^\star}$. But, constraint (21d) yields that $nx_1^\star \geq y^\star$, in which case the minimum is greater than or equal to 1. Then, (23) yields that the price of fairness is always 0, a contradiction. If $y^\star = n$, (21d) suggests that $x_1^\star = 1$. Also, the minimum is equal to $x_1^\star = 1$, a contradiction.

We now show that (21c-21d) are active at $(x_1^\star, x_2^\star, y^\star)$. We argue for $\alpha \geq 1$ and $\alpha < 1$ separately.

$\alpha \geq 1$ : Suppose that (21c) is inactive. Then, a small enough reduction in the value of $x_2^\star$ preserves feasibility (with respect to $T$), and also yields a strictly lower value for the minimum (since $y^\star < n$, by (24)), thus contradicting that the point attains the minimum. Similarly, if (21d) is inactive, a small enough reduction in the value of $x_1^\star$ leads to a contradiction.

$\alpha < 1$ : Suppose that (21d) is inactive at $(x_1^\star, x_2^\star, y^\star)$. Then, we increase $y^\star$ by a small positive value, such that (21d) and (21b) are still satisfied. Constraint (21c) is then relaxed, since $(x_1^\star)^{1-\alpha} > (x_2^\star)^{1-\alpha}$. The minimum then has a strictly lower value, a contradiction. Hence, (21d) is active at any point that attains the minimum. If we solve for $y$ and substitute back, the objective of the minimum becomes

$$x_1 + x_2^\alpha(x_1^{-\alpha} - x_1^{1-\alpha}), \tag{25}$$

and the constraints defining the set $T$ simplify to

$$0 \leq x_2 \leq x_1 \leq 1 \tag{26a}$$

$$x_1^{-\alpha} - x_1^{1-\alpha} + x_2^{1-\alpha} \leq nx_1^{-\alpha}x_2. \tag{26b}$$

In particular, constraint (26b) correspond to constraint (21c). In case (21c) is not active at a minimum, so is (26b). But then, a small enough reduction in the value of $x_2^\star$ leads to a contradiction.

Since for any point that attains the minimum constraints (21c-21d) are active, we can use the corresponding equations to solve for $x_1$ and $x_2$. We get

$$x_1 = \frac{y^{\frac{1}{\alpha}}}{n - y + y^{\frac{1}{\alpha}}}, \tag{27}$$

$$x_2 = \frac{1}{n - y + y^{\frac{1}{\alpha}}}. \tag{28}$$

If we substitute back to (23), we get

$$\text{POF}\,(U;\alpha) \leq 1 - \min_{x\in[1,n]} \frac{x^{1+\frac{1}{\alpha}} + n - x}{x^{1+\frac{1}{\alpha}} + (n-x)x}.$$

The asymptotic analysis is included in Section C.

**Proof of Theorem 2.** We follow similar steps to the ones in the proof of Theorem 1. Thus, assume that $U$ is monotone, the maximum achievable utilities of the players are equal to 1 and that $z_1 \geq z_2 \geq \ldots \geq z_n$ (where z $= z(\alpha) \in U$ is the unique $\alpha$-fair allocation). Then, for the variable $\gamma$ (defined as in (8)), we similarly have

$$\gamma^T u \leq 1, \quad \forall u \in U,$$

and

$$\gamma_1 \leq \gamma_2 \leq \ldots \gamma_n \leq 1.$$

We use the above to bound the maximum value of the fairness metric

$$\max\left\{\min_{j=1,\ldots,n} u_j \,\middle|\, u \in U\right\} \leq \max\left\{\min_{j=1,\ldots,n} u_j \,\middle|\, 0 \leq u \leq \mathbf{1},\, \gamma^T u \leq 1\right\} = \frac{1}{\mathbf{1}^T\gamma},$$

where the equality follows from $z \leq \mathbf{1}$ and $\mathbf{1}^T\gamma \geq 1$.

We bound the price of efficiency using $z_1 \geq \ldots \geq z_n$, $\gamma_n \leq 1$ and the inequality above as follows:

$$\begin{aligned}
\mathrm{POE}\,(U;\alpha) &= \frac{\max\limits_{u\in U}\ \min\limits_{j=1,\ldots,n}\ u_j - \min\limits_{j=1,\ldots,n}\ z_j(\alpha)}{\max\limits_{u\in U}\ \min\limits_{j=1,\ldots,n}\ u_j} \\
&= 1 - \frac{z_n}{\max\limits_{u\in U}\ \min\limits_{j=1,\ldots,n}\ u_j} \\
&\leq 1 - z_n \mathbf{1}^T \gamma \\
&= 1 - \frac{z_n\left(z_1^{-\alpha} + z_2^{-\alpha} + \ldots + z_n^{-\alpha}\right)}{z_1^{1-\alpha} + z_2^{1-\alpha} + \ldots + z_n^{1-\alpha}} \\
&= 1 - f^\star,
\end{aligned}$$

where $f^\star$ is the optimal value of the problem

$$\begin{array}{ll}
\text{minimize} & \dfrac{z_n\left(z_1^{-\alpha} + z_2^{-\alpha} + \ldots + z_n^{-\alpha}\right)}{z_1^{1-\alpha} + z_2^{1-\alpha} + \ldots + z_n^{1-\alpha}} \\
\text{subject to} & 0 \leq z_n \leq z_{n-1} \leq \ldots \leq z_1 \leq 1 \\
& z_n^{-\alpha} \leq z_1^{1-\alpha} + z_2^{1-\alpha} + \ldots + z_n^{1-\alpha}.
\end{array} \tag{29}$$

Let $z^\star$ be an optimal solution of (29) (guaranteed to exist by the Weierstrass Theorem). Then, without loss of generality we can assume that (a) $z_1^\star = z_2^\star = \ldots = z_{n-1}^\star$ and (b) $z_1^\star = 1$. Technical details are included in the Section C. Using those two assumptions, $f^\star$ is then equal to

$$\begin{array}{ll}
\text{minimize} & \dfrac{(n-1)x + x^{1-\alpha}}{n-1+x^{1-\alpha}} \\
\text{subject to} & 0 \leq x \leq 1 \\
& x^{-\alpha} \leq n - 1 + x^{1-\alpha}.
\end{array} \tag{30}$$

Finally, note that for $x \in [0,1]$ the function $x^{-\alpha} - x^{1-\alpha} - n - 1$ is strictly decreasing, is positive for $x$ small and negative for $x = 1$. Hence, for $x \in [0,1]$ the constraint $x^{-\alpha} \leq n - 1 + x^{1-\alpha}$ is equivalent to $x \geq \rho$. As a result,

$$f^\star = \min_{\rho \leq x \leq 1} \frac{(n-1)x + x^{1-\alpha}}{n-1+x^{1-\alpha}}.$$

The asymptotic analysis is similar to the analysis in Theorem 1 and is omitted.

