Online Appendix

Dynamic Pricing under Debt: Spiraling Distortions and Efficiency Losses

A Model with Nonzero Limited Liability

Our base model assumed *zero* liability under debt, so that the DM's objective was given by $\mathbb{E}\big[(\mathscr{R}(\mathbf{p}) - B)^+\big]$. In practice, failing to repay the debt often carries negative consequences for a borrower due to, e.g., recourse by the lender (Wells Fargo, 2016), or bankruptcy/reputation costs when the firm goes into default. To capture this *limited* (but nonzero) liability, we now assume that a DM faced with a debt payment B maximizes the following objective function:

$$\mathbb{E}\big[(\mathscr{R}(\mathbf{p}) - B)^+\big] - k\,\mathbb{E}\big[(B - \mathscr{R}(\mathbf{p}))^+\big], \quad \text{for some } k \in [0,1].$$

The parameter k controls the severity of the penalty when the DM fails to repay the debt. A choice of $k = 0$ recovers our familiar model, and $k = 1$ corresponds to maximizing the total revenues $\mathscr{R}(\mathbf{p})$.

In this context, the recursion characterizing the DM's problem is given by:

$$\tilde{V}_t(b,y) = \max_{p\in\mathcal{P}} \Big\{\lambda(p)\tilde{V}_{t+1}(b-p, y-1) + (1-\lambda(p))\tilde{V}_{t+1}(b,y)\Big\}, \qquad y \geq 1,\ t = 1,...,T \tag{A-1a}$$

$$\tilde{V}_t(b,0) = \tilde{V}_{T+1}(b,y) = (-b)^+ - kb^+, \qquad t = 1,...,T+1,\ y \geq 0. \tag{A-1b}$$

Let $\tilde{p}_t(b,y)$ denote the DM's price in period t. Following similar arguments to our analysis in §3, it can be readily shown that the DM always charges the revenue-maximizing price once the debt is covered, i.e., $\tilde{p}_t(b,y) = p^*(y), \forall\, b \leq 0$. Furthermore, the following proposition provides certain structural properties for the DM's value function.

Lemma A.1. *We have that*

i.) $\tilde{V}_t(b,y)$ *is convex, decreasing in the outstanding debt* b *and decreasing in* t*.*

ii.) $b + \tilde{V}_t(b,y)$ *is positive and increasing in* b*, and* $b + \frac{\tilde{V}_t(b,y)}{k}$ *is positive and decreasing in* b*.*

iii.) $\tilde{V}_t(b,y)$ *is decreasing in* k*.*

Proof of Lemma A.1. Note that $\tilde{V}_{T+1}(b,y) = \max(-b,-kb)$ readily satisfies both properties. Assume by induction that these are also satisfied at time $t+1$. We have that:

$$\tilde{V}_t(b,y) = \max_{p\in\mathcal{P}} \Big\{\lambda(p)\tilde{V}_{t+1}(b-p, y-1) + (1-\lambda(p))\tilde{V}_{t+1}(b,y)\Big\}, \qquad y \geq 1.$$

To show (i), note that since the maximand is convex and decreasing in b for any p, $\tilde{V}_t$ will also retain these properties. Also, $\tilde{V}_t(b,0) \geq \tilde{V}_{t+1}(b,0)$ readily holds for $y = 0$; for any $y \geq 1$, we have $\tilde{V}_{t+1}(b,y) \leq \tilde{V}_{t+1}(b-p,y-1)$ holding for at least some $p \in \mathcal{P}$, so that $\tilde{V}_t(b,y) \geq \tilde{V}_{t+1}(b,y)$ follows. Assuming that $\frac{\partial \tilde{V}_t}{\partial b}$ and $\frac{\partial \tilde{V}_{t+1}}{\partial b}$ are well defined, and that $-1 \leq \frac{\partial \tilde{V}_{t+1}}{\partial b} \leq -k$ holds by induction, we then readily obtain that $-1 \leq \frac{\partial \tilde{V}_t}{\partial b} \leq -k$ by a direct application of the Envelope theorem. Part (iii) readily follows by induction. □

As with our base model, the DM's value function decreases with the outstanding debt b, and additional units of debt reduce his payoff by a diminishing amount. Reflective of the non-zero (but limited) liability, the payoff always decreases at rates faster than k, and can now become negative.

As expected, this new model retains the analytical complexity of our earlier model. To characterize the DM's pricing policy and the impact of the nonzero liability, we thus restrict attention to the special case of asset selling discussed in §5. Furthermore, since $Y = 1$, we limit attention to the interesting case when $b < \bar{p}$, so that the DM has a fractional probability of covering the debt. The following result characterizes the DM's policy in this setting.

Lemma A.2. *Under Assumption 5.1 and for all $b > 0$,*

i.) *there exist thresholds $0 \leq \tilde{B}_T \leq \tilde{B}_{T-1} \leq \cdots \leq \tilde{B}_1$ such that the DM's pricing policy and value function are respectively given by:*

$$\tilde{p}_t(b) = \begin{cases} \pi\big(b + \tilde{V}_{t+1}(b)\big), & \text{if } 0 \leq b \leq \tilde{B}_t \\ p_t^*, & \text{if } \tilde{B}_t < b < \bar{p} \end{cases} \qquad \tilde{V}_t(b) = \begin{cases} \tilde{V}_{t+1}(b) + h\big(b + \tilde{V}_{t+1}(b)\big), & \text{if } 0 \leq b \leq \tilde{B}_t \\ -kb + k\tilde{V}_t(0), & \text{if } \tilde{B}_t < b < \bar{p}, \end{cases}$$

where $\pi(x) := \arg\max_{p \in \mathcal{P}} \lambda(p)(p - x)$ and $h(x) := \max_{p \in \mathcal{P}} \lambda(p)(p - x)$.

ii.) *at low debt values (i.e., $b \leq \tilde{B}_t$), the DM's price $\tilde{p}_t(b)$ is higher than the revenue-maximizing price, and is increasing in the debt b and decreasing with the penalty k.*

iii.) *at large debt values (i.e., $b > \tilde{B}_t$), the DM's price $\tilde{p}_t(b)$ exactly corresponds to the revenue-maximizing price, and is thus unaffected by the debt b or the penalty k.*

Proof of Lemma A.2. The DM's problem in period $t \in \{1, \ldots, T\}$ is given by:

$$\begin{aligned} \tilde{V}_t(b) &= \tilde{V}_{t+1}(b) + \max_{p \in \mathcal{P}} \lambda(p)\big[\tilde{V}_{T+1}(b-p) - \tilde{V}_{t+1}(b)\big] \qquad \big(\text{since } \tilde{V}_{T+1}(b) = \max(-b, -kb)\big) \\ &= \tilde{V}_{t+1}(b) + \max\Big\{\max_{p \geq b} f_t^{\ell}(p,b),\ \max_{p \leq b} f_t^{m}(p,b)\Big\}, \end{aligned}$$

$$\text{where} \qquad f_t^{\ell}(p,b) := \lambda(p)\big[p - b - \tilde{V}_{t+1}(b)\big] \qquad f_t^{m}(p,b) := \lambda(p)\big[k(p-b) - \tilde{V}_{t+1}(b)\big].$$

We analyze each of the problems above separately. First, recall from Proposition 5.1 that for $x \geq 0$, we have $0 \leq \pi'(x) \leq 1$. Therefore, since $b + \tilde{V}_{t+1}(b) \geq 0$ and $b + \tilde{V}_{t+1}(b)/k \geq 0$ by Lemma A.1, we have

$$q_t^{\ell}(b) := \arg\max_{p \in \mathcal{P}} f_t^{\ell}(p, b) = \pi\big(b + \tilde{V}_{t+1}(b)\big), \qquad q_t^{m}(b) := \arg\max_{p \in \mathcal{P}} f_t^{m}(p, b) = \pi\left(b + \frac{\tilde{V}_{t+1}(b)}{k}\right).$$

By Lemma A.1 and since $\pi'(x) \geq 0$, we have that $q_t^{\ell}(b)$ is increasing and $q_t^{m}(\ell)$ is decreasing in b. To determine the DM's optimal price $\tilde{p}_t(b)$, we distinguish two cases, depending on the sign of $\tilde{V}_{t+1}$.

- If $\tilde{V}_{t+1}(b) \geq 0$, then $b \leq q_t^{\ell}(b) \leq q_t^{m}(b)$, so that $\tilde{p}_t(b) = q_t^{\ell}(b)$.
- If $\tilde{V}_{t+1}(b) < 0$, then $q_t^{m}(b) < q_t^{\ell}(b)$. We claim that the optimal policy involves a threshold, such that $q_t^{\ell}(b)$ is charged for b below the threshold, and $q_t^{m}(b)$ is charged for b above the threshold. First, note that $\tilde{p}_t(b) = q_t^{\ell}(b)$ for $b \leq q_t^{m}(b)$, and $\tilde{p}_t(b) = q_t^{m}(b)$ for $b > q_t^{\ell}(b)$. Let us define

$$g_t(b) := f_t^{m}\big(q_t^{m}(b), b\big) - f_t^{\ell}\big(q_t^{\ell}(b), b\big) \equiv k\, h\big(b + \tilde{V}_{t+1}(b)/k\big) - h\big(b + \tilde{V}_{t+1}(b)\big),$$

where $h(x) := \max_{p \in \mathcal{P}} \lambda(p)(p - x)$. To show that the policy is a threshold one, we first argue that g is monotonic increasing. This follows since $h(x)$ is decreasing, and by Lemma A.1, $b + \tilde{V}_{t+1}(b)/k$ is decreasing and $b + \tilde{V}_{t+1}(b)$ is increasing in b. Furthermore, $g_t(0) < 0$ since $\tilde{V}_{t+1}(0) \geq 0$, and $g_t(b) > 0$ if $b > q_t^{\ell}(b)$ (which holds at large b, since $\pi' \leq 1$). Thus, there exists a threshold $\tilde{B}_t \geq 0$ given by:

$$g_t(\tilde{B}_t) = 0$$

price above such that the DM's pricing policy is exactly given by

$$\tilde{p}_t(b) = \begin{cases} q_t^{\ell}(b), & \text{if } b \leq \tilde{B}_t \\ q_t^{m}(b), & \text{if } b > \tilde{B}_t. \end{cases}$$

We also claim that $\tilde{B}_t \geq \tilde{B}_{t+1}, \forall\, t \in \{1, \ldots, T-1\}$. To see this, note that $g_t(b) \leq g_{t+1}(b), \forall\, b \geq 0$, since $\tilde{V}_{t+1}(b) \geq \tilde{V}_{t+2}(b)$, by Lemma A.1. Thus, since g_t and g_{t+1} are increasing, $\tilde{B}_t \geq \tilde{B}_{t+1}$.

To complete the proof, note that the expression for $\tilde{p}_t(b)$ and $\tilde{V}_t(b)$ for the case $b \leq \tilde{B}_t$ follows from the arguments above. For $b > \tilde{B}_t$, we prove by induction that for any $t \in \{1, \ldots, T\}$,

$$\tilde{V}_t(b) = -kb + k\tilde{V}_t(0), \forall\, b > \tilde{B}_t.$$

First, note that this trivially holds at $t = T + 1$ with $\tilde{B}_{T+1} := 0$, since $\tilde{V}_{T+1}(b) = -kb$, $\forall\, b \geq 0$. Assume by induction that the property also holds at time $t+1$. We have:

$$\forall\, b > \tilde{B}_t \quad : \quad q_t^m(b) = \pi\big(b + \tilde{V}_{t+1}(b)/k\big) = (\text{since } \tilde{B}_t \geq \tilde{B}_{t+1}) = \pi\Big(b + \frac{-kb + k\tilde{V}_{t+1}(0)}{k}\Big) = \pi\big(\tilde{V}_{t+1}(0)\big) = p_t^*.$$

Replacing this in the expression for $\tilde{V}_t(b)$, we obtain

$$\tilde{V}_t(b) = -kb + k\tilde{V}_{t+1}(0) + kh\big(\tilde{V}_{t+1}(0)\big) = -kb + k\tilde{V}_t(0),$$

which completes the induction and the proof of parts (i) and (iii).

To prove (ii), it can be readily seen that $\tilde{p}_t(b) = q^\ell(b)$ is increasing in b for $b \leq \tilde{B}_t$. Since $\tilde{V}_t(b)$ is decreasing with k by Lemma A.1, so is □

The result suggests that the presence of a non-zero liability carries certain nontrivial implications on the DM's pricing policy, depending on the required debt repayment b. Two regimes emerge. When the debt is not too large, the DM charges prices that increase with the debt, and exceed the revenue-maximizing price. Qualitatively, this exactly corresponds to the main distortion documented in our base model, whereby the DM shifts risk by charging high(er) prices. However, prices now decrease with the magnitude of k, which confirms the intuition that transferring more liability to the DM successfully reduces his risk shifting incentives.

Interestingly, when debt is sufficiently high, the DM's price exactly equals the revenue-maximizing price. This occurs when the debt exceeds a certain time-dependent threshold $\tilde{B}_t$, and the switch is sudden: the DM's price exhibits a downward jump, from a value that exceeds $\tilde{B}_t$ to $p_t^\star < \tilde{B}_t$. Once the switch occurs, the DM then continues to follow the revenue-maximizing policy for the remaining planning horizon (since $\tilde{B}_t \geq \tilde{B}_\tau,\ \forall\, \tau \geq t$). Qualitatively, in this regime the DM effectively acts *as if* he were unable to repay the debt, and therefore relies on a strategy that seeks to minimize his losses, or equivalently maximize revenues. (In fact, since $b > p_t^\star$ holds here, following the revenue-maximizing policy actually yields a "self-fulfilling prophecy," *guaranteeing* bankruptcy.) This regime is new, and is entirely caused by the non-zero liability, which acts as a severe threat for the DM. We note that the debt required to generate this regime is very high: $\tilde{B}_t > \tilde{V}_t(0)$, so that the debt repayment would exceed the expected revenues that could be generated over the remaining horizon.[15]

Regarding the time-dynamics of the pricing policy, similar arguments to those in Proposition 5.2 can be used to confirm that when the DM still relies on the risk-shifting strategy, he would reduce prices over time, but the markdowns would always be lower than the revenue-maximizing ones, and would decrease with the debt. Thus, the pricing distortions and efficiency losses would again compound

[15]Since loans might not be issued under such unfortunate circumstances, it would be interesting to study whether this regime survives in equilibrium, once the debt repayment is endogenized. We leave this interesting analysis for future research.

over time, leading to a spiraling behavior.

To summarize our findings, the presence of the non-zero (limited) liability successfully reduces some of risk-shifting incentives driving the DM's decisions. However, unless the entire liability is transferred to the DM (i.e., $k = 1$), the pricing distortions and the associated efficiency losses persist, albeit with a diminished magnitude.

B Numerical Experiments on Multi-unit Case

In our experiments, we considered several demand functions that all yielded consistent findings—including ones that do not satisfy the requirements in Assumption 5.1. Below, we present the case of a logit demand function $\lambda(p) = e^{1-p}/(1 + e^{1-p})$.