## B. More on Near Worst-case Examples for the Price of Fairness

We demonstrate how one can construct near worst-case examples, for which the price of fairness is very close to the bounds implied by Theorem 1, for any values of the problem parameters; the number of players $n$ and the value of the inequality aversion parameter $\alpha$. We then provide details about the bandwidth allocation problem in Section 3.1.1.

For any $n \in \mathbf{N} \setminus \{0,1\}$, $\alpha > 0$, we create a utility set using Procedure 1.

**Procedure 1** Creation of near worst-case utility set

**Input:** $n \in \mathbf{N} \setminus \{0,1\}$, $\alpha > 0$
**Output:** utility set $U$

1: compute $y := \operatorname*{argmin}_{x \in [1,n]} \dfrac{x^{1+\frac{1}{\alpha}} + n - x}{x^{1+\frac{1}{\alpha}} + (n-x)x}$
2: $x_1 \leftarrow \frac{y^{\frac{1}{\alpha}}}{n-y+y^{\frac{1}{\alpha}}}$ (as in (27))
3: $x_2 \leftarrow \frac{1}{n-y+y^{\frac{1}{\alpha}}}$ (as in (28))
4: $\ell \leftarrow \min\{\text{round}(y), n-1\}$
5: $\gamma_i \leftarrow \frac{x_i^{-\alpha}}{yx_1^{1-\alpha}+(n-y)x_2^{1-\alpha}}$ for $i = 1, 2$
6: $U \leftarrow \left\{u \in \mathbf{R}^n_+ \,\middle|\, \gamma_1 u_1 + \ldots + \gamma_1 u_\ell + \gamma_2 u_{\ell+1} + \ldots + \gamma_2 u_n \le 1, \quad u \le \mathbf{1}\ \forall j\right\}$

The following proposition demonstrates why Procedure 1 creates utility sets that achieve a price of fairness very close to the bounds implied by Theorem 1.

**Proposition 1.** *For any $n \in \mathbf{N} \setminus \{0,1\}$, $\alpha > 0$, the output utility set $U$ of Procedure 1 satisfies the conditions of Theorem 1. If $y \in \mathbf{N}$, the output utility set $U$ satisfies the bound of Theorem 1 with equality.*

**Proof.** The output utility set $U$ is a bounded polyhedron, hence convex and compact. Boundedness follows from positivity of $\gamma_1$ and $\gamma_2$.

Note that the selection of $x_1$, $x_2$ and $y$ in Procedure 1 corresponds to a point that attains the minimum of (23), hence all properties quoted in the proof of Theorem 1 apply. In particular, by (18d) we have $\gamma_2 \le 1$ and (21d) is tight, $y\gamma_1 = 1$. Moreover, the bound from Theorem 1 can be expressed as

$$\text{POF}\,(U;\alpha) \le 1 - \frac{yx_1 + (n-y)x_2}{y}.$$

The maximum achievable utility of the $j$th player is equal to 1. To see this, note that the definition of $U$ includes the constraint $u_j \le 1$, so it suffices to show that $e_j \in U$. For $j \le \ell$, we have $\gamma_1 \le \gamma_1 y = 1$. For $j > \ell$, we have $\gamma_2 \le 1$. It follows that $U$ satisfies the conditions of Theorem 1.

Suppose that $y \in \mathbf{N}$. By (24) and the choice of $\ell$ in Procedure 1, we get $\ell = y$. Consider the vector $z \in \mathbf{R}^n$ with $z_1 = \ldots = z_\ell = x_1$ and $z_{\ell+1} = \ldots = z_n = x_2$. Then, the sufficient first order optimality condition for $z$ to be the $\alpha$-fair allocation of $U$ is satisfied, as for any $u \in U$

$$\sum_{j=1}^{n} z_j^{-\alpha}(u_j - z_j) = x_1^{-\alpha}(u_1 + \ldots + u_\ell) + x_2^{-\alpha}(u_{\ell+1} + \ldots + u_n) - yx_1^{1-\alpha} - (n-y)x_2^{1-\alpha} \le 0,$$

since $\gamma_1(u_1 + \ldots + u_\ell) + \gamma_2(u_{\ell+1} + \ldots + u_n) \leq 1$. Hence,

$$\text{FAIR}\,(U;\alpha) = \mathbf{1}^T z = yx_1 + (n-y)x_2.$$

For the efficiency-maximizing solution, since $y\gamma_1 = 1$, we get

$$\text{SYSTEM}\,(U) = y.$$

Then,

$$\text{POF}\,(U;\alpha) = 1 - \frac{yx_1 + (n-y)x_2}{y},$$

which is exactly the bound from Theorem 1. ■

The above result demonstrates why one should expect Procedure 1 to generate examples that have a price of fairness very close to the established bounds. In particular, Proposition 1 shows that the source of error between the price of fairness for the utility sets generated by Procedure 1 and the bound is the (potential) non-integrality of $y$. In case that error is "large", one can search in the neighborhood of parameters $\gamma_1$ and $\gamma_2$ for an example that achieves a price closer to the bound, for instance by using finite-differencing derivatives and a gradient descent method (respecting feasibility).

**Near worst-case bandwidth allocation**

We utilize Proposition 1 and Procedure 1 to construct near worst-case network topologies. In particular, one can show that the line-graph discussed in Section 3.1.1, actually corresponds to a worst-case topology in this setup.

Suppose that we fix the number of players $n \geq 2$, the desired inequality aversion parameter $\alpha > 0$, and follow Procedure 1. Further suppose that $y \in \mathbf{N}$, as in Proposition 1. Consider then a network with $y$ links of unit capacity, in a line-graph topology: the routes of the first $y$ flows are disjoint and they all occupy a single (distinct) link. The remaining $n-y$ flows have routes that utilize all $y$ links. Each flow derives a utility equal to its assigned nonnegative rate, which we denote $u_1, \ldots, u_n$. We next show that the price of fairness for this network is equal to the bound of Theorem 1.

The output utility set of Procedure 1 achieves the bound, by Proposition 1, since $y \in \mathbf{N}$. Moreover, we also get that $y\gamma_1 = 1$ and $\gamma_2 = 1$. Hence, the output utility set that achieves the bound can be formulated as

$$U = \{u \geq 0 \,|\, u_1 + \ldots + u_y + y\,(u_{y+1} + \ldots + u_n) \leq y,\, u \leq \mathbf{1}\}\,.$$

The utility set corresponding to the line-graph example above can be expressed using the non-negativity constraints of the flow rates, and the capacity constraints on each of the $y$ links as follows,

$$\overline{U} = \{u \geq 0 \,|\, u_j + u_{y+1} + \ldots + u_n \leq 1,\, j = 1, \ldots, y\}\,.$$

Clearly, the maximum sum of utilities under both sets is equal to $y$, simply by setting the first $y$ components of $u$ to 1. It suffices then to show that the two sets also share the same $\alpha$-fair allocation. In particular, by symmetry of $U$ and strict concavity of $W_\alpha$, if $u^F$ is its $\alpha$ fair allocation, then $u^F_1 = \ldots = u^F_y$, and $u^F_{y+1} = \ldots = u^F_n$. As a result, it follows that $u^F \in \overline{U}$. Finally, noting that all inequalities in the definition of $U$ are also valid for $\overline{U}$, it follows that $\overline{U} \subset U$ and that $u^F$ is also the $\alpha$-fair allocation of $\overline{U}$.

# C. Auxiliary Results

**Proposition 2.** *For a point $(d, \lambda, x) \in S$ that attains the minimum of (19),*

*(a) if $\lambda + 1 < n$, then without loss of generality*

$$\underline{x}_{\lambda+1} = x_{\lambda+2} = \ldots = x_n, \text{ and,}$$

*(b) without loss of generality*

$$x_1 = \ldots = x_\lambda = \overline{x}_{\lambda+1}.$$

**Proof.** (a) We drop the underline notation for $\underline{x}_{\lambda+1}$ to simplify notation. Suppose that $x_j > x_{j+1}$, for some index $j \in \{\lambda+1, \ldots, n-1\}$. We will show that there always exists a new point, $(d, \lambda, x') \in S$, for which $x'_i = x_i$, for all $i \in \{1, \ldots, n\} \setminus \{j, j+1\}$, and which either achieves the same objective with $x'_j = x'_{j+1}$, or it achieves a strictly lower objective.

If $j = \lambda + 1$ and $d = 1$, we set $x'_j = x'_{j+1} = x_{j+1}$. The new point is feasible, and the objective attains the same value.

Otherwise, let $x'_j = x_j - \epsilon$, for some $\epsilon > 0$. We have two cases.

$\alpha \geq 1$: Let $x'_{j+1} = x_{j+1}$ and pick $\epsilon$ small enough, such that $x'_j \geq x'_{j+1}$. Moreover, for the new point (compared to the feasible starting point) the left-hand sides of (18d) and (18e) are unaltered, whereas the right-hand sides are either unaltered (for $\alpha = 1$) or greater, since $x_j^{1-\alpha} < (x_j - \epsilon)^{1-\alpha}$ for $\alpha > 1$. Hence, the new point is feasible. It also achieves a strictly lower objective value.

$\alpha < 1$: Let $x'_{j+1} = x_{j+1} + \rho b\epsilon$, where

$$b = \begin{cases} 1 - d, & \text{if } j = \lambda + 1, \\ 1, & \text{otherwise,} \end{cases}$$
$$\rho \in \left( \frac{x_j^{-\alpha}}{x_{j+1}^{-\alpha}}, 1 \right).$$

For $\epsilon$ small enough, we have $x'_j \geq x'_{j+1}$. For the new point, the left-hand side of (18d) either decreases (if $j + 1 = n$), or remains unaltered. The left-hand side of (18e) remains also unaltered. For the right-hand sides, since the only terms that change are those involving $x_j$ and $x_{j+1}$, we use a first order Taylor series expansion to get

$$\begin{aligned} b\left(x'_j\right)^{1-\alpha} + \left(x'_{j+1}\right)^{1-\alpha} &= b\left(x_j - \epsilon\right)^{1-\alpha} + \left(x_{j+1} + \rho b\epsilon\right)^{1-\alpha} \\ &= bx_j^{1-\alpha} - b\epsilon(1-\alpha)x_j^{-\alpha} + x_{j+1}^{1-\alpha} + \rho b\epsilon(1-\alpha)x_{j+1}^{-\alpha} + O(\epsilon^2) \\ &= \left(bx_j^{1-\alpha} + x_{j+1}^{1-\alpha}\right) + b(1-\alpha)\left(\rho x_{j+1}^{-\alpha} - x_j^{-\alpha}\right)\epsilon + O(\epsilon^2). \end{aligned}$$

By the selection of $\rho$, the coefficient of the first order term (with respect to $\epsilon$) above is positive, and hence, for small enough $\epsilon$ we get

$$b\left(x'_j\right)^{1-\alpha} + \left(x'_{j+1}\right)^{1-\alpha} > bx_j^{1-\alpha} + x_{j+1}^{1-\alpha}.$$

That shows that the right hand side increases, and the new point is feasible. Finally, the difference in the objective value is $-b\epsilon + \rho b\epsilon$, and thus negative.

(b) We drop the overline notation for $\overline{x}_{\lambda+1}$ to simplify notation. Suppose that $x_j > x_{j+1}$, for some index $j \in \{1, \ldots, \lambda\}$.

We will show that there always exists a new point, $(d, \lambda, x') \in S$, for which $x'_i = x_i$, for all $i \in \{1, \ldots, n\} \setminus \{j, j+1\}$, and which either achieves the same objective with $x'_j = x'_{j+1}$, or it achieves a strictly lower objective.

If $j + 1 = \lambda + 1$ and $d = 0$, we set $x'_j = x'_{j+1} = x_j$. The new point is feasible, and the objective attains the same value.

Otherwise, let

$$x'_j = x_j - \epsilon$$
$$x'_{j+1} = x_{j+1} + \rho c\epsilon,$$

for some $\epsilon > 0$, where

$$\rho \in \left( \frac{x_{j+1}}{x_j}, \frac{x_{j+1}^{-\alpha}}{x_j^{-\alpha}} \right)$$
$$c = \frac{x_j^{-\alpha}}{b x_{j+1}^{-\alpha}}$$
$$b = \begin{cases} d, & \text{if } j+1 = \lambda + 1, \\ 1, & \text{otherwise.} \end{cases}$$

For $\epsilon$ small enough, we have $x'_j \geq x'_{j+1}$. For the new point, the left-hand side of (18d) remains unaltered. For the left-hand side of (18e) we use a first order Taylor series expansion (similarly as above) to get

$$\begin{aligned} \left(x'_j\right)^{-\alpha} + b\left(x'_{j+1}\right)^{-\alpha} &= (x_j - \epsilon)^{-\alpha} + b\left(x_{j+1} + \rho c \epsilon\right)^{-\alpha} \\ &= x_j^{-\alpha} + \epsilon \alpha x_j^{-\alpha-1} + b x_{j+1}^{-\alpha} - b\rho c \epsilon \alpha x_{j+1}^{-\alpha-1} + O(\epsilon^2) \\ &= \left(x_j^{-\alpha} + b x_{j+1}^{-\alpha}\right) + \epsilon \alpha x_j^{-\alpha-1} - \rho \epsilon \alpha x_j^{-\alpha} x_{j+1}^{-1} + O(\epsilon^2) \\ &= \left(x_j^{-\alpha} + b x_{j+1}^{-\alpha}\right) + \alpha x_j^{-\alpha-1} \left(1 - \rho \frac{x_j}{x_{j+1}}\right) \epsilon + O(\epsilon^2). \end{aligned}$$

By the selection of $\rho$, the coefficient of the first order term (with respect to $\epsilon$) above is negative, and hence, for small enough $\epsilon$ we get that the left-hand side decreases.