Pricing policy. In Figure B-1, we depict the DM's price $p^{\dagger}_{T-4}$ and the revenue-maximizing price $p^{\star}_{T-4}$ as functions of the outstanding debt, for two starting inventory levels, $y = 5$ and $y = 3$. Consistent

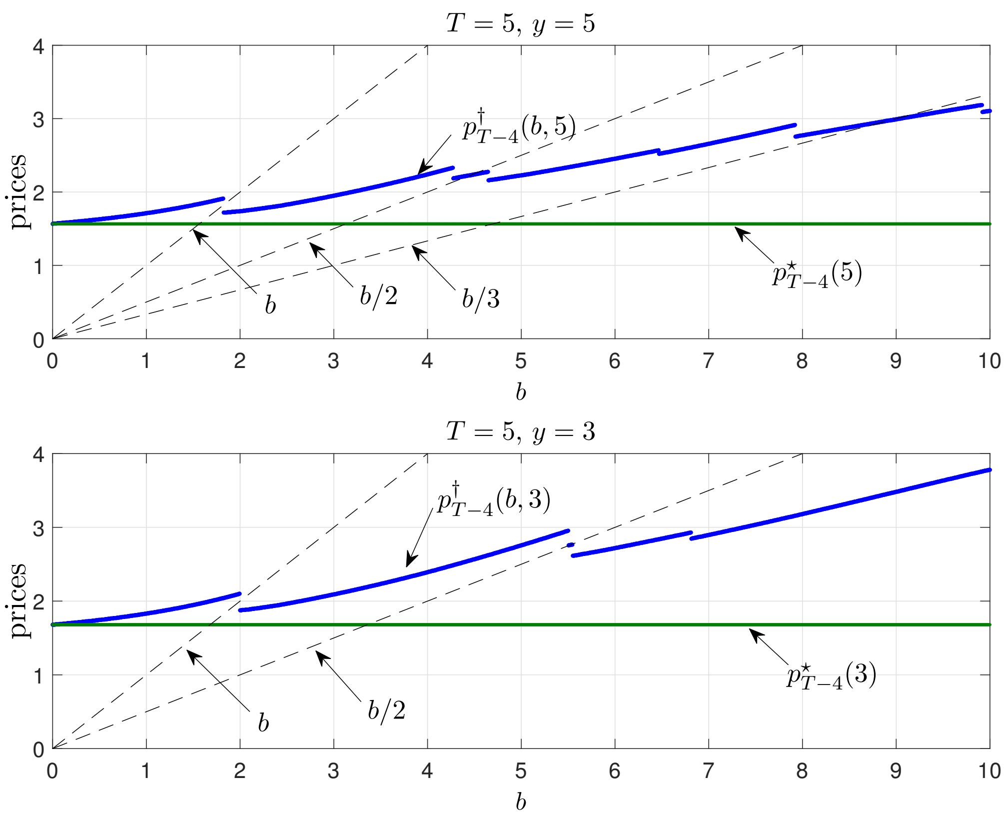

Figure B-1: **The DM's** pricing policy structure. The demand curve is $\lambda(p) = e^{1-p}/(1 + e^{1-p})$, and the time horizon is $T = 5$.

with our findings in §4, we observe that the DM's prices are piecewise increasing in the debt; but with many units in inventory and many periods, there are now multiple discontinuity points, as we alluded to in our earlier discussion.

In contrast with the two-period case depicted in Figure 1, it is no longer possible to exactly associate strategies that rely on selling k units to cover the debt with prices that lie between $b/(k-1)$ and b/k. The reason is that, with more than two periods to go, strategies become increasingly complex, as they also need to account for multiple *future* contingent strategies that possibly rely on more or less units, depending on sales realizations. This precisely illustrates how the underlying combinatorial structure dramatically increases the complexity of the DM's pricing policies in the general case, defying an analytical characterization.

In Figure B-2, we depict the time-evolution of prices on sample paths where no sale occurs, for different levels of debt. Similar to the analytical results in §4, the DM's policy always entails slower markdowns than the revenue-maximizing policy, and actually may prescribe *markups*. Furthermore, price distortions increase monotonically over time. However, due to the discontinuities in the DM's price as a function of debt, price distortions are not monotonic in the debt level.

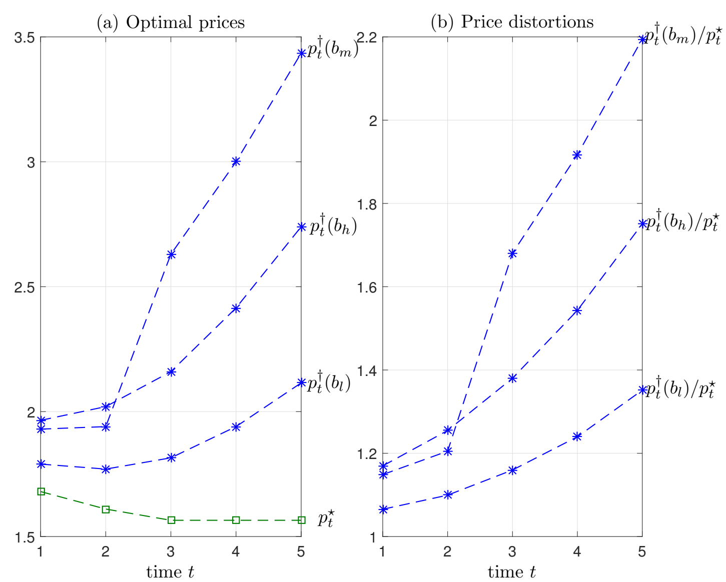

Figure B-2: (a) Price evolution for the revenue-maximizing policy and the DM's policy under $\lambda(p) = e^{1-p}/(1+e^{1-p})$, for $T = 5$ and $y = 3$, and for different levels of debt: 'low' $b_l = p_T^\star/2$, 'medium' $b_m = p_T^\star$, and 'high' $b_h = 1.5 \times p_T^\star$. (b) Evolution of price distortions over time.

Finally, Figure B-3 depicts all possible sample paths corresponding to the evolution of the DM's price over time, as well as the corresponding expected prices. We observe that when inventory is ample,

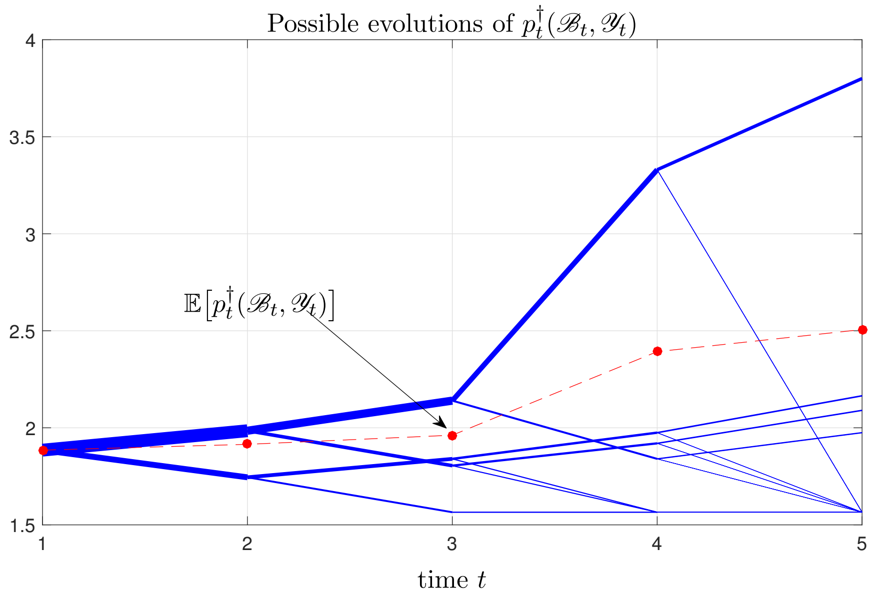

Figure B-3: Efficiency loss evolution under $\lambda(p) = e^{1-p}/(1+e^{1-p})$, for $T = 5$, $y = 5$ and $b = 1.75 \times p_T^\star$. Each edge in the tree depicts a possible evolution of $p_t^\dagger(\mathscr{B}_t, \mathscr{Y}_t)$. The width of the edge is proportional to the probability of the edge. The dots on the dashed line depict $\mathbb{E}\big[p_t^\dagger(\mathscr{B}_t, \mathscr{Y}_t)\big]$.

as in the case depicted, the expected price increases over time. Furthermore, the range of possible price values also significantly expands, particularly on the paths on which few sales to date occur, where the DM becomes more and more aggressive with pricing decisions as time progresses.

Efficiency losses. In Figure B-4, we explore the evolution of $\mathscr{L}_t$ and the various paths that may be taken. In particular, consistent with all our analytical results thus far, we observe that the expected efficiency loss L_t is increasing over time, and the "variability" of $\mathscr{L}_t$ also increases over time, with a wide range of possible outcomes in the last period $t = 5$.

C Endogenizing Debt

In this section, we formulate a model where debt is endogenously determined. We base our analysis on the setup in §4, but assume that the DM is no longer endowed with inventory; instead he can purchase an inventory bundle of $Y = 2$ units at a cost of c, before the start of the selling season. The DM has limited available equity $\bar{e}$ that he can use to pay for the purchase. To exclude uninteresting cases, we assume that the optimal revenues that could be generated from this purchase would exceed the costs, i.e., $J^\star \geq c$, and that the DM's available equity is insufficient to cover the purchase, i.e., $\bar{e} < c$.

If the DM decides to proceed with the purchase by investing $e \leq \bar{e}$ of his own equity, he may be able to obtain a loan for the remaining amount $c - e$ from a lender. We make the standard assumption that

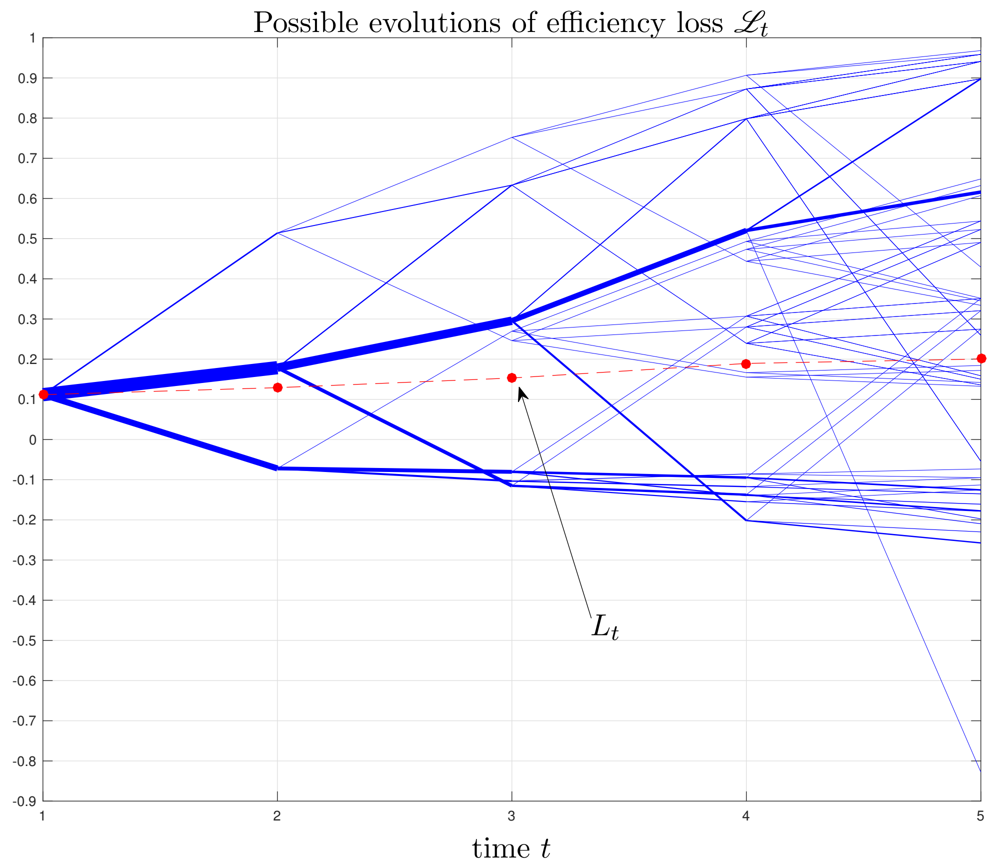

Figure B-4: Efficiency loss evolution under $\lambda(p) = e^{1-p}/(1+e^{1-p})$, for $T = 5$, $y = 5$, and $b = 1.75 \times p_T^\star$. Each edge in the tree depicts a possible evolution of $\mathscr{L}_t$. The width of the edge is proportional to the probability of the edge. The dots and dashed lines depicts L_t.

the debt market is perfectly competitive Tirole (2006, page 115). That is, when the borrowed amount is $c-e$, the lender would set the required repayment B (of principal plus interest) at the loan's maturity so as to break even. More precisely, using the terminology and notation of §2, the lender would anticipate that if the required repayment were set to B, the borrower would follow a pricing policy $p^\dagger(B,Y)$, which would yield an expected repayment (i.e., a debt value) of $D(B,Y) := \mathbb{E}\big[\min\{B, \mathscr{R}(\mathbf{p}^\dagger(B,Y))\}\big]$. The lender would then set B so that

$$D(B,Y) = c - e. \tag{C-2}$$

Note that this equation may not have a solution B for particular values of e, in which case the lender would be unwilling to extend a loan. Intuitively, this could occur if the expected revenues are low and the borrowed amount $c-e$ is high.

By not pursuing the inventory purchase, the DM would achieve a profit of zero. Alternatively, by injecting equity e, borrowing $c-e$, and facing a repayment of B, his profit would be given by $V_1(B,Y)-e$. His decision problem before the start of the selling season can then be formulated as:

$$\max\left\{0, \max_{\substack{0\le e\le \bar{e} \\ D(B,Y)=c-e}} V_1(B,Y)-e\right\} \tag{C-3}$$

Several outcomes are possible. If the inner optimization problem is infeasible (i.e., when (C-2) is infeasible for any e), we say that *lenders refuse to lend.* If the inner optimization is feasible, but has a negative optimal value, we say that *the DM finds the purchase unprofitable.* In both of these cases, the DM generates zero profit. Finally, if the optimal value in (C-3) is strictly positive, we say that *the inventory purchase goes through.*

C.1 One-Period Case

We first analyze the one-period case under a linear demand model, i.e., $T=1$ and $\lambda(p)=\alpha-\beta p$, for some $\alpha\in(0,1]$ and $\beta>0$. Using our analysis from the proof of Proposition 4.1 (with $b\equiv B$ to retain the familiar notation), it can readily be seen that the lender's expected collected payment is equal to

$$D(b)=\lambda(p_T^\dagger(b,1))b=\frac{1}{2}b\lambda(b).$$

The break-even equation (C-2) yields $\beta b^2-\alpha b+2(c-e)=0$. Consequently, lenders refuse to lend unless

$$e\ge c-\frac{\alpha^2}{8\beta},$$

where the right-hand side can be interpreted as the minimum equity level that lenders expect the DM to inject. When the DM injects more than this minimum level, lenders set the repayment amount to

$$b(e)=\frac{\alpha-\sqrt{\alpha^2-8(c-e)\beta}}{2\beta},$$

and the DM's profit can be expressed as

$$V_1(b(e),1)-e=\frac{\alpha\left(\alpha+\sqrt{\alpha^2-8(c-e)\beta}\right)-4(c+e)\beta}{8\beta}.$$

It can be readily checked that $V_1(b(e),1)-e$ is increasing in e. Thus, the DM injects all available equity $\bar{e}$. The inventory purchase is then profitable for the DM as long as $V_1(b(\bar{e}),1)-\bar{e}\ge 0$.