For the right-hand side of (18d) and (18e), we similarly get that

$$\begin{aligned} \left(x'_j\right)^{1-\alpha} + b\left(x'_{j+1}\right)^{1-\alpha} &= (x_j - \epsilon)^{1-\alpha} + b\left(x_{j+1} + \rho c \epsilon\right)^{1-\alpha} \\ &= x_j^{1-\alpha} - \epsilon(1-\alpha) x_j^{-\alpha} + b x_{j+1}^{1-\alpha} + b\rho c \epsilon (1-\alpha) x_{j+1}^{1-\alpha} + O(\epsilon^2) \\ &= \left(x_j^{1-\alpha} + b x_{j+1}^{1-\alpha}\right) + (1-\alpha) x_j^{-\alpha} \left(\rho - 1\right) \epsilon + O(\epsilon^2). \end{aligned}$$

If for $\alpha > 1$ we pick $\rho < 1$, and for $\alpha < 1$ we pick $\rho > 1$, the first order term (with respect to $\epsilon$) above is positive, and hence, for small enough $\epsilon$ we get that the right-hand side increases for $\alpha \neq 1$. For $\alpha = 1$, the right-hand side remains unaltered.

In all cases, the new point is feasible, and the difference in the objective value is

$$-\epsilon + \rho c b \epsilon = (\rho c b - 1)\,\epsilon = \left(\rho \frac{x_j^{-\alpha}}{x_{j+1}^{-\alpha}} - 1\right) \epsilon,$$

and thus negative (by the selection of $\rho$). ■

**Proposition 3.** *Let $n \in \mathbf{N} \setminus \{0,1\}$ and $f : [1,n] \to \mathbf{R}$ be defined as*

$$f(x;\alpha,n) = \frac{x^{1+\frac{1}{\alpha}} + n - x}{x^{1+\frac{1}{\alpha}} + (n-x)x}.$$

*For any $\alpha > 0$,*

*(a) $-f$ is unimodal over $[1,n]$, and thus has a unique minimizer $\xi^\star \in [1,n]$.*

*(b) $\min_{x\in[1,n]} f(x;\alpha,n) = f(\xi^\star;\alpha,n) = \Theta\left(n^{-\frac{\alpha}{\alpha+1}}\right)$.*

**Proof.** (a) The derivative of $f$ is

$$f'(x;\alpha,n) = \frac{g(x)}{\left(x^{1+\frac{1}{\alpha}} + (n-x)x\right)^2},$$

where

$$g(x) = \left(1 - \frac{1}{\alpha}\right) x^{2+\frac{1}{\alpha}} + \frac{n+1}{\alpha} x^{1+\frac{1}{\alpha}} - n\left(1 + \frac{1}{\alpha}\right) x^{\frac{1}{\alpha}} - (x-n)^2.$$

Note that the sign of the derivative is determined by $g(x)$, since the denominator is positive for $1 \le x \le n$, that is,

$$\operatorname{sgn} f'(x;\alpha,n) = \operatorname{sgn} g(x). \tag{31}$$

We will show that $g$ is strictly increasing over $[1,n]$. To this end, we have

$$g'(x) = x^{\frac{1}{\alpha}-1} q(x) + 2(n-x),$$

where

$$q(x) = \left(2 + \frac{1}{\alpha}\right)\left(1 - \frac{1}{\alpha}\right) x^2 + \left(1 + \frac{1}{\alpha}\right)\left(\frac{n+1}{\alpha}\right) x - \frac{n}{\alpha}\left(1 + \frac{1}{\alpha}\right).$$

Since we are interested in the domain $[1,n]$, it suffices to show that $q(x) > 0$ over it. For $\alpha > 1$, $q$ is a convex quadratic, with its minimizer being equal to

$$-\frac{\left(1 + \frac{1}{\alpha}\right)\left(\frac{n+1}{\alpha}\right)}{2\left(2 + \frac{1}{\alpha}\right)\left(1 - \frac{1}{\alpha}\right)} < 0.$$

Hence, $q(x) \ge q(1)$ for $x \in [1,n]$. Similarly, for $\alpha < 1$, $q$ is a concave quadratic, and as such, for $x \in [1,n]$ we have $q(x) \ge \min\{q(1), q(n)\}$. For $\alpha = 1$, $q(x) = 2(n+1)x - 2n$, which is positive for $x \ge 1$. Then, $q(x) > 0$ in $[1,n]$ for all $\alpha > 0$, if and only if $q(1) > 0$ and $q(n) > 0$. Note that for $r = 1$, we get $q(1) = 2$ and $q(n) = 2n^2$, and

$$\frac{dq(1)}{dr} = 2 > 0, \quad \frac{dq(n)}{dr} = 2n^2 > 0,$$

which demonstrates that $q(1)$ and $q(n)$ are positive. Furthermore,

$$g(n) = n^{1+\frac{1}{\alpha}}(n-1) > 0.$$

Using the above, the fact that $g$ is continuous and strictly increasing over $[1, n]$ and (31), we deduce that if $g(1) < 0$, there exists a unique $m \in (1, n)$ such that

$$\operatorname{sgn} f'(x; \alpha, n) \begin{cases} < 0, & \text{if } 1 \leq x < m, \\ > 0, & \text{if } m < x \leq n. \end{cases}$$

Similarly, if $g(1) \geq 0$, $f$ is strictly increasing for $1 \leq x \leq n$. It follows that $-f$ is unimodal.

(b) Let $\theta_n = n^{\frac{\alpha}{\alpha+1}}$. Using the mean value Theorem, for every $n \geq 2$, there exists a $\psi_n \in [\theta_n, \xi^\star]$ (or $[\xi^\star, \theta_n]$, depending on if $\theta_n \leq \xi^\star$), such that

$$f(\theta_n; \alpha, n) = f(\xi^\star; \alpha, n) + f'(\psi_n; \alpha, n)(\theta_n - \xi^\star),$$

or, equivalently,

$$\frac{f(\xi^\star; \alpha, n)}{f(\theta_n; \alpha, n)} = 1 - \frac{f'(\psi_n; \alpha, n)(\theta_n - \xi^\star)}{f(\theta_n; \alpha, n)}.$$

We will show that, for a sufficiently small $\epsilon > 0$

(I.) $f'(\psi_n; \alpha, n) = O\left(n^{-\frac{\min\{1,\alpha\}+2\alpha}{\alpha+1}+2\epsilon}\right)$,

(II.) $\theta_n - \xi^\star = O\left(n^{\frac{\alpha}{\alpha+1}+\epsilon}\right)$,

(III.) $f(\theta_n; \alpha, n) = \Theta\left(n^{-\frac{\alpha}{\alpha+1}}\right)$.