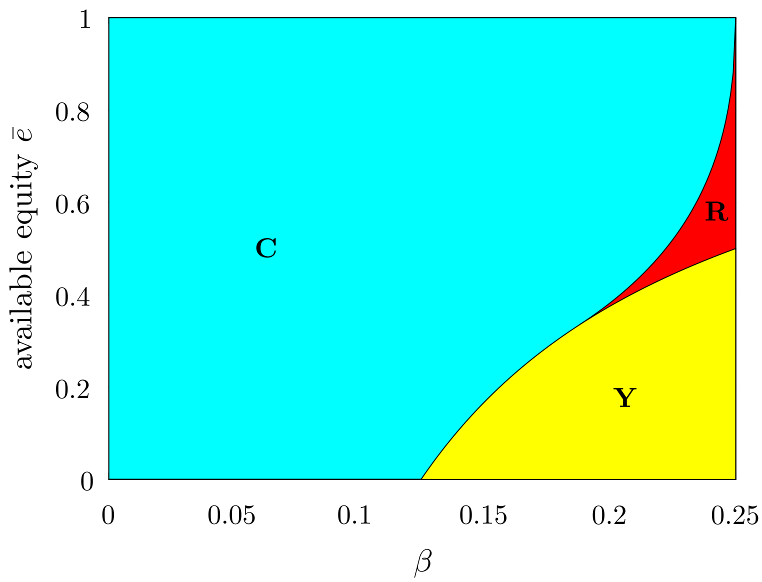

Figure C-5: Outcomes for $T = 1$, $c = 1$, $\alpha = 1$ and different values of β and $\bar{e}$. **C**yan means that the inventory purchase goes through; **Y**ellow means that lenders refuse to lend; **R**ed means that the DM finds the purchase unprofitable.

To summarize our findings for the case of $T = 1$: *the DM always injects $\bar{e}$ and achieves a profit of $V_1(b(\bar{e}), 1) - \bar{e}$, unless his available equity $\bar{e}$ is low. In particular, if $\bar{e} < c - \frac{\alpha^2}{8\beta}$, the lenders refuse to lend and if $V_1(b(\bar{e}), 1) - \bar{e} < 0$, the DM finds the purchase unprofitable.*

We illustrate the above cases graphically in Figure C-5, for fixed values of $c = 1$ and $\alpha = 1$, and all possible values of $\bar{e}$ and β, namely $0 \le \bar{e} < 1$ and $0 < \beta \le \frac{1}{4}$.[16] We observe that for small values of β (namely $\le \frac{1}{8}$), no minimum equity is required for the inventory purchase to go through. For intermediate values of β and above (namely $\ge \frac{1}{8}$), lenders always require some minimum equity to be injected in order for them to lend. For higher values of β (namely $\ge \frac{3}{16}$), even if the DM has the minimum equity that lenders require, he might still find the inventory purchase unprofitable. Intuitively, this is because as β increases, the revenues that can be extracted decrease. Thus, for intermediate β's and above, lenders refuse to lend high amounts. For high β's, even if they agree to lend, it is possible that they charge a prohibitively high interest that makes the inventory purchase unprofitable for the DM.

C.2 Two-Period Case

We now consider the two period case we analyzed in §4, i.e., $T = 2$ and $Y = 2$ and $\lambda(p) = \alpha - \beta p$, for some $\alpha \in (0, 1]$ and $\beta > 0$. In this case, the break-even equation (C-2) becomes a fourth-order

[16] The upper bound on β follows from $J^\star \ge c$.

polynomial equation in b, when the DM prices so as to cover the debt in one period. In case he prices q_m so as to cover the debt in two periods, it also involves square root terms as in the definition of q_m, see (E-6). Consequently, we were unable to obtain closed-form expressions and tackle the DM's problem (C-3) analytically. Instead, we performed a numerical study for fixed values of $c = 2$ and $\alpha = 1$, where we considered various possible values for $0 \leq \bar{e} < 2$ and $0 < \beta \leq \frac{1}{4}$. Figure C-6(a) depicts the outcomes. We distinguish the following cases, depending on the value of β:

- For small values of β's (namely ≤ 0.133), lenders set a low enough repayment amount b, so that the DM prices to cover it in one period, for any value of $\bar{e}$.
- For intermediate values of β's (namely $0.133 \leq \beta \leq 0.211$), if $\bar{e}$ is too low, lenders might refuse to lend (yellow region). If $\bar{e}$ is slightly higher, then lenders set a high enough repayment amount b, and the DM prices to cover it in two periods (magenta region). If $\bar{e}$ is even higher, then lenders set a lower repayment amount b, and the DM prices to cover it in one period (cyan region).
- For high values of β's (namely ≥ 0.211), if $\bar{e}$ is too low, lenders refuse to lend (yellow region). If $\bar{e}$ is slightly higher, then lenders set a high enough repayment amount b, and the DM finds the purchase unprofitable (red region). If $\bar{e}$ is even higher, then lenders set a lower repayment amount b, and the DM prices to cover it in one period (cyan region).

These results are in line with our findings in the one period case above, and bear a similar interpretation. Importantly, however, they demonstrate that *both pricing strategies of covering the debt in one or two periods (discussed in §4) could arise.* Put differently, the discontinuity in the DM's pricing strategy we elicited in §4 (see Figure 1) could arise. This is further illustrated in Figure C-7 where we plot the optimal price $p^{\star}_{T-1}(2)$ (green) and the DM's price $p^{\dagger}_{T-1}(b, 2)$ (blue) for fixed $\bar{e} = 0.4$, as we vary β. We observe that for high enough values of β, the repayment amount increases to the extent that the DM switches his pricing strategy, resulting in a discontinuity point. The dashed line corresponds to the repayment amount b, and helps to highlight the strategy switch.

An important feature that arises in the two period case is that the DM's profit $V_1(b, 2) - e$ is no longer monotonic in the equity injected e. In other words, the DM may find it profitable to only invest a *fraction* of his initial equity, which would lead to larger profits than choosing *whether* to invest the entire equity.

To appreciate this point, it is useful to compare Figure C-6(a) with Figure C-6(b). In Figure C-6(a), the DM can choose what amount of equity to inject, i.e., $e \in [0, \bar{e}]$. In Figure C-6(b), the DM is only allowed to choose *whether* to inject all his available equity, i.e., $e \in \{0, \bar{e}\}$. The critical difference between the two figures occurs at intermediate values of β, namely $0.15 \leq \beta \leq 0.211$. In this range, there are particular values of the initial equity $\bar{e}$ such that the inventory purchase goes through when

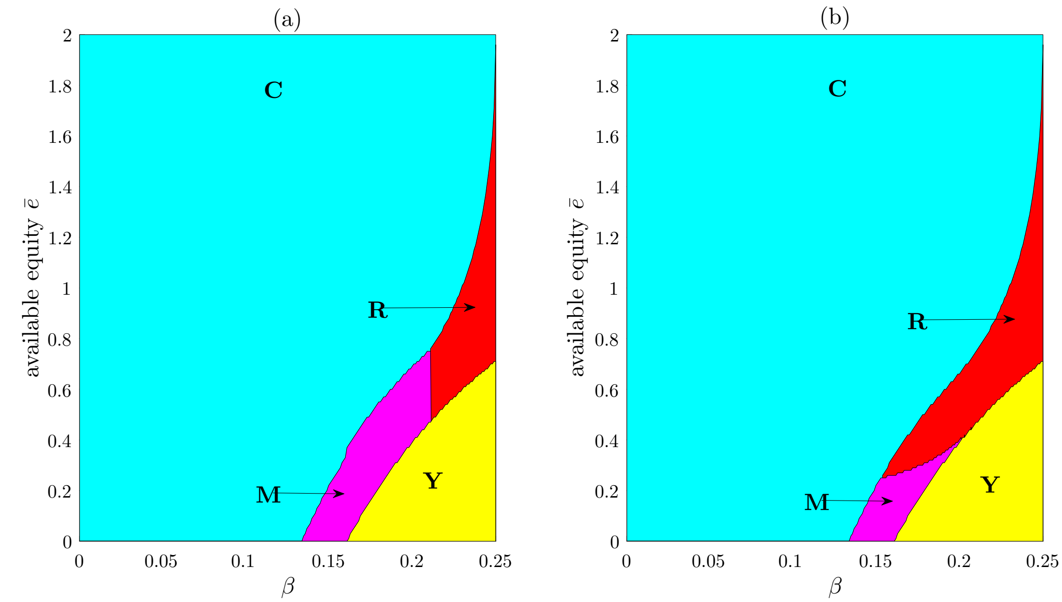

Figure C-6: Outcomes for $T = 2$, $Y = 2$, $c = 2$, $\alpha = 1$ and different values of β and available equity $\bar{e}$. In (a), the DM chooses how much equity to invest ($e \in [0, \bar{e}]$). In (b), the DM chooses whether to invest all equity or not ($e \in \{0, \bar{e}\}$). **C**yan (**M**agenta) means that the inventory purchase goes through, and the DM prices so as to cover the debt in one (two) periods; **Y**ellow means that lenders refuse to lend; **R**ed means that the DM finds the purchase unprofitable.

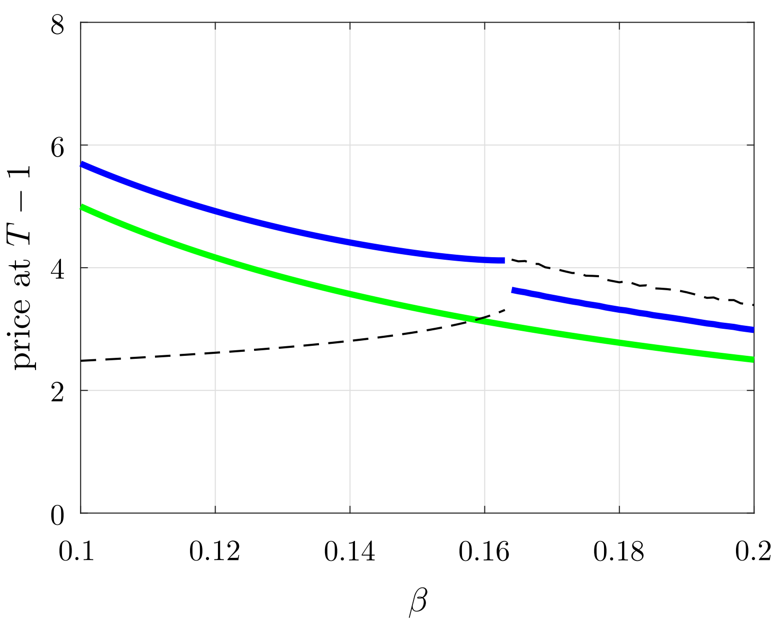

Figure C-7: Optimal price $p^{\star}_{T-1}(2)$ (green) and DM's price $p^{\dagger}_{T-1}(b, 2)$ (blue) as a function of β, for $T = 2$, $c = 1$, $\alpha = 1$, $\bar{e} = 0.4$. The dashed line depicts the associated repayment amount b.

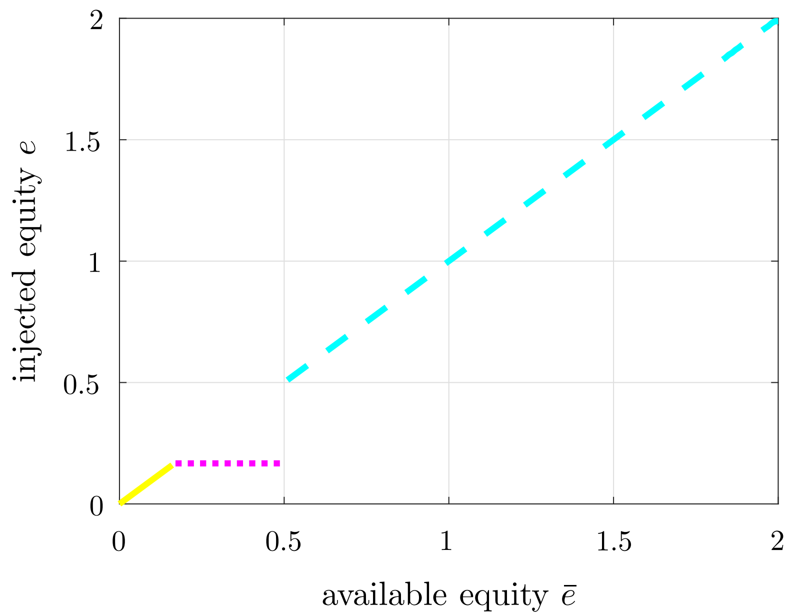

Figure C-8: Optimal injected equity e as a function of the available equity $\bar{e}$, for $T = 2$, $c = 1$, $\alpha = 1$, $\beta = 0.175$. Solid yellow means that lenders end up refusing to lend; Dotted magenta (dashed cyan) means that the inventory purchase goes through, and the DM prices so as to cover the debt in two (one) periods.

the DM can inject a fraction of the equity—corresponding to the magenta region in Figure C-6(a)—but the inventory purchase does *not* go through when the DM only chooses *whether* to inject the entire equity—corresponding to the red region in Figure C-6(b). To further illustrate this, Figure C-8 considers the case $\beta = 0.175$, and plots the optimal equity $e \in [0, \bar{e}]$ that the DM would invest as a function of the available equity $\bar{e}$. Note that three regions emerge:

- For $\bar{e} < \frac{1}{6}$, the DM injects all his equity, but lenders refuse to lend (solid yellow line).
- For $\bar{e} > \frac{1}{2}$, the DM injects all his available equity, $e = \bar{e}$, and the debt he raises induces him to price so as to cover the debt in one period (dashed cyan line).
- For $\frac{1}{6} \leq \bar{e} \leq \frac{1}{2}$, the DM chooses to inject only $e = \frac{1}{6}$, i.e., he does not use all available equity and $e < \bar{e}$. In this case, the amount of debt he raises induces him to price so as to cover the debt in two periods (dotted magenta line).

This highlights the same phenomenon as our earlier discussion: at intermediate values of equity—when $\frac{1}{6} \leq \bar{e} \leq \frac{1}{2}$—the DM would prefer not retain some of his equity, and only invest $e = \frac{1}{6}$. To understand this preference, note that investing a lower equity requires the DM to raise a high debt; in turn, this higher debt allows him to credibly pre-commit to a pricing strategy that clears the debt in two periods.

Such a strategy is less risky, by relying on lower prices, as we discussed in §4. Consequently, this allows the lenders to charge a lower interest, with the DM eventually benefiting.

This discussion illustrates the strategic role that debt could play in our setting. Thus, the appropriate selection of capital structure leads to a better alignment of incentives, so that the DM implements an ex-ante higher-value pricing policy. This strategic use of debt as a pre-commitment mechanism is akin to other papers in the finance literature, e.g., see Titman (1984). More broadly, a positive effect of debt on firm value has also been documented in several other papers in the corporate finance literature, see, e.g., Brander and Lewis (1986) and Chemla and Faure-Grimaud (2001).

D Proofs for Section 3

Proof of Lemma 3.1. *i.)* If $y = 0$, $V_t(b, y) = (-b)^+$ is clearly convex, decreasing in b. If $y \geq 1$, $V_{T+1}(b, y)$ is similarly convex, decreasing in b. Assuming that so is $V_{t+1}(b, y)$ for some $t = 1, \ldots, T$, then the recursion (2) yields that

$$V_t(b, y) = \max_{p \in \mathcal{P}} \left\{ \lambda(p) V_{t+1}(b - p, y - 1) + (1 - \lambda(p)) V_{t+1}(b, y) \right\}.$$

For any $p \in \mathcal{P}$, the maximand above is convex, decreasing in b because it is a convex combination of two such functions. Thus, $V_t(b, y)$ is also convex, decreasing in b.