Using the above facts, it is easy to see that

$$\frac{f(\xi^\star; \alpha, n)}{f(\theta_n; \alpha, n)} = 1 - \frac{f'(\psi_n; \alpha, n)(\theta_n - \xi^\star)}{f(\theta_n; \alpha, n)} = 1 - O\left(n^{-\frac{\min\{1,\alpha\}}{\alpha+1}+3\epsilon}\right) \to 1,$$

and thus $f(\xi^\star; \alpha, n) = \Theta\left(n^{-\frac{\alpha}{\alpha+1}}\right)$.

(I.) We first show that for any sufficiently large $n$,

$$n^{\frac{\alpha}{\alpha+1}-\epsilon} \leq \xi^\star \leq n^{\frac{\alpha}{\alpha+1}+\epsilon}. \tag{32}$$

By part (a), $\xi^\star$ is the unique root of $g$ in the interval $[1, n]$. Moreover, $g$ is strictly increasing.

The dominant term of

$$g\left(n^{\frac{\alpha}{\alpha+1}-\epsilon}\right) = \left(1-\frac{1}{\alpha}\right) n^{\left(2+\frac{1}{\alpha}\right)\left(\frac{\alpha}{\alpha+1}-\epsilon\right)} + \frac{1}{\alpha} n^{1-\frac{\alpha+1}{\alpha}\epsilon} + \frac{1}{\alpha} n^{2-\frac{\alpha+1}{\alpha}\epsilon} - \left(1+\frac{1}{\alpha}\right) n^{1+\frac{1}{\alpha+1}-\frac{1}{\alpha}\epsilon} - n^2 - n^{\frac{2\alpha}{\alpha+1}-2\epsilon} + 2n^{1+\frac{\alpha}{\alpha+1}-\epsilon},$$

is $-n^2$, and hence, for sufficiently large $n$ we have $g\left(n^{\frac{\alpha}{\alpha+1}-\epsilon}\right) < 0$. Similarly, the dominant term of $g\left(n^{\frac{\alpha}{\alpha+1}+\epsilon}\right)$ is $\frac{1}{\alpha}n^{2+\frac{\alpha+1}{\alpha}\epsilon}$, and for sufficiently large $n$ we have $g\left(n^{\frac{\alpha}{\alpha+1}+\epsilon}\right) > 0$. The claim then follows. Using the above bound, for sufficiently large $n$, we also get that $\psi_n \geq n^{\frac{\alpha}{\alpha+1}-\epsilon}$. We now provide a bound for the denominator of $f'(\psi_n;\alpha,n)$. In particular, for sufficiently large $n$, we get that for $x \leq n^{\frac{\alpha}{\alpha+1}+\epsilon}$,

$$\frac{d}{dx}\left(x^{1+\frac{1}{\alpha}} + nx - x^2\right) = \left(1+\frac{1}{\alpha}\right) x^{\frac{1}{\alpha}} + n - 2x > 0,$$

which shows that the denominator is strictly increasing. Hence, using the lower bound on $\psi_n$,

$$\frac{1}{\left(\psi_n^{1+\frac{1}{\alpha}} + n\psi_n - \psi_n^2\right)^2} \leq \frac{1}{\left(n^{\left(\frac{\alpha}{\alpha+1}-\epsilon\right)\left(1+\frac{1}{\alpha}\right)} + n^{1+\frac{\alpha}{\alpha+1}-\epsilon} - n^{\frac{2\alpha}{\alpha+1}-2\epsilon}\right)^2} \leq \frac{n^{-2-\frac{2\alpha}{\alpha+1}+2\epsilon}}{\left(n^{-\frac{\alpha}{\alpha+1}-\frac{1}{\alpha}\epsilon} + 1 - n^{-\frac{1}{\alpha+1}}\right)^2} = O\left(n^{-2-\frac{2\alpha}{\alpha+1}+2\epsilon}\right).$$

We now provide a bound for the numerator. Since $g$ is strictly increasing and $\xi^\star$ is a root, we get

$$\begin{aligned}|g(\psi_n)| &\leq |g(\theta_n)| \\ &= \left|\left(1-\frac{1}{\alpha}\right)\alpha^{\frac{2\alpha+1}{\alpha+1}} n^{-\frac{1}{\alpha+1}+2} + n - \left(1+\frac{1}{\alpha}\right)\alpha^{\frac{1}{\alpha+1}} n^{-\frac{\alpha}{\alpha+1}+2} - \alpha^{\frac{2\alpha}{\alpha+1}} n^{-\frac{2}{\alpha+1}+2} + 2\alpha^{\frac{\alpha}{\alpha+1}} n^{-\frac{1}{\alpha+1}+2}\right| \\ &= O\left(n^{-\frac{\min\{1,\alpha\}}{\alpha+1}+2}\right).\end{aligned}$$

If we combine the above results, we get $f'(\psi_n;\alpha,n) = O\left(n^{-\frac{\min\{1,\alpha\}+2\alpha}{\alpha+1}+2\epsilon}\right)$.

(II.) Follows from (32).

(III.) We have

$$f(\theta_n;\alpha,n) = \frac{n + n - n^{\frac{\alpha}{\alpha+1}}}{n + n^{1+\frac{\alpha}{\alpha+1}} - n^{\frac{2\alpha}{\alpha+1}}} = \frac{n\left(2 - n^{-\frac{1}{\alpha+1}}\right)}{n^{1+\frac{\alpha}{\alpha+1}}\left(n^{-\frac{\alpha}{\alpha+1}} + 1 - n^{-\frac{1}{\alpha+1}}\right)} = \Theta\left(n^{-\frac{\alpha}{\alpha+1}}\right).$$ ■

**Proposition 4.** *There exists a point $z \in \mathbf{R}^n$ that attains the minimum of (29), for which*

$$z_1 = \ldots = z_{n-1} = 1.$$

**Proof.** For $\alpha = 1$, problem (29) is written as

$$\begin{array}{ll} \text{minimize} & \frac{1}{n}\left(\frac{z_n}{z_1} + \frac{z_n}{z_2} + \ldots + \frac{z_n}{z_{n-1}} + 1\right) \\ \text{subject to} & \frac{1}{n} \leq z_n \leq z_{n-1} \leq \ldots \leq z_1 \leq 1. \end{array}$$

If $z$ is an optimal solution of the above, then clearly $z_1 = \ldots = z_{n-1} = 1$.