To show that $V_t(b, y)$ is decreasing in t, note that it is immediate for $y = 0$, since $V_t(b, y) = (-b)^+$. For $y \geq 1$, by the recursion (2) it suffices to show that $V_t(b, y) \leq V_t(b - p, y - 1)$, for some $p \in \mathcal{P}$, for all $t = 1, \ldots, T + 1$. We shall show it for $p = \overline{p}$. At $T + 1$, we get $V_{T+1}(b, y) = (-b)^+ \leq (\overline{p} - b)^+ = V_{T+1}(b - \overline{p}, y - 1)$. Suppose that it is true at $t + 1$. Then, for $y \geq 2$

$$\begin{aligned} V_t(b - \overline{p}, y - 1) &= \max_{p \in \mathcal{P}} \left\{ \lambda(p) V_{t+1}(b - \overline{p} - p, y - 2) + (1 - \lambda(p)) V_{t+1}(b - \overline{p}, y - 1) \right\} \\ &\geq \max_{p \in \mathcal{P}} \left\{ \lambda(p) V_{t+1}(b - p, y - 1) + (1 - \lambda(p)) V_{t+1}(b, y) \right\} \\ &= V_t(b, y). \end{aligned}$$

A similar argument can be employed for $y = 1$.

ii.) For $b \leq 0$, it suffices to show that $V_t(b, y) = -b + V_t(0, y)$, $t = 1, \ldots, T + 1$. At $T + 1$, $V_{T+1}(b, y) =$

$(-b)^+ = -b = -b + V_{T+1}(0, y)$. Suppose now that it is true at some $t+1$. Then, (2) yields

$$\begin{aligned} V_t(b,y) &= \max_{p\in\mathcal{P}} \{\lambda(p)\,(p-b+V_{t+1}(0,y-1)) + (1-\lambda(p))\,(-b+V_{t+1}(0,y))\} \\ &= -b + \max_{p\in\mathcal{P}} \{\lambda(p)V_{t+1}(-p,y-1) + (1-\lambda(p))V_{t+1}(0,y)\} \\ &= -b + V_t(0,y). \end{aligned}$$

For $b > 0$, it suffices to show *iii.)* below.

iii.) At $T+1$, or for $y = 0$, $V_t(b,y) = \frac{\partial}{\partial b}V_t(b,y) = 0$ and the probability of covering the debt is 0.

For $y \geq 1$, at T the DM generates no revenue and fails to cover the debt with probability 1, unless he charges $p^\dagger(b,y) \geq b$. Consequently, if $\overline{p} < b$, we have that $p^\dagger(b,y) \leq \overline{p} < b$ and $V_T(b,y) = \frac{\partial}{\partial b}V_T(b,y) = 0$. Otherwise, if $\overline{p} \geq b$ the DM charges $p^\dagger(b,y) \in [b, \overline{p}]$ and covers the debt only if he makes a sale, i.e., with probability $\lambda(p_T^\dagger(b,y))$. Also, by the Envelope Theorem

$$-\frac{\partial}{\partial b}V_T(b,y) = -\frac{\partial}{\partial b}\left\{\lambda(p)(p-b)\right\}\Big|_{p=p_T^\dagger(b,y)} = \lambda(p_T^\dagger(b,y)).$$

Thus, V_T is differentiable with respect to b and $-\frac{\partial}{\partial b}V_T(b,y)$ is the probability of covering the debt. Assuming that V_{t+1} has the same properties, we can apply the Envelope Theorem to the recursion for t to obtain

$$\begin{aligned} -\frac{\partial}{\partial b}V_t(b,y) &= -\frac{\partial}{\partial b}\Big\{\lambda(p)V_{t+1}(b-p,y-1) + (1-\lambda(p))V_{t+1}(b,y)\Big\}\Big|_{p=p_t^\dagger(b,y)} \\ &= -\lambda(p_t^\dagger(b,y))\frac{\partial V_{t+1}}{\partial b}(b-p_t^\dagger(b,y),y-1) - (1-\lambda(p_t^\dagger(b,y)))\frac{\partial V_{t+1}}{\partial b}(b,y). \end{aligned}$$

By the law of total probability (applied depending on whether a sale occurred at t), it follows that $-\frac{\partial}{\partial b}V_t(b,y)$ is the probability of covering the debt at t. □

E Proofs for Section 4

Proof of Proposition 4.1. To facilitate exposition, we first attend to parts *ii.)* and *iii.)*.

ii.) Since there is only one period left $p_T^\dagger(b,y) = p_T^\dagger(b,1)$, which is shown to be increasing in b in the analysis of the one-unit case, Proposition 5.1.

iii.) The fact that $p_{T-1}^\dagger(b,1)$ is increasing in b follows again from Proposition 5.1.

Next, we analyze $p_{T-1}^\dagger(b,2)$, which involves the solution of the optimization problem in (2), for $t = T-1$ and $y = 2$. To this end, a characterization of $V_T(b,y)$ is required.

Consider first the case of $b \leq \frac{\alpha}{\beta}$. Clearly, $V_T(b,0) = 0$. For $y \geq 1$, $V_T(b,y) = V_T(b,1)$. Thus, by the analysis of the one-unit case, for $b \geq 0$ (F-22) yields that $p^{\dagger}_T(b,1) = \frac{1}{2}\left(\frac{\alpha}{\beta} + b\right)$. Substituting into (F-25), we get that $V_T(b,1) = \frac{\lambda^2(b)}{4\beta}$. For $b < 0$, as we argued in the proof of Lemma 3.1*ii.*), $V_T(b,1) = -b + V_T(0,1)$. Summarizing then,

$$V_T(b,y) = \begin{cases} \frac{\lambda^2(b)}{4\beta} & b \geq 0,\ y \geq 1 \\ -b + V_T(0,1) & b < 0,\ y \geq 1 \\ 0 & y = 0. \end{cases}$$

Substituting for $V_T(b,y)$ using the above equation, the maximand of (2) for $t = T-1$ and $y = 2$ becomes

$$V_T(b,1) + \begin{cases} f_\ell(p) := \lambda(p)\left(p - b + V_T(0,1) - V_T(b,1)\right) & p \in (b, \frac{\alpha}{\beta}] \\ f_m(p) := \lambda(p)\left(V_T(b-p,1) - V_T(b,1)\right) & p \in [0,b]. \end{cases}$$

We next analyze the problems $\max_{p\in(b,\frac{\alpha}{\beta}]} f_\ell(p)$ and $\max_{p\in[0,b]} f_m(p)$ separately. We show that they have unique optimal solutions, denoted by $p_\ell(b)$ and $p_m(b)$, and optimal values denoted by $F_\ell(b)$ and $F_m(b)$ respectively. If we let $\Delta F(b) := F_\ell(b) - F_m(b)$ we have

$$p^{\dagger}_{T-1}(b,2) = \begin{cases} p_\ell(b) & \text{if } \Delta F(b) \geq 0 \\ p_m(b) & \text{otherwise.} \end{cases} \tag{E-4}$$

- For $\max_{p\in(b,\frac{\alpha}{\beta}]} f_\ell(p)$, note that f_ℓ is concave, quadratic attaining its maximum at

$$q_\ell(b) := \frac{1}{2}\left(b + V_T(b,1) - V_T(0,1) + \frac{\alpha}{\beta}\right). \tag{E-5}$$

  The value $q_\ell(b)$, which is quadratic in b, is bigger than b if and only if $b \leq b_\ell := \frac{\alpha}{\beta} - \frac{\sqrt{\alpha^2+4}-2}{\beta}$. Thus,

$$p_\ell(b) = \begin{cases} q_\ell(b) & b \in [0, b_\ell] \\ b & b \in [b_\ell, \frac{\alpha}{\beta}]. \end{cases}$$

- For $\max_{p\in[0,b]} f_m(p)$, note that f_m is cubic. By solving the quadratic equation $(f_m)'(p) = 0$ we obtain the stationary points $\frac{2\beta b - \alpha \pm \sqrt{4\beta^2 b^2 - 10\alpha\beta b + 7\alpha^2}}{3\beta}$. It can be readily checked that for $b \in [0, \frac{\alpha}{\beta})$ the point

$$q_m(b) := \frac{2\beta b - \alpha + \sqrt{4\beta^2 b^2 - 10\alpha\beta b + 7\alpha^2}}{3\beta} \tag{E-6}$$

  is non-negative and a local maximizer, whereas the other point is non-positive and a local minimizer. Thus, f_m is increasing in $[0, q_m(b)]$ and decreasing in $[q_m(b), \infty)$. Since we are interested in $p \in [0,b]$, note that $q_m(b) < b \Leftrightarrow 3\beta^2 b^2 - 12\alpha\beta b + 6\alpha^2 < 0 \Leftrightarrow b > b_m := (2-\sqrt{2})\frac{\alpha}{\beta}$. Combining the last two observations,

$$p_m(b) = \begin{cases} b & b \in [0, b_m] \\ q_m(b) & b \in (b_m, \frac{\alpha}{\beta}]. \end{cases}$$

We use these results to simplify (E-4). In particular, we consider different values for b.

- For $0 \le b \le b_m < b_\ell$,[17] we have $\Delta F(b) = F_\ell(b) - F_m(b) = f_\ell(q_\ell(b)) - f_m(b) > f_\ell(b) - f_m(b) = 0$.
- For $b_m < b_\ell \le b \le \frac{\alpha}{\beta}$, we similarly get $\Delta F(b) < 0$.
- For $b_m < b < b_\ell$ we have that $\Delta F(b) = f_\ell(q_\ell(b)) - f_m(q_m(b))$. Using tedious algebra, one can show that $(\Delta F)'$ is increasing in b and negative at b_ℓ. Thus, ΔF is decreasing in b. Since $\Delta F(b_m) > 0$ and $\Delta F(b_\ell) < 0$, there exists a unique $\widehat{b} \in (b_m, b_\ell)$ such that $\Delta F(b) \ge (<)0$ for $b \le (>)\widehat{b}$.

By combining the above we obtain that

$$p^\dagger_{T-1}(b,2) = \begin{cases} q_\ell(b) & b \in [0, \widehat{b}] \\ q_m(b) & b \in (\widehat{b}, \frac{\alpha}{\beta}]. \end{cases}$$

For $b \in (\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}]$, we have that $V_T(b,y) = 0$ and thus

$$p^\dagger_{T-1}(b,2) \in \arg\max_{p \in [0, \frac{\alpha}{\beta}]} \left\{ \lambda(p) V_T(b-p, 1) \right\}.$$

The objective function above, denoted by f_h, evaluates to 0 for $b - p > \frac{\alpha}{\beta}$. Thus, we consider only prices $p \ge b - \frac{\alpha}{\beta}$. Then, f_h is cubic. Its stationary points are $\frac{-\lambda(b)}{\beta}$, which is a local minimizer, and

$$q_h(b) := \frac{b}{3} + \frac{\alpha}{3\beta},$$

which is a local maximizer. It can be readily checked that $q_h(b) \in [b - \frac{\alpha}{\beta}, \frac{\alpha}{\beta}]$ and as a result $p^\dagger_{T-1}(b,2) = q_h(b)$ for $b \in (\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}]$.

Having characterized the three pricing regimes for $p^\dagger_{T-1}(b,2)$, it suffices to show that q_ℓ, q_m, and q_h are all increasing. In particular, q_ℓ is increasing by Lemma 3.1*ii.*). To show that q_m is increasing note that

$$\frac{dq_m}{db}(b) = \frac{2}{3} + \frac{4\beta b - 5\alpha}{3\sqrt{4\beta^2 b^2 - 10\alpha\beta b + 7\alpha^2}}, \quad \frac{d^2 q_m}{db^2}(b) = \frac{\alpha^2 \beta}{(4\beta^2 b^2 - 10\alpha\beta b + 7\alpha^2)^{\frac{3}{2}}} > 0.$$

[17] It can be readily checked that $b_m < b_\ell$ for $\alpha \le 1$.

Thus, $\frac{dq_m}{db}(b) \geq \frac{dq_m}{db}(0) = \frac{2}{3} - \frac{5}{3\sqrt{7}} > 0$. q_h is linear and clearly increasing.
i.) By parts *ii.)* and *iii.)*, we get that for $b > 0$, $p_T^\dagger(b, y) > p_T^\dagger(0, y) = p_T^\star(y)$ and $p_{T-1}^\dagger(b, 1) > p_{T-1}^\dagger(0, 1) = p_T^\star(1)$. For $p_{T-1}^\dagger(b, 2)$, we consider the three regimes. For $0 < b \leq \widehat{b}$, $p_{T-1}^\dagger(b, 2) = q_\ell(b) > q_\ell(0) = p_{T-1}^\dagger(0, 2) = p_T^\star(2)$. For $\widehat{b} < b \leq \frac{\alpha}{\beta}$, $p_{T-1}^\dagger(b, 2) = q_m(b) \geq q_m(0) = \frac{\sqrt{7}-1}{3}\frac{\alpha}{\beta} > \frac{\alpha}{2\beta} = p_{T-1}^\star(2)$. Finally, for $\frac{\alpha}{\beta} < b \leq 2\frac{\alpha}{\beta}$, note that $p_{T-1}^\dagger(b, 2) = q_h(b) \geq q_h(\frac{\alpha}{\beta}) = q_m(\frac{\alpha}{\beta}) > p_{T-1}^\star(2)$ since q_h is increasing, and the proof is complete.

We also note that the DM's price $p_{T-1}^\dagger(b, 2)$ is higher than the debt b for $0 \leq b \leq \widehat{b}$, and lower than the debt for $\widehat{b} < b \leq 2\frac{\alpha}{\beta}$. To see this, note that for $b \leq \widehat{b}$ we have that $p_{T-1}^\dagger(b, 2) = q_\ell(b)$, which by its definition is greater than b. For $b \in (\widehat{b}, \frac{\alpha}{\beta}]$ we have that $p_{T-1}^\dagger(b, 2) = q_m(b)$, which by its definition is less than b. Finally, our claim follows for $b > \frac{\alpha}{\beta}$ since prices are less than $\frac{\alpha}{\beta}$. □

Proof of Lemma 4.1. We consider three cases, depending on b.

Case 1. When $b \in [0, \widehat{b}]$, the DM's pricing policy in period $T-1$ is given by $p_{T-1}^\dagger(b) = q_\ell(b)$, where q_ℓ is given by (E-5) (refer to the proof of Proposition 4.1). Expressing $L_1 = 1 - \frac{J_{T-1}^\dagger}{J_{T-1}^\star}$, it can be readily checked that:

$$
\begin{aligned}
\frac{\partial^2 L_1}{\partial b^2} &= \frac{\alpha^2[(4+3\beta b)^2 - 12(1+\beta b)\alpha + 2\alpha^2]}{8\alpha^2} \geq 0 \\
\frac{\partial^2 L_1}{\partial \alpha^2} &= \frac{\beta^2 b^2[\beta b(48 + 9\beta b - 8\alpha) + 24(4-\alpha)]}{16\alpha^4} \geq 0 \\
\frac{\partial^2 L_1}{\partial \beta^2} &= \frac{b^2[(4+3\beta b)^2 - 12(1+\beta b)\alpha + 2\alpha^2]}{8\alpha^2} \geq 0 \\
\frac{\partial L_1}{\partial b} &= \frac{\beta^2 b[3\beta^2 b^2 + 6\beta b(2-\alpha) + 2(4-\alpha)(2-\alpha)]}{8\alpha^2} \geq 0 \\
\frac{\partial L_1}{\partial \alpha} &= -\frac{\beta^2 b[\beta b + 4(4-\alpha) + 4(8-3\alpha)]}{16\alpha^3} \leq 0 \\
\frac{\partial L_1}{\partial \beta} &= \frac{\beta b^2[3\beta^2 b^2 + 6\beta b(2-\alpha) + 2(4-\alpha)(2-\alpha)]}{8\alpha^2} \geq 0.
\end{aligned}
$$

The inequalities readily follow in each case, by recognizing that $\alpha, \beta, b \geq 0$ and $\alpha \leq 1$. This shows that L_1 is component-wise convex in b, α, and β, and has the desired monotonicity.

To prove the lower bound on L_1, we let $y := \frac{b}{p^\star}$, and define

$$
g_\ell(y, \alpha) := \frac{L_1}{y^2} = \frac{128 - 32(3-y)\alpha + (4-y)(4-3y)\alpha^2}{512}.
$$

Note that g_ℓ is convex and quadratic in y, reaching its minimum at $\frac{-8(2-\alpha)}{3\alpha} < 0$. Therefore,

$$g_\ell(y,\alpha) \geq g_\ell(0,\alpha) = \frac{(4-\alpha)(2-\alpha)}{32}.$$

The latter function is convex and quadratic in α, reaching its minimum at $\alpha = 3$. Thus, it is decreasing on $[0,1]$, so we can conclude that $L_1/y^2 = g_\ell(y,\alpha) \geq g_\ell(0,1) = \frac{3}{32} \approx 0.093$.

Case 2. When $b \in (\widehat{b}, \frac{\alpha}{\beta}]$, by the proof of Proposition 4.1 we have that $p^\dagger_{T-1}(b) = q_m(b)$, which is given by (E-6). The efficiency loss can be written as

$$L_1(b,\alpha,\beta) = \frac{1}{54\alpha^2}\Big[123\beta^2 b^2 + 16\beta^3 b^3 - 240\alpha\beta b - 28\beta^3 b^2\alpha + 106\beta\alpha^2 - 2\beta^2\alpha^2 b + 18\beta\alpha^3 + \left(41\beta^2 b + 8\beta^3 b^2 - 40\beta\alpha - 4\beta^2\alpha b - 9\beta\alpha^2\right)\sqrt{4\beta^2 b^2 - 10\beta b\alpha + 7\alpha^2}\Big].$$

As such, it can be readily checked that testing the positivity or negativity of a first-order or second-order partial derivative of L_1 with respect to b, α, or β is equivalent to showing that

$$f_0(\alpha,\beta,b) \geq 0, \, \forall\, (\alpha,\beta,b) \in \mathcal{X} := \{(\alpha,\beta,b) \in \mathbb{R}^3 \,:\, f_i(\alpha,\beta,b) \geq 0, i = 1,\ldots,m\},$$

where $\{f_i\}_{i=0}^m$ are polynomial functions in the variables b,α,β. This problem falls in the general class of polynomial optimization problems, which require testing the positivity of a polynomial objective on a feasible set given by a finite number of polynomial equalities and inequalities. Exact computational methods are available to produce certificates in such problems, using *sum-of-squares* (SOS) methods (see Parrilo, 2003 and references therein for details). We use these for every derivative above, and confirm that $\frac{\partial^2 L_1}{\partial b^2} \geq 0$, $\frac{\partial^2 L_1}{\partial \alpha^2} \geq 0$, $\frac{\partial^2 L_1}{\partial \beta^2} \geq 0$, $\frac{\partial L_1}{\partial b} \geq 0$, $\frac{\partial L_1}{\partial \alpha} \leq 0$, $\frac{\partial L_1}{\partial \beta} \geq 0$ always hold. Details are omitted for space considerations, but are available upon request.

To prove the bounds on L_1, we can again write $L_1 = g_m(y,\alpha)$, where $y := \frac{b}{p^\star}$ and

$$g_m(y,\alpha) := \frac{1}{216}\Big[636 - 480y + 123y^2 + 184\alpha - 108y\alpha - 12y^2\alpha + 8y^3\alpha + (-240 + 96y - 64\alpha + 8y\alpha + 8y^2\alpha)\sqrt{y^2 - 5y + 7}\Big].$$

Before proceeding with the argument, it is useful to derive a set of bounds on the value of y. To this end, note that $b \geq \widehat{b}$, and by the proof of Proposition 4.1, $\widehat{b} \geq b_m := (2-\sqrt{2})\frac{\alpha}{\beta}$. Thus, we have $y := \frac{2\beta b}{\alpha} \geq 4 - 2\sqrt{2}$. Furthermore, since $b \leq \frac{\alpha}{\beta}$, we also have $y \leq 2$. Using SOS techniques, it can then be readily checked that for any $(y, \alpha) \in [4 - 2\sqrt{2}, 2] \times [0, 1]$,

$$\frac{\partial g_m}{\partial \alpha} \leq 0 \tag{E-7a}$$

$$\frac{\partial^2 g_m}{\partial y^2} \geq 0. \tag{E-7b}$$

By letting $\underline{y} := 4 - 2\sqrt{2}$, we can therefore conclude that

$$g_m(y, \alpha) \overset{\text{(E-7a)}}{\geq} g_m(y, 1) \overset{\text{(E-7b)}}{\geq} g_m(\underline{y}, 1) + \left.\frac{\partial g_m(y, 1)}{\partial y}\right|_{y=\underline{y}} \cdot (y - \underline{y}), \quad \forall\, y \in [\underline{y}, 2],$$

which yields the desired bound when substituting the values.

Case 3. When $b \in (\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}]$, by the proof of Proposition 4.1 we have that $p^{\dagger}_{T-1}(b) = q_h(b) = \frac{\beta b + \alpha}{3\beta}$. The efficiency loss can be written as

$$L_1(b, \alpha, \beta) = \frac{-2\beta^3 b^3 - 6\alpha\beta b + (15 - 8\alpha)\alpha^2 + 6(1 + \alpha)\beta^2 b^2}{27\alpha^2}.$$

As such, we have:

$$\begin{aligned}
\frac{\partial L_1}{\partial b} &= \frac{2\beta[\beta b(2 + 2\alpha - \beta b) - \alpha]}{9\alpha^2} \geq 0 \\
\frac{\partial L_1}{\partial \alpha} &= -\frac{8\alpha^3 + 6(2 + \alpha)\beta^2 b^2 - 4\beta^3 b^3 - 6\alpha\beta b}{27\alpha^3} \leq 0 \\
\frac{\partial L_1}{\partial \beta} &= \frac{2b[\beta b(2 + 2\alpha - \beta b) - \alpha]}{9\alpha^2} \geq 0 \\
\frac{\partial^2 L_1}{\partial b^2} &= \frac{4\beta^2(1 + \alpha - \beta b)}{9\alpha^2} \geq 0 \\
\frac{\partial^2 L_1}{\partial \alpha^2} &= \frac{4\beta b[\beta b(3 + \alpha - \beta b) - \alpha]}{9\alpha^4} \geq 0 \\
\frac{\partial^2 L_1}{\partial \beta^2} &= \frac{4b^2(1 + \alpha - \beta b)}{9\alpha^2},
\end{aligned}$$

where each of the inequalities follows by using the fact that $\alpha \leq \beta b \leq 2\alpha \leq 2$.

To prove the bound on L_1, we can again write $L_1 = g_h(y, \alpha)$, where $y := \frac{b}{p^\star}$ and

$$g_h(y, \alpha) := \frac{1}{108}\left[6(y^2 - 2y + 10) - \alpha(y + 2)(4 - y)^2\right].$$

Since $b \in (\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}]$, we always have $y \in (2,4]$, and it can then be readily checked that for any such y,

$$\frac{\partial g_h}{\partial \alpha} = -\frac{(4-y)^2(2+y)}{108} \leq 0. \tag{E-8a}$$

$$\frac{\partial^2 g_h}{\partial y^2} = \frac{2-\alpha(y-2)}{18} \geq 0. \tag{E-8b}$$

Therefore, we always have:

$$g_h(y,\alpha) \overset{\text{(E-8a)}}{\geq} g_h(y,1) \overset{\text{(E-8b)}}{\geq} g_h(2,1) + \left.\frac{\partial g_h(y,1)}{\partial y}\right|_{y=2} \cdot (y-2),\ \forall\, y \in (2,4]$$

and the proof is complete. □

Proof of Proposition 4.2. *i.)* We use the expressions for $p_t^\dagger(b,2)$, $t = T-1, T$ derived in the proof of Proposition 4.1. We deal with the three regimes separately.

For $0 \leq b \leq \widehat{b}$, we have that

$$\begin{aligned}\mathbb{E}\left[p_T^\dagger(\mathscr{B}_T, \mathscr{Y}_T)\right] - p_{T-1}^\dagger(b,2) &= \lambda(q_\ell(b))p_T^\dagger(0,1) + (1-\lambda(q_\ell(b)))p_T^\dagger(b,2) - q_\ell(b) \\ &= \frac{\beta b^2(2(1-\alpha)+\beta b)}{16},\end{aligned}$$

which is clearly positive and increasing in b.

For $\widehat{b} < b \leq \frac{\alpha}{\beta}$, we have that

$$\mathbb{E}\left[p_T^\dagger(\mathscr{B}_T, \mathscr{Y}_T)\right] - p_{T-1}^\dagger(b,2) = \lambda(q_m(b))p_T^\dagger(b-q_m(b),1) + (1-\lambda(q_m(b)))p_T^\dagger(b,2) - q_m(b),$$

and for $\frac{\alpha}{\beta} < b \leq 2\frac{\alpha}{\beta}$, we have that

$$\mathbb{E}\left[p_T^\dagger(\mathscr{B}_T, \mathscr{Y}_T)\right] - p_{T-1}^\dagger(b,2) = \lambda(q_h(b))p_T^\dagger(b-q_h(b),1) + (1-\lambda(q_m(b)))p_T^\dagger(b,2) - q_h(b).$$

Substituting for q_m, q_h, and $p_T^\dagger$, and using similar arguments as in the proof of Lemma 4.1, one can show that the above differences are positive and increasing in b. Details are omitted for space considerations, but are available upon request.

ii.) The result follows from analysis of the one-unit case, Proposition 5.2.

iii.) We use the expressions for $p_t^\dagger(b,2)$, $t = T-1, T$ derived in the proof of Proposition 4.1. We show that $g(b) := p_{T-1}^\dagger(b,2) - p_T^\dagger(b,2)$ is decreasing. For $b \in [0, \widehat{b}]$, $g'(b) = (q_\ell)'(b) - \frac{1}{2} = \frac{1}{2}\frac{\partial}{\partial b}V_T(b,1) \leq 0$, where the inequality follows from Lemma 3.1*iii.)*. For $b \in (\widehat{b}, \frac{\alpha}{\beta}]$, $g'(b) = (q_m)'(b) - \frac{1}{2} \leq (q_m)'(\frac{\alpha}{\beta}) - \frac{1}{2} = \frac{1}{3} - \frac{1}{2} < 0$, where the first inequality follows from ${q_m}'$ being increasing and positive, as argued in the proof of Proposition 4.1. For $b \in (\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}]$, $g'(b) = (q_h)'(b) - \frac{1}{2} = \frac{1}{3} - \frac{1}{2} < 0$. We finally need to check the difference at the point of discontinuity $\widehat{b}$. We have $\lim_{b \to \widehat{b}+} g(b) = q_m(\widehat{b}) - p^\dagger_T(\widehat{b}, 2) < q_\ell(\widehat{b}) - p^\dagger_T(\widehat{b}, 2) = \lim_{b \to \widehat{b}-} g(b)$, where the inequality follows from Proposition 4.1. To complete the proof, note that for $y > T - t$ it can be readily seen that the revenue-maximizing policy is to charge the same price in both $T - 1$ and T, thus $p^\dagger_{T-1}(b, y) - p^\dagger_T(b, y) \le p^\star_{T-1}(y) - p^\star_T(y) = 0$. □

Proof of Proposition 4.3. For $y = 1$ the result follows from analysis of the one-unit case, Proposition 5.3. For $y = 2$, according to Proposition 4.1, the price $p^\dagger_{T-1}(b, 2)$ takes different expressions for b in $[0, \widehat{b}]$, $(\widehat{b}, \frac{\alpha}{\beta}]$ and $(\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}]$. We argue for these cases separately. Note that in all cases, it can be readily shown that following the revenue-maximizing policy would result in pricing at $\arg\max_{p \in \mathcal{P}}\{p\lambda(p)\} = \frac{\alpha}{2\beta}$ in both periods. Thus $J^\star = 2\frac{\alpha}{2\beta}\lambda(\frac{\alpha}{2\beta}) = \frac{\alpha^2}{2\beta}$.

- For $b \in [0, \widehat{b}]$, according to Proposition 4.1, $p^\dagger_{T-1}(b, 2) = q_\ell(b) > b$. Thus, in case of a sale at $T-1$, the DM covers his debt and charges $p^\dagger_T(0, 1)$ at T. Otherwise, he charges $p^\dagger_T(b, 1)$. Combining these observations we get

$$J^\dagger = \lambda(q_\ell(b))\left(q_\ell(b) + \lambda\left(p^\dagger_T(0,1)\right)p^\dagger_T(0,1)\right) + (1 - \lambda(q_\ell(b)))\lambda\left(p^\dagger_T(b,1)\right)p^\dagger_T(b,1).$$

We now derive an expression for the expectation $\mathbb{E}\left[\frac{\mathscr{I}^\dagger_T}{\mathscr{I}^\star_T}\right]$. Using the law of total expectation, in a similar fashion as in the proof of Proposition 5.3, we get

$$\begin{aligned}\mathbb{E}\left[\frac{\mathscr{I}^\dagger_T}{\mathscr{I}^\star_T}\right] &= \lambda(q_\ell(b))\frac{q_\ell(b) + \lambda\left(p^\dagger_T(0,1)\right)p^\dagger_T(0,1)}{p^\star_{T-1}(2) + \lambda\left(p^\star_T(1)\right)p^\star_T(1)} \\ &\quad + \left(\lambda\left(p^\star_{T-1}(2)\right) - \lambda(q_\ell(b))\right)\frac{\lambda\left(p^\dagger_T(b,1)\right)p^\dagger_T(b,1)}{p^\star_{T-1}(2) + \lambda\left(p^\star_T(1)\right)p^\star_T(1)} \\ &\quad + \lambda\left(p^\star_{T-1}(2)\right)\frac{\lambda\left(p^\dagger_T(b,1)\right)p^\dagger_T(b,1)}{\lambda\left(p^\star_T(1)\right)p^\star_T(1)}.\end{aligned}$$

By substituting for all the prices in the expressions above and setting $x := \beta b \in [0, \beta\widehat{b}]$, after some tedious algebra we get

$$J^\dagger - \mathbb{E}\left[\frac{\mathscr{I}^\dagger_T}{\mathscr{I}^\star_T}\right]J^\star = \underbrace{-\frac{x^2(2-\alpha)}{64\beta(2+\alpha)}}_{<0}\underbrace{\left(3x^2 + 8(2-\alpha)x - 4\alpha(6-\alpha)\right)}_{<0} > 0.$$

To see that the second multiplier above is negative, note that it is increasing for $x \geq 0$. Since $\beta\widehat{b} < \alpha$, we can upper bound the multiplier by evaluating it for $x = \alpha$, which yields $-\alpha(8+\alpha) < 0$.