We now deal with the case of $\alpha \neq 1$. We first show that if $z$ is an optimal solution of (29), then $z_1 = \ldots = z_{n-1}$. We analyze the cases $0 < \alpha < 1$ and $\alpha > 1$ separately.

For $0 < \alpha < 1$, the function $z_1^{1-\alpha} + \ldots + z_{n-1}^{1-\alpha}$ is strictly concave, and the function $z_1^{-\alpha} + \ldots + z_{n-1}^{-\alpha}$ is strictly convex. If $z$ is an optimal solution of (29) for which $z_1 = \ldots = z_{n-1}$ is violated, we construct a point $\bar{z} \in \mathbf{R}^n$, such that its first $n-1$ components are all equal to the mean of $z_1, \ldots, z_{n-1}$ and $\bar{z}_n = z_n$. We show that $\bar{z}$ is feasible for (29) and it achieves a strictly lower objective value compared to $z$, a contradiction. Note that by strict concavity/ convexity we get

$$\bar{z}_1^{1-\alpha} + \ldots + \bar{z}_{n-1}^{1-\alpha} > z_1^{1-\alpha} + \ldots + z_{n-1}^{1-\alpha},$$

and

$$\bar{z}_1^{-\alpha} + \ldots + \bar{z}_{n-1}^{-\alpha} < z_1^{-\alpha} + \ldots + z_{n-1}^{-\alpha},$$

respectively. For feasibility, $0 \leq \bar{z}_n \leq \ldots \leq \bar{z}_1 \leq 1$ is immediate and

$$\bar{z}_n^{-\alpha} = z_n^{-\alpha} \leq z_1^{1-\alpha} + \ldots + z_{n-1}^{1-\alpha} + z_n^{1-\alpha} < \bar{z}_1^{1-\alpha} + \ldots + \bar{z}_{n-1}^{1-\alpha} + \bar{z}_n^{1-\alpha}.$$

Finally, compared to $z$, if we evaluate the objective of (29) at $\bar{z}$, the numerator strictly decreases and the denominator strictly increases, hence the objective value strictly decreases.

For $\alpha > 1$, let $z$ be an optimal solution of (29) for which $z_{j+1} < z_j$ for some $j = 1, \ldots, n-2$. We similarly construct a feasible point $\bar{z}$ for (29) that achieves a strictly lower objective value than $z$. Let $\bar{z}_i = z_i$ for all $i \neq j, j+1$, $\bar{z}_j = z_j - \epsilon$ and $\bar{z}_{j+1} = z_{j+1} + \delta\epsilon$, where $\epsilon > 0$ and

$$\delta = \frac{z_j^{-\alpha} - \mu}{z_{j+1}^{-\alpha}}, \qquad \mu \in \left(0, z_j^{-\alpha}\left(\frac{z_j - z_{j+1}}{z_j}\right)\right).$$

For small enough $\epsilon$, $0 \leq \bar{z}_n \leq \ldots \leq \bar{z}_1 \leq 1$ is immediate. Using a first order Taylor series expansion,

$$\begin{aligned}\bar{z}_j^{1-\alpha} + \bar{z}_{j+1}^{1-\alpha} &= z_j^{1-\alpha} + z_{j+1}^{1-\alpha} + (z_j^{-\alpha} - \delta z_{j+1}^{-\alpha})(\alpha - 1)\epsilon + O(\epsilon^2) \\ &> z_j^{1-\alpha} + z_{j+1}^{1-\alpha}\end{aligned}$$

for small enough $\epsilon$, since $z_j^{-\alpha} > \delta z_{j+1}^{-\alpha} \Leftrightarrow \mu > 0$. As a result,

$$\bar{z}_1^{1-\alpha} + \ldots + \bar{z}_{n-1}^{1-\alpha} + \bar{z}_n^{1-\alpha} > z_1^{1-\alpha} + \ldots + z_{n-1}^{1-\alpha} + z_n^{1-\alpha},$$

and $\bar{z}$ is feasible. Moreover, the denominator of the objective strictly increases. Thus it suffices to show that the numerator decreases. To this end, we have

$$\begin{aligned}\bar{z}_j^{-\alpha} + \bar{z}_{j+1}^{-\alpha} &= z_j^{-\alpha} + z_{j+1}^{-\alpha} + (z_j^{-\alpha-1} - \delta z_{j+1}^{-\alpha-1})\alpha\epsilon + O(\epsilon^2) \\ &< z_j^{-\alpha} + z_{j+1}^{-\alpha}\end{aligned}$$

for small enough $\epsilon$, since $z_j^{-\alpha-1} < \delta z_{j+1}^{-\alpha-1} \Leftrightarrow \mu < z_j^{-\alpha}\left(\frac{z_j - z_{j+1}}{z_j}\right)$.

Since for every optimal solution of (29), we have $z_1 = \ldots = z_{n-1}$, problem (29) can be written equivalently as

$$\begin{aligned} \text{minimize} \quad & g(z_1, z_2) = \frac{(n-1)z_1^{-\alpha} z_2 + z_2^{1-\alpha}}{(n-1)z_1^{1-\alpha} + z_2^{1-\alpha}} \\ \text{subject to} \quad & 0 \leq z_2 \leq z_1 \leq 1 \\ & z_2^{-\alpha} \leq (n-1)z_1^{1-\alpha} + z_2^{1-\alpha}. \end{aligned} \tag{33}$$

It suffices to show that there exists an optimal solution $z$ of (33) for which $z_1 = 1$.

Let $z$ be an optimal solution of (33).

If $0 < \alpha < 1$, assume that $z_1 < 1$. Then, increase $z_1$ by a small enough amount such that it remains less than 1. The quantity $z_1^{1-\alpha}$ increases, so the new point we get is feasible. Also, the quantity $z_1^{-\alpha}$ decreases. Hence, the new point is feasible and achieves a strictly lower objective value, a contradiction.

If $\alpha > 1$, the point $z$ lies on the boundary of the feasible set or is a stationary point of the objective. Suppose that $z$ is not a stationary point, *i.e.*, $\nabla g(z_1, z_2) \neq 0$. If $z_1 = z_2$, the objective evaluates to 1 for any such $z$, so we can assume $z_1 = 1$. We next rule out the possibility of $z$ lying on the $z_2^{-\alpha} = (n-1)z_1^{1-\alpha} + z_2^{1-\alpha}$ boundary with $z_1 < 1$. Suppose that it does. We will demonstrate that we can always find a feasible direction along which the objective decreases. We have

$$\begin{aligned}\frac{\partial g}{\partial z_1} &= \frac{(n-1)z_1^{-\alpha} z_2}{\left((n-1)z_1^{1-\alpha} + z_2^{1-\alpha}\right)^2}\left(-(n-1)z_1^{-\alpha} - \alpha z_1^{-1} z_2^{1-\alpha} + (\alpha - 1)z_2^{-\alpha}\right), \\ \frac{\partial g}{\partial z_2} &= -\frac{z_1}{z_2}\frac{\partial g}{\partial z_1}.\end{aligned}$$

Note that we assumed that $\nabla g(z) \neq 0$, hence $\frac{\partial g}{\partial z_1}(z) \neq 0$. Suppose that $\frac{\partial g}{\partial z_1}(z) > 0$. Then, $(1, \delta)$ is a direction along which the objective decreases, for large enough $\delta > 0$, since

$$\frac{\partial g}{\partial z_1}(z) + \delta \frac{\partial g}{\partial z_2}(z) = \frac{\partial g}{\partial z_1}(z) \left(1 - \delta \frac{z_1}{z_2}\right) < 0.$$

It is also a feasible direction, since for $\epsilon > 0$ small enough, $0 \leq z_2 + \delta\epsilon \leq z_1 + \epsilon \leq 1$, and is also a direction along which $(n-1)z_1^{1-\alpha} + z_2^{1-\alpha} + z_2^{-\alpha}$ increases, since

$$\begin{aligned}(n-1)z_1^{-\alpha} + \delta\left((1-\alpha)z_2^{-\alpha} + \alpha z_2^{-\alpha-1}\right) &= (1-\alpha)(z_2^{-\alpha} - z_2^{1-\alpha}) + \delta\left((1-\alpha)z_2^{-\alpha} + \alpha z_2^{-\alpha-1}\right) \\ &= z_2^{-\alpha}((1-\alpha)(1-z_2) + \delta\left(\frac{a}{z_2} - (\alpha - 1)\right) > 0\end{aligned}$$

for large enough $\delta$. Similarly, if $\frac{\partial g}{\partial z_1}(z) < 0$, one can show that $(1, \delta)$ is again a feasible direction along which the objective decreases, for

$$\frac{(\alpha-1)(1-z_2)z_2}{\alpha - (\alpha-1)z_2} < \delta < \frac{z_2}{z_1},$$

if one can select such $\delta$. Otherwise, one can show that $(-1, -\delta)$ is a feasible direction along which the objective decreases, for

$$\frac{z_2}{z_1} < \delta < \frac{(\alpha-1)(1-z_2)z_2}{\alpha - (\alpha-1)z_2}.$$

We have thus established that if $z$ is not a stationary point, then there also exists an optimal solution for which $z_1 = 1$. We next show that the same holds true if $z$ is a stationary point.

Suppose that $z$ is a stationary point, *i.e.*, $\nabla g(z_1, z_2) = 0$. Then, we have

$$(n-1)z_1^{1-\alpha} + \alpha z_2^{1-\alpha} - (\alpha-1)z_1 z_2^{-\alpha} = 0.$$

Using the above, the objective evaluates to

$$g(z_1, z_2) = \frac{\alpha}{\alpha - 1}\frac{z_2}{z_1}.$$

Moreover, if $z_1 = \lambda z_2$ for some $\lambda \geq 1$, the stationarity condition yields

$$(n-1)\lambda^{1-\alpha} - (\alpha-1)\lambda + \alpha = 0,$$

an equation that has a unique solution in $[1, \infty)$. Let $\bar{\lambda}$ be the solution. Then, the problem (33) constrained on the stationary points of its objective can be expressed as

$$\begin{aligned} \text{minimize} \quad & \tfrac{\alpha}{\alpha-1}\tfrac{z_2}{z_1} \\ \text{subject to} \quad & z_1 = \bar{\lambda} z_2, \quad z_1 \le 1 \\ & z_2^{-\alpha} \le (n-1) z_1^{1-\alpha} + z_2^{1-\alpha}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \text{minimize} \quad & \tfrac{\alpha}{\alpha-1}\tfrac{1}{\bar{\lambda}} \\ \text{subject to} \quad & z_1 = \bar{\lambda} z_2 \\ & \tfrac{1}{(\alpha-1)(\bar{\lambda}-1)} \le z_2 \le \tfrac{1}{\bar{\lambda}}. \end{aligned}$$

In case the above problem is feasible, we pick $z_2 = \frac{1}{\bar{\lambda}}$, and $z_1 = 1$ and the proof is complete. ■

**Proposition 5.** *Consider a resource allocation problem with $n$ players, $n \ge 2$. Let the utility set, denoted by $U \subset \mathbf{R}^n$, be compact and convex. If the players have equal maximum achievable utilities (greater than zero),*

$$\operatorname{POF}(U;1) \le 1 - \frac{2\sqrt{n}-1}{n}. \quad \textit{(price of proportional fairness)}$$

*Let $\{\alpha_k \in \mathbf{R} \mid k \in \mathbf{N}\}$ be a sequence such that $\alpha_k \to \infty$ and $\alpha_k \ge 1$, $\forall k$. Then,*

$$\limsup_{k\to\infty} \operatorname{POF}(U;\alpha_k) \le 1 - \frac{4n}{(n+1)^2}. \quad \textit{(price of max-min fairness)}$$

**Proof.** Let $f$ be defined as in Proposition 3. Using Theorem 1 for $\alpha = 1$ we get

$$\begin{aligned} \operatorname{POF}(U;1) &\le 1 - \min_{x\in[1,n]} f(x;1,n) \\ &= 1 - \min_{x\in[1,n]} \frac{x^2+n-x}{nx} \\ &= 1 - \frac{2\sqrt{n}-1}{n}. \end{aligned}$$

Similarly, for any $k \in \mathbf{N}$ and $\alpha = \alpha_k$

$$\operatorname{POF}(U;\alpha_k) \le 1 - \min_{x\in[1,n]} f(x;\alpha_k,n),$$

which implies that

$$\begin{aligned} \limsup_{k\to\infty} \operatorname{POF}(U;\alpha_k) &\le \limsup_{k\to\infty} \left(1 - \min_{x\in[1,n]} f(x;\alpha_k,n)\right) \\ &\le 1 - \liminf_{k\to\infty} \min_{x\in[1,n]} f(x;\alpha_k,n). \end{aligned} \tag{34}$$