- For $b \in \left(\widehat{b}, \frac{\alpha}{\beta}\right]$, according to Proposition 4.1, $p^{\dagger}_{T-1}(b, 2) = q_m(b) < b$. Thus, in case of a sale at $T-1$, the DM still fails to cover his debt and charges $p^{\dagger}_{T}(b - q_m(b), 1)$ at T. Otherwise, he charges $p^{\dagger}_{T}(b, 1)$. Combining these observations we get

$$J^{\dagger} = \lambda(q_m(b)) \left(q_m(b) + \lambda\left(p^{\dagger}_{T}(b - q_m(b), 1)\right) p^{\dagger}_{T}(b - q_m(b), 1)\right) + (1 - \lambda(q_m(b)))\lambda\left(p^{\dagger}_{T}(b, 1)\right) p^{\dagger}_{T}(b, 1).$$

Using the law of total expectation as above we get

$$\begin{aligned}\mathbb{E}\left[\frac{\mathscr{I}^{\dagger}_{T}}{\mathscr{I}^{\star}_{T}}\right] &= \lambda(q_m(b)) \frac{q_m(b) + \lambda\left(p^{\dagger}_{T}(b - q_m(b), 1)\right) p^{\dagger}_{T}(b - q_m(b), 1)}{p^{\star}_{T-1}(2) + \lambda\left(p^{\star}_{T}(1)\right) p^{\star}_{T}(1)} \\ &\quad + \left(\lambda\left(p^{\star}_{T-1}(2)\right) - \lambda(q_m(b))\right) \frac{\lambda\left(p^{\dagger}_{T}(b, 1)\right) p^{\dagger}_{T}(b, 1)}{p^{\star}_{T-1}(2) + \lambda\left(p^{\star}_{T}(1)\right) p^{\star}_{T}(1)} \\ &\quad + \lambda\left(p^{\star}_{T-1}(2)\right) \frac{\lambda\left(p^{\dagger}_{T}(b, 1)\right) p^{\dagger}_{T}(b, 1)}{\lambda\left(p^{\star}_{T}(1)\right) p^{\star}_{T}(1)}.\end{aligned}$$

By substituting for all the prices in the expressions above and setting $x := \beta b \in (\beta\widehat{b}, \alpha]$, after some tedious algebra we get

$$J^{\dagger} - \mathbb{E}\left[\frac{\mathscr{I}^{\dagger}_{T}}{\mathscr{I}^{\star}_{T}}\right] J^{\star} = \underbrace{-\frac{(2-\alpha)}{108\beta(2+\alpha)}}_{<0} \left(\underbrace{g_1(x)}_{<0} \sqrt{4x^2 - 10\alpha x + 7\alpha^2} + g_2(x)\right),$$

where $g_1(x) := 8x^2 + (4\alpha + 48)x - 4\alpha(4\alpha + 15)$ and $g_2(x) := 16x^3 + 4(\alpha + 12)x^2 - 2\alpha(27\alpha + 120)x + \alpha^2(46\alpha + 159)$. To see that $g_1(x)$ is negative, note that it is increasing for $x \geq 0$ and evaluates to $-4\alpha(\alpha + 3) < 0$ for $x = \alpha$. Thus, it suffices to show that

$$g_2^2(x) - g_1^2(x)(4x^2 - 10\alpha x + 7\alpha^2) = \underbrace{27(\alpha - x)}_{\geq 0} w(x) \leq 0,$$

or equivalently that $w(x) \leq 0$, where $w(x) = 32x^4 + (8\alpha(1 + 2\alpha) + 165)x^3 - \alpha(4\alpha + 315)x^2 + \alpha^2(3 - 44\alpha(\alpha + 4))x + \alpha^3(4\alpha(11 + 3\alpha) + 3)$. The derivative of w is cubic in x and can be readily maximized over $[0, \alpha]$ to obtain that $w'(x) \leq 0$. Since w is then decreasing, we can upper bound it as follows

$$w(x) \leq w(\widehat{b}\beta) \leq w(b_m\beta) = w((2 - \sqrt{2})\alpha) < 0.$$

Note that $w((2-\sqrt{2})\alpha)$ depends only on α and be readily maximized over $(0,1]$ to obtain the last inequality above, which concludes the proof for this case.

- For $b \in \left(\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}\right]$, according to Proposition 4.1, $p^{\dagger}_{T-1}(b,2) = q_h(b) \leq \frac{\alpha}{\beta} < b$. Thus, in case of a sale at $T-1$, the DM still fails to cover his debt and charges $p^{\dagger}_T(b-q_h(b),1)$ at T. Otherwise, it becomes infeasible for him to cover the debt. Combining these observations we get

$$J^{\dagger} = \lambda(q_h(b))\left(q_h(b) + \lambda\left(p^{\dagger}_T(b-q_h(b),1)\right)p^{\dagger}_T(b-q_h(b),1)\right).$$

Using the law of total expectation as above we get

$$\mathbb{E}\left[\frac{\mathscr{J}^{\dagger}_T}{\mathscr{J}^{\star}_T}\right] = \lambda(q_h(b))\frac{q_h(b) + \lambda\left(p^{\dagger}_T(b-q_h(b),1)\right)p^{\dagger}_T(b-q_h(b),1)}{p^{\star}_{T-1}(2) + \lambda\left(p^{\star}_T(1)\right)p^{\star}_T(1)}.$$

By substituting for all the prices in the expression above we get

$$J^{\dagger} - \mathbb{E}\left[\frac{\mathscr{J}^{\dagger}_T}{\mathscr{J}^{\star}_T}\right]J^{\star} = \frac{(2-\alpha)(\beta b+\alpha)(\beta b-2\alpha)(\beta b-2\alpha-3)}{27\beta(2+\alpha)} \geq 0,$$

since $\beta b - 2\alpha - 3 < \beta b - 2\alpha \leq 0$ for $b \leq 2\frac{\alpha}{\beta}$. ☐

F Proofs for Section 5

Proof of Proposition 5.1. To show *i.*), we follow the steps below.

Step 1: We first show that for all $0 \leq x < \overline{p}$, $\lambda(p)(p-x)$ admits a unique maximizer in p over $\mathcal{P}$, equal to $\pi(x)$, such that $\pi(x) \geq x$.

To this end, note that no $p < x$ can be a maximizer of $\lambda(p)(p-x)$ in p over $\mathcal{P}$. This is clear if $x \leq \underline{p}$. For $x > \underline{p}$ and for all p such that $\underline{p} < p < x < \overline{p}$, we get that $\lambda(p)(p-x) < 0 \leq \lambda(\overline{p})(\overline{p}-x)$. Our claim that $\pi(x) \geq x$ follows.

To show that $\lambda(p)(p-x)$ admits a unique maximizer over $[x \wedge \underline{p}, \overline{p}]$, it suffices to show that it is unimodal. If it has no stationary points in $(x \wedge \underline{p}, \overline{p})$, this is clearly the case. Otherwise, let $\hat{p} \in (x \wedge \underline{p}, \overline{p})$ be a stationary point, i.e., $\hat{p}$ solves the FOC

$$\frac{d}{dp}\{\lambda(p)(p-x)\}\bigg|_{\hat{p}} = \lambda'(\hat{p})(\hat{p}-x) + \lambda(\hat{p}) = 0. \tag{F-9}$$

The second derivative at $\hat{p}$ evaluates to

$$\frac{d^2}{dp^2}\{\lambda(p)(p-x)\}\Big|_{\hat{p}} = \lambda''(\hat{p})(\hat{p}-x) + 2\lambda'(\hat{p}) = \underbrace{-\frac{1}{\lambda'(\hat{p})}}_{>0}\underbrace{\left(\lambda''(\hat{p})\lambda(\hat{p}) - 2(\lambda'(\hat{p}))^2\right)}_{<0} < 0, \tag{F-10}$$

where the second equality follows by substituting for $\hat{p}-x$ from the FOC (note that if $\lambda'(\hat{p})=0$, we get $\lambda(\hat{p})=0$ from the FOC, a contradiction since $\hat{p} \in (x \wedge \underline{p}, \overline{p})$). The negativity of the second multiplier above follows from log-concavity of λ, which yields $0 \geq \lambda''(\hat{p})\lambda(\hat{p}) - (\lambda'(\hat{p}))^2 > \lambda''(\hat{p})\lambda(\hat{p}) - 2(\lambda'(\hat{p}))^2$. As such, all stationary points are local maxima. Thus, there exists a unique local maximum, which has to be the unique global maximizer.

Step 2: We show that $0 \leq \pi'(x) \leq 1$. To this end, note that from (F-9) for $p \to x$ the derivative of $\lambda(p)(p-x)$ becomes positive. Thus, if it has no stationary points in $(x \wedge \underline{p}, \overline{p})$, it is increasing and $\pi(x) = \overline{p}$, which trivially satisfies our claim. Otherwise, $\pi(x)$ is equal to the unique solution of the FOC. We calculate its derivatives using the Implicit Function Theorem. In particular, by taking the derivative of the FOC with respect to x we get

$$\pi'(x) = \frac{\lambda'(\pi(x))}{\lambda''(\pi(x))(\pi(x)-x) + 2\lambda'(\pi(x))} = \frac{-(\lambda'(\pi(x)))^2}{\lambda''(\pi(x))\lambda(\pi(x)) - 2(\lambda'(\pi(x)))^2} \geq 0, \tag{F-11}$$

where the second equality and the inequality follow from (F-10). Showing $\pi'(x) \leq 1$ is equivalent to $\lambda''(\pi(x))\lambda(\pi(x)) - 2(\lambda'(\pi(x)))^2 \leq 0$, which follows from log-concavity of λ.

Step 3: We now complete the proof of *i.*) by deriving the DM's price. For $b \geq \overline{p}$, as we argued in Section 5.1 we can take $p_t^\dagger(b) = \overline{p}$ without loss.

For $b < \overline{p}$, we have that $p_t^\dagger(b) \in \underset{p \in \mathcal{P},\, p \geq b}{\arg\max}\{\lambda(p)\,[p - (b + V_{t+1}(b))]\}$ for all $t = 1, \ldots, T$. Note first that posting a price $\underline{p} < b + \epsilon < \overline{p}$ (for $\epsilon > 0$ appropriately chosen) ensures a non-zero probability of covering the debt since $\lambda(b+\epsilon) > 0$. Thus, $-V_t'(b) < 1$ for all $t = 1, \ldots, T$ by Lemma 3.1*iii.*). Consequently, $b + V_{t+1}(b)$ is strictly increasing and $b + V_{t+1}(b) < \overline{p} + V_{t+1}(\overline{p}) = \overline{p}$. By our result in Step 1, we get that $p_t^\dagger(b) = \pi(b + V_{t+1}(b))$.

Finally, we also remark that for $b < \overline{p}$, we have that $p_t^\dagger(b) > b$. To see this, note that if $b < \underline{p}$, clearly $p_t^\dagger(b) > b$. Otherwise, for $\underline{p} \leq b < \overline{p}$ note that for $p = b$, the maximand above is less than equal to zero, whereas for $p = b + V_{t+1}(b) + \epsilon' < \overline{p}$ (for $\epsilon' > 0$ small enough) it is positive. Hence, again $p_t^\dagger(b) > b$ and the proof is complete.

To show *ii.*) and *iii.*), we argue as follows. For $b \geq \overline{p}$, *i.*) suggests that $p_t^\dagger(b) = \overline{p}$, which is trivially increasing in b. For $b < \overline{p}$, we have that $p_t^\dagger(b) = \pi(b + V_{t+1}(b))$, which is increasing in b since $b + V_{t+1}(b)$ is increasing in b (by Lemma 3.1*ii.*)) and so is π (by *i.*)). The proof is complete by recalling that $p_t^\dagger(0) = p_t^\star$. □

Proof of Proposition 5.2. We first show that $\pi(x)$ is concave. To that end, using (F-11) and differentiating, we obtain:

$$\begin{aligned}\pi''(x) &= \frac{([\lambda''(\pi(x))]^2 - \lambda'(\pi(x))\lambda'''(\pi(x)))(\pi(x)-x)\pi'(x) + (1-\pi'(x))\lambda'(\pi(x))\lambda''(\pi(x))}{(\lambda''(\pi(x))(\pi(x)-x) + 2\lambda'(\pi(x)))^2} \\ &\propto \underbrace{([\lambda''(\pi(x))]^2 - \lambda'(\pi(x))\lambda'''(\pi(x)))}_{\leq 0,\text{ by } -\lambda' \text{ log-convex}} \underbrace{(\pi(x)-x)\pi'(x)}_{\geq 0,\text{ by Proposition 5.1}} + \underbrace{(1-\pi'(x))}_{\geq 0,\text{ by Thm. 5.1}} \underbrace{\lambda'(\pi(x))\lambda''(\pi(x))}_{\leq 0,\text{ by } \lambda \text{ convex}} \leq 0.\end{aligned}$$

To show *i.*), first note that for $b \geq \overline{p}$, Proposition 5.1 suggests that $p_t^\dagger(b) = p_{t+1}^\dagger(b) = \overline{p}$, and thus their difference is trivially decreasing in b. For $b < \overline{p}$, we express the prices $p_t^\dagger(b)$, $p_{t+1}^\dagger(b)$ using the function π and take the derivative of their difference. We obtain

$$\frac{d}{db}\left(p_t^\dagger(b) - p_{t+1}^\dagger(b)\right) = \left(1 + V_{t+1}'(b)\right)\pi'(b + V_{t+1}(b)) - \left(1 + V_{t+2}'(b)\right)\pi'(b + V_{t+2}(b)).$$

Note that $V_t(b)$ is decreasing in t (Lemma 3.1), and π' is positive and decreasing. Thus,

$$\pi'(b + V_{t+2}(b)) \geq \pi'(b + V_{t+1}(b)) \geq 0.$$

Also, since $-V_t'(b)$ is less that one and decreasing in t, we get

$$1 + V_{t+2}'(b) \geq 1 + V_{t+1}'(b) \geq 0.$$

Based on these two inequalities and the expression for the derivative above, we conclude that $p_t^\dagger(b) - p_{t+1}^\dagger(b)$ is decreasing in b.

To show *ii.*), simply recall that $p_t^\dagger(0) = p_t^\star$ and that $p_t^\dagger(b) \geq p_t^\star$. For *iii.*), we have that

$$\frac{p_t^\dagger(b)}{p_t^\star} = \frac{(p_t^\dagger(b) - p_{t+1}^\dagger(b)) + p_{t+1}^\dagger(b)}{(p_t^\star - p_{t+1}^\star) + p_{t+1}^\star} \overset{(*)}{\leq} \frac{(p_t^\star - p_{t+1}^\star) + p_{t+1}^\dagger(b)}{(p_t^\star - p_{t+1}^\star) + p_{t+1}^\star} \overset{(**)}{\leq} \frac{p_{t+1}^\dagger(b)}{p_{t+1}^\star},$$

where $(*)$ follows from part *i*), and $(**)$ is true since $0 \leq p_{t+1}^\star \leq p_{t+1}^\dagger(b)$, by Proposition 5.1. □

Proof of Proposition 5.3. Similarly to $J_t^\dagger$, we define $J_t^\star$ as the expected revenues under the revenue-maximizing policy at the beginning of period t, conditional on no sale in periods $1, \ldots, t-1$. For clarity, we explicitly highlight the dependence of $J_t^\dagger$ on the debt, i.e., we write $J_t^\dagger(b)$. We first show that $L_1 \leq L_2$, or equivalently that

$$\frac{J_1^\dagger(b)}{J_1^\star} \geq \mathbb{E}\left[\frac{\mathscr{J}_2^\dagger}{\mathscr{J}_2^\star}\right] = \mathbb{E}\left[\frac{\mathbb{E}[\mathscr{R}(\mathbf{p}^\dagger)|\sigma(\mathscr{W}_1)]}{\mathbb{E}[\mathscr{R}(\mathbf{p}^\star)|\sigma(\mathscr{W}_1)]}\right]. \tag{F-12}$$

We condition on the value of $\mathscr{W}_1$ in order to express the right-hand side above using the law of total expectation. In particular, since $p_1^\dagger(b) \geq p_1^\star$ by Proposition 5.1, we consider the events $\mathcal{E}_1 = \{\mathscr{W}_1 < p_1^\star\}$, $\mathcal{E}_2 = \{p_1^\star \leq \mathscr{W}_1 < p_1^\dagger(b)\}$ and $\mathcal{E}_3 = \{p_1^\dagger(b) \leq \mathscr{W}_1\}$. Under $\mathcal{E}_1$, no sale occurs under either the DM's or the revenue-maximizing policies. Under $\mathcal{E}_2$, a sale occurs only under the revenue-maximizing policy, whereas under $\mathcal{E}_3$ a sale occurs under both policies. For the revenue-maximizing (DM's) policy, expected revenue equal $J_2^\star$ ($J_2^\dagger(b)$) when no sale occurs, and $p_1^\star$ ($p_1^\dagger(b)$) when a sale occurs. Combining all these facts, we get that

$$
\begin{aligned}
\mathbb{E}\left[\frac{\mathbb{E}[\mathscr{R}(\mathbf{p}^\dagger)|\sigma(\mathscr{W}_1)]}{\mathbb{E}[\mathscr{R}(\mathbf{p}^\star)|\sigma(\mathscr{W}_1)]}\right] &= \sum_{i=1}^{3} \mathbb{E}\left[\frac{\mathbb{E}[\mathscr{R}(\mathbf{p}^\dagger)|\sigma(\mathscr{W}_1)]}{\mathbb{E}[\mathscr{R}(\mathbf{p}^\star)|\sigma(\mathscr{W}_1)]} \,\middle|\, \mathcal{E}_i\right] \mathbb{P}(\mathcal{E}_i) \\
&= \frac{J_2^\dagger(b)}{J_2^\star}(1-\lambda(p_1^\star)) + \frac{J_2^\dagger(b)}{p_1^\star}(\lambda(p_1^\star) - \lambda(p_1^\dagger(b))) + \frac{p_1^\dagger(b)}{p_1^\star}\lambda(p_1^\dagger(b)) \\
&= (1-\lambda(p_1^\star))\left(\frac{1}{J_2^\star} - \frac{1}{p_1^\star}\right) J_2^\dagger(b) + \frac{1}{p_1^\star} J_1^\dagger(b),
\end{aligned}
$$

where the last equality follows from $J_1^\dagger(b) = \lambda(p_1^\dagger(b))p_1^\dagger(b) + (1-\lambda(p_1^\dagger(b)))J_2^\dagger(b)$. Thus,

$$
\frac{J_1^\dagger(b)}{J_1^\star} - \mathbb{E}\left[\frac{\mathscr{J}_2^\dagger}{\mathscr{J}_2^\star}\right] = J_2^\dagger(b)\left[\left(\frac{1}{J_1^\star} - \frac{1}{p_1^\star}\right)\frac{J_1^\dagger(b)}{J_2^\dagger(b)} - (1-\lambda(p_1^\star))\left(\frac{1}{J_2^\star} - \frac{1}{p_1^\star}\right)\right].
$$

Thus, the inequality in (F-12) is equivalent with $\left(\frac{1}{J_1^\star} - \frac{1}{p_1^\star}\right)\frac{J_1^\dagger(b)}{J_2^\dagger(b)} - (1-\lambda(p_1^\star))\left(\frac{1}{J_2^\star} - \frac{1}{p_1^\star}\right) \geq 0$. For $b = 0$ the inequality holds with equality, since the two policies become the same. For $b > 0$, it suffices to show that the left-hand side is increasing in b. This is indeed the case since $\frac{J_1^\dagger(b)}{J_2^\dagger(b)}$ is increasing in b, and $J_1^\star < p_1^\star$, $p_1^\star$ being the largest price charged under the revenue-maximizing policy.

Using identical arguments and the fact that $\frac{J_t^\dagger(b)}{J_{t+1}^\dagger(b)}$ is increasing in b, one can show that for all $t = 2, \ldots, T-1$

$$
\frac{J_t^\dagger(b)}{J_t^\star} \;\geq\; \mathbb{E}\left[\frac{\mathscr{J}_{t+1}^\dagger}{\mathscr{J}_{t+1}^\star} \,\middle|\, \mathscr{W}_1 = \ldots = \mathscr{W}_{t-1} = 0\right], \tag{F-13}
$$

where the conditioning event in the right-hand side ensures that no sale has occurred under either policy up until and including period $t-1$.

We now show that for any $t = 2, \ldots, T-1$ we have that $L_t \leq L_{t+1}$, or equivalently that

$$
\mathbb{E}\left[\frac{\mathscr{J}_t^\dagger}{\mathscr{J}_t^\star}\right] \;\geq\; \mathbb{E}\left[\frac{\mathscr{J}_{t+1}^\dagger}{\mathscr{J}_{t+1}^\star}\right]. \tag{F-14}
$$

We condition again on the events $\{\mathcal{E}_i\}_{i=1,2,3}$ to express the left-hand side as follows

$$\mathbb{E}\left[\frac{\mathscr{I}_t^\dagger}{\mathscr{I}_t^\star}\right] = \frac{p_1^\dagger(b)}{p_1^\star}\lambda(p_1^\dagger(b)) + \frac{J_2^\dagger(b)}{p_1^\star}(\lambda(p_1^\star) - \lambda(p_1^\dagger(b))) + \mathbb{E}\left[\frac{\mathscr{I}_t^\dagger}{\mathscr{I}_t^\star}\,\middle|\, \mathscr{W}_1 < p_1^\star\right](1-\lambda(p_1^\star)),$$

where we used the fact that $\mathbb{E}\left[\mathscr{I}_t^\dagger \,\middle|\, \mathscr{W}_1 < p_1^\star\right]$ is equal to $J_2^\dagger(b)$ by definition. We can now use the same approach in order to express the expectation in the last term above, by conditioning on the value of $\mathscr{W}_2$. In particular, we get

$$\begin{aligned}\mathbb{E}\left[\frac{\mathscr{I}_t^\dagger}{\mathscr{I}_t^\star}\right] &= \frac{p_1^\dagger(b)}{p_1^\star}\lambda(p_1^\dagger(b)) + \frac{J_2^\dagger(b)}{p_1^\star}(\lambda(p_1^\star) - \lambda(p_1^\dagger(b))) \\ &+ (1-\lambda(p_1^\star))\left(\frac{p_2^\dagger(b)}{p_2^\star}\lambda(p_2^\dagger(b)) + \frac{J_3^\dagger(b)}{p_2^\star}(\lambda(p_2^\star) - \lambda(p_2^\dagger(b))) + \mathbb{E}\left[\frac{\mathscr{I}_t^\dagger}{\mathscr{I}_t^\star}\,\middle|\, \mathscr{W}_1 < p_1^\star, \mathscr{W}_2 < p_2^\star\right](1-\lambda(p_2^\star))\right).\end{aligned}$$

By applying the same approach recursively we obtain

$$\mathbb{E}\left[\frac{\mathscr{I}_t^\dagger}{\mathscr{I}_t^\star}\right] = \sum_{\tau=1}^{t-1}\phi_{\tau-1}\left[\frac{p_\tau^\dagger(b)}{p_\tau^\star}\lambda(p_\tau^\dagger(b)) + \frac{J_{\tau+1}^\dagger(b)}{p_\tau^\star}(\lambda(p_\tau^\star) - \lambda(p_\tau^\dagger(b)))\right] + \phi_{t-1}\frac{J_t^\dagger(b)}{J_t^\star},$$

where we used that $\mathbb{E}\left[\frac{\mathscr{I}_t^\dagger}{\mathscr{I}_t^\star}\,\middle|\, \mathscr{W}_1 < p_1^\star, \ldots \mathscr{W}_{t-1} < p_{t-1}^\star\right] = \frac{J_t^\dagger(b)}{J_t^\star}$, and $\phi_0 := 1$, $\phi_\tau := \prod_{i=1}^{\tau}(1-\lambda(p_\tau^\star))$ for $\tau \geq 1$. If we use the same expression for the right-hand side of the inequality (F-14) we want to show, we can express the difference of the two sides as

$$\begin{aligned}\mathbb{E}\left[\frac{\mathscr{I}_{t+1}^\dagger}{\mathscr{I}_{t+1}^\star} - \frac{\mathscr{I}_t^\dagger}{\mathscr{I}_t^\star}\right] &= \phi_{t-1}\left(\frac{p_t^\dagger(b)}{p_t^\star}\lambda(p_t^\dagger(b)) + \frac{J_{t+1}^\dagger(b)}{p_t^\star}(\lambda(p_t^\star) - \lambda(p_t^\dagger(b)))\right) + \phi_t\frac{J_{t+1}^\dagger(b)}{J_{t+1}^\star} - \phi_{t-1}\frac{J_t^\dagger(b)}{J_t^\star} \\ &= \phi_{t-1}\left(\frac{p_t^\dagger(b)}{p_t^\star}\lambda(p_t^\dagger(b)) + \frac{J_{t+1}^\dagger(b)}{p_t^\star}(\lambda(p_t^\star) - \lambda(p_t^\dagger(b))) + (1-\lambda(p_t^\star)\frac{J_{t+1}^\dagger(b)}{J_{t+1}^\star} - \frac{J_t^\dagger(b)}{J_t^\star}\right) \\ &= \phi_{t-1}\left(\mathbb{E}\left[\frac{\mathscr{I}_{t+1}^\dagger}{\mathscr{I}_{t+1}^\star}\,\middle|\, \mathscr{W}_1 = \ldots = \mathscr{W}_{t-1} = 0\right] - \frac{J_t^\dagger(b)}{J_t^\star}\right) \leq 0,\end{aligned}$$

where the inequality follows from (F-13) and the proof is complete. □

Proposition F.1. *Suppose that the demand function is either linear or exponential. Then,* $J_t^\dagger/J_{t+1}^\dagger$ *is increasing in* b *for any* $t = 1, ..., T-1$.

Proof of Proposition F.1. Throughout this proof, x' will denote the derivative $\frac{d}{db}x$. To ease notation, we suppress the superscript † and the dependence on b, e.g., $p_t = p_t^\dagger(b)$, etc. Also, let

$$\lambda_t := \lambda(p_t^\dagger(b)), \quad t = 1, \dots, T.$$

We also define ω_t to be the probability of failing to cover the debt under the DM's policy at the beginning of period t, given that no sale has occurred until then. That is,

$$\omega_{T+1} := 1 \quad \text{and} \quad \omega_t := (1 - \lambda_t)\omega_{t+1}, \quad t = 1, \dots, T. \tag{F-15}$$

Using our notation, Lemma 3.1*iii.*) can be expressed as

$$1 - \omega_t = -V_t', \quad t = 1, \dots, T+1. \tag{F-16}$$

Also, the expected revenue can be expressed as

$$J_t = \lambda_t p_t + (1 - \lambda_t)J_{t+1} = V_t + (1 - \omega_t)b, \quad t = 1, \dots, T+1. \tag{F-17}$$

By differentiating (F-17) and using (F-16) we get

$$J_t' = -b\omega_t', \quad t = 1, \dots, T+1. \tag{F-18}$$

We treat the two cases separately.

Case (1): $\lambda(p) = e^{-\alpha p}$, $\alpha > 0$, $p \in [0, \infty)$.[18] Using our notation, Proposition 5.1 yields that $p_t = \pi(b + V_{t+1})$, for all $t = 1, \dots, T$. Using the fact that $\arg\max_{p \geq 0} e^{-\alpha p}(p - x) = \frac{1}{\alpha} + x$ and (F-17), we get that

$$p_t = \frac{1}{\alpha} + b + V_{t+1} = \frac{1}{\alpha} + b\omega_{t+1} + J_{t+1}, \quad t = 1, \dots, T. \tag{F-19}$$

In conjunction with Lemma 3.1*i.*), this also shows that p_t is decreasing in t. By diffentiating (F-19) and using (F-16), we get that

$$p_t' = \omega_{t+1}, \quad t = 1, \dots, T, \tag{F-20}$$

which also yields

$$\lambda_t' = (e^{-\alpha p_t})' = -\alpha p_t' \lambda_t = -\alpha \omega_{t+1} \lambda_t, \quad t = 1, \dots, T. \tag{F-21}$$

[18]The proof can be generalized to the case where $\lambda(p) = \alpha_0 e^{-\alpha p}$, $\alpha_0 \in (0, 1)$, in a straightforward manner.

These allow us to express

$$\begin{aligned}
\left(\frac{J_{t-1}}{J_t}\right)' &= \left(\frac{\lambda_{t-1}p_{t-1} + (1-\lambda_{t-1})J_t}{J_t}\right)' \\
&= \left(\frac{\lambda_{t-1}p_{t-1} - \lambda_{t-1}J_t}{J_t}\right)' \\
&= \frac{\lambda_{t-1}}{J_t^2}\left((p_{t-1} - J_t)\frac{\lambda'_{t-1}}{\lambda_{t-1}}J_t + p'_{t-1}J_t - p_{t-1}J'_t\right) \\
&= \frac{\lambda_{t-1}}{J_t^2}\left(-(p_{t-1} - J_t)\alpha p'_{t-1}J_t + p'_{t-1}J_t - p_{t-1}J'_t\right) && \text{[by (F-21)]} \\
&= \frac{\lambda_{t-1}}{J_t^2}\left(-\alpha(-\frac{1}{\alpha} + p_{t-1} - J_t)p'_{t-1}J_t - p_{t-1}J'_t\right) \\
&= \frac{\lambda_{t-1}}{J_t^2}\left(-\alpha b\omega_t p'_{t-1}J_t - p_{t-1}J'_t\right) && \text{[by (F-19)]} \\
&= \frac{\lambda_{t-1}}{J_t^2}\left(-\alpha b\omega_t^2 J_t + b\omega'_t p_{t-1}\right) && \text{[by (F-20) and (F-18)]} \\
&= \frac{\lambda_{t-1}\omega_t b}{J_t^2}\left(-\alpha\omega_t J_t + \frac{\omega'_t}{\omega_t}p_{t-1}\right),
\end{aligned}$$

for all $t = 2, \ldots, T$. Thus, it suffices to show that the inequality $\frac{\omega'_t}{\omega_t} \geq \frac{\alpha\omega_t J_t}{p_{t-1}}$ holds for all $t = 2, \ldots, T+1$. We will use induction. It is trivially true for $T+1$, since $\omega_{T+1} = 1$ and $J_{T+1} = 0$. We hypothesize now that it is true for some $t+1$. Then,

$$\begin{aligned}
\frac{\omega'_t}{\omega_t} &= -\frac{\lambda'_t}{1-\lambda_t} + \frac{\omega'_{t+1}}{\omega_{t+1}} && \text{[by differentiating (F-15)]} \\
&\geq -\frac{\lambda'_t}{1-\lambda_t} + \frac{\alpha\omega_{t+1}J_{t+1}}{p_t} && \text{[by the induction hypothesis]} \\
&\geq \alpha\omega_{t+1}\left(\frac{\lambda_t}{1-\lambda_t} + \frac{J_{t+1}}{p_t}\right) && \text{[by (F-21)]} \\
&= \alpha\frac{\omega_t}{1-\lambda_t}\frac{\lambda_t p_t + (1-\lambda_t)J_{t+1}}{(1-\lambda_t)p_t} && \text{[by (F-15)]} \\
&= \frac{\alpha\omega_t J_t}{(1-\lambda_t)^2 p_t} && \text{[by (F-17)]} \\
&\geq \frac{\alpha\omega_t J_t}{p_{t-1}} && \text{[since } 1-\lambda_t < 1 \text{ and } p_t \text{ is decreasing in } t\text{]}
\end{aligned}$$

and the proof for this case is complete.