Consider the set of (real-valued) functions $\{f(\,.\,;\alpha_k,n)\,|\,k\in\mathbf{N}\}$ defined over the compact set $[1,n]$. We show that the set is equicontinuous, and that the closure of the set $\{f(x;\alpha_k,n)\,|\,k\in\mathbf{N}\}$ is bounded for any $x\in[1,n]$. Boundedness follows since $0\le f(x;\alpha,n)\le 1$ for any $\alpha>0$ and $x\in[1,n]$. The set of functions $\{f(\,.\,;\alpha_k,n)\,|\,k\in\mathbf{N}\}$ shares the same Lipschitz constant, as for any $k\in\mathbf{N}$, $\alpha_k\ge 1$ and $x\in[1,n]$ we have

$$\begin{aligned}|f'(x;\alpha_k,n)| &= \left|\frac{\left(1-\frac{1}{\alpha_k}\right)x^{2+\frac{1}{\alpha_k}}+\frac{n+1}{\alpha_k}x^{1+\frac{1}{\alpha_k}}-n\left(1+\frac{1}{\alpha_k}\right)x^{\frac{1}{\alpha_k}}-(x-n)^2}{\left(x^{1+\frac{1}{\alpha_k}}+(n-x)x\right)^2}\right|\\
&\le\left|\left(1-\frac{1}{\alpha_k}\right)x^{2+\frac{1}{\alpha_k}}+\frac{n+1}{\alpha_k}x^{1+\frac{1}{\alpha_k}}-n\left(1+\frac{1}{\alpha_k}\right)x^{\frac{1}{\alpha_k}}-(x-n)^2\right|\\
&\le\left(1-\frac{1}{\alpha_k}\right)x^{2+\frac{1}{\alpha_k}}+\frac{n+1}{\alpha_k}x^{1+\frac{1}{\alpha_k}}+n\left(1+\frac{1}{\alpha_k}\right)x^{\frac{1}{\alpha_k}}+(x-n)^2\\
&\le n^3+(n+1)n^2+2n^2+n^2=2(n^3+2n^2).\end{aligned}$$

As a result, the set of functions $\{f(\,.\,;\alpha_k,n)\,|\,k\in\mathbf{N}\}$ is equicontinuous.

Using the above result,

$$\lim_{k\to\infty}\min_{x\in[1,n]}f(x;\alpha_k,n)=\min_{x\in[1,n]}\lim_{k\to\infty}f(x;\alpha_k,n).$$

Thus, (34) yields

$$\begin{aligned}\limsup_{k\to\infty}\mathrm{POF}\,(U;\alpha_k) &\le 1-\liminf_{k\to\infty}\min_{x\in[1,n]}f(x;\alpha_k,n)\\
&=1-\min_{x\in[1,n]}\lim_{k\to\infty}f(x;\alpha_k,n)\\
&=1-\min_{x\in[1,n]}\lim_{k\to\infty}\frac{x^{1+\frac{1}{\alpha_k}}+n-x}{x^{1+\frac{1}{\alpha_k}}+(n-x)x}\\
&=1-\min_{x\in[1,n]}\frac{n}{x+(n-x)x}\\
&=1-\frac{4n}{(n+1)^2}.\end{aligned}$$

■

## D. A Model for Air Traffic Flow Management

The following is a model for air traffic flow management due to Bertsimas and Stock-Patterson (1998). Consider a set of flights, $\mathscr{F}=\{1,\ldots,F\}$, that are operated by airlines over a (discretized) time period in a network of airports, utilizing a capacitated airspace that is divided into sectors. Let $\mathscr{F}_a\subset\mathscr{F}$ be the set of flights operated by airline $a\in\mathscr{A}$, where $\mathscr{A}=\{1,\ldots,A\}$ is the set of airlines. Similarly, $\mathscr{T}=\{1,\ldots,T\}$ is the set of time steps, $\mathscr{K}=\{1,\ldots,K\}$ the set of airports, and $\mathscr{J} = \{1, \ldots, J\}$ the set of sectors. Flights that are continued are included in a set of pairs, $\mathscr{C} = \{(f', f) : f' \text{ is continued by flight } f\}$. The model input data, the main decision variables, and a description of the feasibility set are described below:

**Data.**

$$
\begin{aligned}
N_f &= \text{number of sectors in flight } f\text{'s path,}\\
P(f,i) &= \begin{cases} \text{the departure airport, if } i = 1,\\ \text{the } (i-1)\text{th sector in flight } f\text{'s path, if } 1 < i < N_f,\\ \text{the arrival airport, if } i = N_f, \end{cases}\\
P_f &= (P(f,i) : 1 \le i \le N_f),\\
D_k(t) &= \text{departure capacity of airport } k \text{ at time } t,\\
A_k(t) &= \text{arrival capacity of airport } k \text{ at time } t,\\
S_j(t) &= \text{capacity of sector } j \text{ at time } t,\\
d_f &= \text{scheduled departure time of flight } f,\\
r_f &= \text{scheduled arrival time of flight } f,\\
s_f &= \text{turnaround time of an airplane after flight } f,\\
l_{fj} &= \text{number of time steps that flight } f \text{ must spend in sector } j,\\
T_f^j &= \text{set of feasible time steps for flight } f \text{ to arrive to sector } j = \{\underline{T}_f^j, \ldots, \overline{T}_f^j\},\\
\underline{T}_f^j &= \text{first time step in the set } T_f^j \text{, and}\\
\overline{T}_f^j &= \text{last time step in the set } T_f^j.
\end{aligned}
$$

**Decision Variables.**

$$
w_{ft}^j = \begin{cases} 1, & \text{if flight } f \text{ arrives at sector } j \text{ by time step } t,\\ 0, & \text{otherwise.} \end{cases}
$$

**Feasibility Set.** The variable $w$ is feasible if it satisfies the constraints:

$$
\begin{aligned}
&\textstyle\sum_{f:P(f,1)=k}(w_{ft}^k - w_{f,t-1}^k) \le D_k(t) \;\; \forall k \in \mathscr{K}, t \in \mathscr{T},\\
&\textstyle\sum_{f:P(f,N_f)=k}(w_{ft}^k - w_{f,t-1}^k) \le A_k(t) \;\; \forall k \in \mathscr{K}, t \in \mathscr{T},\\
&\textstyle\sum_{f:P(f,i)=j,P(f,i+1)=j',i<N_f}(w_{ft}^j - w_{ft}^{j'}) \le S_j(t) \;\; \forall j \in \mathscr{J}, t \in \mathscr{T},\\
&w_{f,t+l_{fj}}^{j'} - w_{ft}^j \le 0 \;\; \forall f \in \mathscr{F}, t \in T_f^j, j = P(f,i), j' = P(f,i+1), i < N_f,\\
&w_{ft}^k - w_{f,t-s_f}^k \le 0 \;\; \forall (f',f) \in \mathscr{C}, t \in T_f^k, k = P(f,i) = P(f',N_f),\\
&w_{ft}^j - w_{f,t-1}^j \ge 0 \;\; \forall f \in \mathscr{F}, j \in P_f, t \in T_f^j,\\
&w_{ft}^j \in \{0,1\} \;\; \forall f \in \mathscr{F}, j \in P_f, t \in T_f^j.
\end{aligned}
$$

The constraints correspond to capacity constraints for airports and sectors, connectivity between sectors and airports, and connectivity in time (for more details, see Bertsimas and Stock-Patterson (1998)).