Case (2): $\lambda(p) = \alpha - \beta p$, $\alpha \in (0,1]$, $\beta > 0$, $p \in [0, \frac{\alpha}{\beta}]$. Let $b < \frac{\alpha}{\beta}$. As before,

$$p_t = \pi(b + V_{t+1}) = \frac{1}{2}\left(\frac{\alpha}{\beta} + b + V_{t+1}\right) = \frac{1}{2}\left(\frac{\alpha}{\beta} + b\omega_{t+1} + J_{t+1}\right), \quad t = 1, \ldots, T, \tag{F-22}$$

which also shows that p_t is decreasing in t and that

$$p_t' = \frac{1}{2}\omega_{t+1}, \quad t = 1, \ldots, T \tag{F-23}$$

$$\lambda_t' = -\beta p_t' = -\frac{\beta}{2}\omega_{t+1}, \quad t = 1, \ldots, T \tag{F-24}$$

By substituting for p_t in the main recursion $V_t = \lambda_t(p_t - b) + (1 - \lambda_t)V_{t+1}$, we obtain

$$V_t = V_{t+1} + \frac{\lambda_t}{2}\left(\frac{\lambda(b)}{\beta} - V_{t+1}\right), \quad t = 1, \ldots, T, \tag{F-25}$$

which in conjuction with Lemma 3.1*i.*) also shows that

$$\frac{\lambda(b)}{\beta} - V_{t+1} \geq 0, \quad t = 1, \ldots, T, \tag{F-26}$$

Also, for all $t = 2, \ldots, T$

$$\begin{aligned}
(1-\lambda_t)^2\lambda_t &\leq (1-\lambda_t)\lambda_t && \text{[since } 1-\lambda_t < 1\text{]}\\
&= \lambda_{t-1} - (\lambda_{t-1} - \lambda_t) - \lambda_t^2\\
&= \lambda_{t-1} - \frac{\beta}{2}\left(V_{t+1} - V_t\right) - \lambda_t^2 && \text{[since } \lambda_t = \alpha - \beta p_t \text{ and (F-22)]}\\
&= \lambda_{t-1} - \frac{\beta\lambda_t}{4}\left(V_{t+1} - \frac{\lambda(b)}{\beta}\right) - \lambda_t^2 && \text{[by (F-25)]}\\
&= \lambda_{t-1} - \frac{\beta\lambda_t}{4}\left(V_{t+1} - \frac{\lambda(b)}{\beta} + \frac{4}{\beta}\lambda_t\right)\\
&= \lambda_{t-1} - \frac{\beta\lambda_t}{4}\left(V_{t+1} - 2V_{t+1} + \frac{\lambda(b)}{\beta}\right) && \text{[since } \lambda_t = \alpha - \beta p_t \text{ and (F-22)]}\\
&= \lambda_{t-1} - \frac{\beta\lambda_t}{4}\left(\frac{\lambda(b)}{\beta} - V_{t+1}\right)\\
&\leq \lambda_{t-1}. && \text{[by (F-26)]}
\end{aligned}$$

These allow us to express as above

$$\begin{aligned}
\left(\frac{J_{t-1}}{J_t}\right)' &= \frac{1}{J_t^2}\left((p_{t-1}-J_t)\lambda_{t-1}'J_t + \lambda_{t-1}p_{t-1}'J_t - \lambda_{t-1}p_{t-1}J_t'\right) \\
&= \frac{1}{J_t^2}\left(-\frac{\beta}{2}(p_{t-1}-J_t)\omega_t J_t + \frac{1}{2}\lambda_{t-1}\omega_t J_t + \lambda_{t-1}p_{t-1}b\omega_t'\right) \quad \text{[by (F-24), (F-23) and (F-18)]} \\
&= \frac{\lambda_{t-1}p_{t-1}b\omega_t}{J_t^2}\left(\left[1-\frac{\beta(p_{t-1}-J_t)}{\lambda_{t-1}}\right]\frac{J_t}{2p_{t-1}b} + \frac{\omega_t'}{\omega_t}\right) \\
&= \frac{\lambda_{t-1}p_{t-1}b\omega_t}{J_t^2}\left(\left[1-\frac{\beta(\frac{1}{\beta}\lambda_{t-1}+b\omega_t)}{\lambda_{t-1}}\right]\frac{J_t}{2p_{t-1}b} + \frac{\omega_t'}{\omega_t}\right) \quad \text{[by (F-22) and (F-17)]} \\
&= \frac{\lambda_{t-1}p_{t-1}b\omega_t}{J_t^2}\left(-\frac{\beta\omega_t J_t}{2\lambda_{t-1}p_{t-1}} + \frac{\omega_t'}{\omega_t}\right),
\end{aligned}$$

for all $t = 2,\ldots,T$. Thus, it suffices to show that the inequality $\omega_t' \geq \frac{\beta\omega_t^2 J_t}{2\lambda_{t-1}p_{t-1}}$ holds for all $t = 2,\ldots,T+1$. We will use induction. It is trivially true for $T+1$, since $\omega_{T+1} = 1$ and $J_{T+1} = 0$. We hypothesize now that it is true for some $t+1$. Then,

$$\begin{aligned}
\omega_t' &= (V_t')' && \text{[by (F-16)]} \\
&= (-\lambda_t + (1-\lambda_t)V_{t+1}')' && \text{[by the Envelope Theorem applied to (4)]} \\
&= -\lambda_t'\omega_{t+1} + (1-\lambda_t)\omega_{t+1}' && \text{[by (F-16)]} \\
&\geq -\lambda_t'\omega_{t+1} + (1-\lambda_t)\frac{\beta\omega_{t+1}^2 J_{t+1}}{2\lambda_t p_t} && \text{[by the induction hypothesis]} \\
&= \frac{\beta}{2}\omega_{t+1}^2 + (1-\lambda_t)\frac{\beta\omega_{t+1}^2 J_{t+1}}{2\lambda_t p_t} && \text{[by (F-23)]} \\
&= \frac{\beta\omega_{t+1}^2 J_t}{2\lambda_t p_t} && \text{[by (F-17)]} \\
&= \frac{\beta\omega_t^2 J_t}{2(1-\lambda_t)^2\lambda_t p_t} && \text{[by (F-15)]} \\
&= \frac{\beta\omega_t^2 J_t}{2\lambda_{t-1}p_{t-1}}, && \text{[since } (1-\lambda_t)^2\lambda_t \leq \lambda_{t-1} \text{ and } p_t \text{ is decreasing in } t\text{]}
\end{aligned}$$

and the proof is complete. □

G Proofs for Section 6

Proof of Proposition 6.1. **Claim 1:** $J^\dagger = J^E$. We first deal with contracts with early repayment option. The DM's problem at T is the same with the one we analyzed in Section 4. Thus, the DM charges $p_T^\dagger(b,1)$, and the value function $V_T(b,y)$ is as in Proposition 4.1. At $T-1$, the DM can follow a strategy of covering the debt either in one, or two periods. We analyze these separately.

- When the DM plans on covering the debt in one period, his problem can be expressed as

$$\max_{p \in (\gamma b, \frac{\alpha}{\beta}]} f_\ell^E(p, \gamma),$$

where $f_\ell^E(p,\gamma) := \lambda(p)\,(p - \gamma b + V_T(0,1) - V_T(b,1))$ in this case. The price he charges then is $q_\ell^E(b,\gamma) := \frac{1}{2}\left(\gamma b + V_T(b,1) - V_T(0,1) + \frac{\alpha}{\beta}\right)$.

Having characterized the DM's pricing policy, we can now express the debt value

$$D(\{B,\gamma\}) = \lambda(q_\ell^E(B,\gamma)) \cdot \gamma B + (1 - \lambda(q_\ell^E(B,\gamma))) \cdot \lambda(p_T^\dagger(B,1)) \cdot B,$$

where the first term corresponds to the expected (discounted) debt payment the debtholders receive if the DM makes a sale at $T-1$, and the second term in case he makes a sale only at T.

Similarly, expected revenues are

$$J(\{B,\gamma\}) = \lambda(q_\ell^E(B,\gamma))(q_\ell^E(B,\gamma) + V_T(0,1)) + (1 - \lambda(q_\ell^E(B,\gamma))) \cdot \lambda(p_T^\dagger(B,1)) \cdot p_T^\dagger(B,1),$$

where the terms have similar interpretation as above (see also the proof of Proposition 4.3 for similar derivations).

By equation (6b), note that B is essentially a function of γ. In case this equation has multiple positive roots, we assume that the smallest one is always preferred; this is because higher debt repayment B will lead to higher efficiency losses for the regime where the DM charges a price to cover the debt in one period (see, for example, Figure 2 and the discussion in that Section). It can also be readily checked that

$$D(\{0,\gamma\}) = D\left(\left\{2\frac{\alpha}{\beta},\gamma\right\}\right) = 0,$$

which implies that

$$\frac{\partial D(\{B,\gamma\})}{\partial B} \geq 0$$

is a necessary condition for the smallest positive root of (6b).

We now argue that expected revenues $J(\{B,\gamma\})$ are non-decreasing in γ, implying that $\gamma = 1$ would be an optimal early payment discount. In particular note that

$$\frac{dJ(\{B,\gamma\})}{d\gamma} = \frac{\partial J(\{B,\gamma\})}{\partial \gamma} + \frac{\partial J(\{B,\gamma\})}{\partial B}\frac{dB}{d\gamma},$$

where the derivative of B with respect to γ can be obtained using the Implicit Function Theorem for (6b)

$$\frac{\partial D(\{B,\gamma\})}{\partial \gamma} + \frac{\partial D(\{B,\gamma\})}{\partial B}\frac{dB}{d\gamma} = 0.$$

One can then readily verify, using sum-of-square techniques, that the set

$$\left\{(\beta, B, \gamma, d) \,:\, \frac{dJ(\{B,\gamma\})}{d\gamma} \leq 0,\, D(\{B,\gamma\}) = d,\, \frac{\partial D(\{B,\gamma\})}{\partial B} \geq 0,\, \beta \in (0,1],\, B \geq 0,\, \gamma \in (0,1],\, d > 0\right\} \tag{G-27}$$

is empty.

- When the DM plans on covering the debt in two periods, we follow a similar approach. In particular, his problem can be expressed as

$$\max_{p \in [0,\gamma b]} f_m^E(p,\gamma),$$

where $f_m^E(p,\gamma) := \lambda(p)\left(V_T(B - \frac{p}{\gamma}, 1) - V_T(b,1)\right)$ in this case. The price he charges then is

$$q_m^E(b,\gamma) := \frac{2\beta b\gamma + \alpha(1-2\gamma) + \sqrt{\alpha^2 + 2\alpha(\alpha - \beta b)\gamma + 4(\alpha-\beta b)^2\gamma^2}}{3\beta}.$$

Having characterized the DM's pricing policy, we can now express the debt value

$$D(\{B,\gamma\}) = \lambda(q_m^E(B,\gamma)) \cdot \left[q_m^E(B,\gamma) + \lambda\left(p_T^\dagger\left(b - \frac{q_m^E(B,\gamma)}{\gamma}, 1\right)\right)\left(b - \frac{q_m^E(B,\gamma)}{\gamma}\right)\right] + (1 - \lambda(q_m^E(B,\gamma))) \cdot \lambda(p_T^\dagger(B,1)) \cdot B.$$

Similarly, expected revenues are

$$J(\{B,\gamma\}) = \lambda(q_m^E(B,\gamma)) \cdot \left[q_m^E(B,\gamma) + \lambda\left(p_T^\dagger\left(b - \frac{q_m^E(B,\gamma)}{\gamma}, 1\right)\right) p_T^\dagger\left(b - \frac{q_m^E(B,\gamma)}{\gamma}, 1\right)\right] + (1 - \lambda(q_m^E(B,\gamma))) \cdot \lambda(p_T^\dagger(B,1)) \cdot p_T^\dagger(B,1).$$

We now argue that expected revenues $J(\{B,\gamma\})$ are non-decreasing in γ, implying that $\gamma = 1$ would be an optimal early payment discount in this case as well. In particular, one can then readily verify, using sum-of-square techniques, that the set (G-27) is empty.

Since $\gamma = 1$ is without loss an optimal early payment discount in all cases, we have $J^\dagger = J^E$.

Claim 2: $J^\dagger \leq J^R$. We now deal with contracts with debt relief. If $r = 0$, we recover the plain contract. If $r = 1$ debtholders have the option of adjusting the debt repayment at the beginning of period T so as to maximize debt value. That is, if the outstanding debt is $b_T > 0$, debtholders adjust debt repayment by solving

$$\max_{0 \leq b \leq b_T} \lambda(\pi(b)) \cdot b.$$

Thus, debt is adjusted to $\min\{p_T^\star, b_T\}$. Consequently, no debt relief will take place if the outstanding debt is low enough, $b_T \leq p_T^\star$.

As we shall prove later, the DM's price at $T-1$ under an optimal contract with debt relief with $r = 1$ is always such that it leads to no debt relief when a sale takes place. That is, the DM's price at $T-1$ is always higher than $B - p_T^\star$.

Using the above observation as a fact, we now analyze the cases where the DM follows a strategy of covering the debt in one or two periods at $T-1$ separately. Note also that if $r = 0$, or if $r = 1$ and $B \leq p_T^\star$, debtholders would never adjust the debt. In other words, the contract $\{B, 0\}$ is essentially a plain contract with debt repayment B, and so $\{B, 1\}$ when $B \leq p_T^\star$. We henceforth deal with contracts for which $r = 1$ and $B > p_T^\star$:

- When the DM plans on covering the debt in one period, his problem can be expressed as

$$\max_{p \in (b, \frac{\alpha}{\beta}]} f_\ell^R(p),$$

  where $f_\ell^R(p) := \lambda(p)\left(p - b + V_T(0,1) - V_T(p_T^\star, 1)\right)$ in this case. The price he charges then is $q_\ell^R(b) := \frac{1}{2}\left(b + V_T(p_T^\star, 1) - V_T(0,1) + \frac{\alpha}{\beta}\right)$.

  Having characterized the DM's pricing policy, we can now express the debt value

$$D(\{B,1\}) = \lambda(q_\ell^R(B)) \cdot B + (1 - \lambda(q_\ell^R(B))) \cdot \lambda(p_T^\dagger(p_T^\star, 1)) \cdot p_T^\star, \qquad \text{(G-28)}$$

  where the first term corresponds to the debt payment the debtholders receive if the DM makes a sale at $T-1$, and the second term in case he makes a sale only at T—after his debt is adjusted.

  Similarly, expected revenues are

$$J(\{B,1\}) = \lambda(q_\ell^R(B))(q_\ell^R(B) + V_T(0,1)) + (1 - \lambda(q_\ell^R(B))) \cdot \lambda(p_T^\dagger(p_T^\star, 1)) \cdot p_T^\dagger(p_T^\star, 1). \qquad \text{(G-29)}$$

- When the DM plans on covering the debt in two periods, we follow a similar approach. In particular, his problem can be expressed as

$$\max_{p \in [0,b]} f_m^R(p),$$

where $f^R_m(p) := \lambda(p)\left(V_T(B-p,1) - V_T(p^\star_T,1)\right)$ in this case. The price he charges then is

$$q^E_m(b) := \frac{4\beta b - 2\alpha + \sqrt{19\alpha^2 - 16\alpha\beta b + 4\beta^2 b}}{6\beta}.$$

Having characterized the DM's pricing policy, we can now express the debt value

$$D(\{B,1\}) = \lambda(q^R_m(B)) \cdot \left[q^R_m(B) + \lambda\left(p^\dagger_T\left(b - q^R_m(B),1\right)\right)\left(b - q^R_m(B)\right)\right] \\ + (1 - \lambda(q^R_m(B))) \cdot \lambda(p^\dagger_T(p^\star_T,1)) \cdot p^\star_T. \tag{G-30}$$

Similarly, expected revenues are

$$J(\{B,1\}) = \lambda(q^R_m(B)) \cdot \left[q^R_m(B) + \lambda\left(p^\dagger_T\left(b - q^R_m(B),1\right)\right) p^\dagger_T\left(b - q^R_m(B),1\right)\right] \\ + (1 - \lambda(q^R_m(B)) \cdot \lambda(p^\dagger_T(p^\star_T,1)) \cdot p^\dagger_T(p^\star_T,1). \tag{G-31}$$

We now show that $J^\dagger \le J^R$ need not always hold with equality, nor with strict inequality. To this end, consider the following instances:

- For $d = 2$, $\alpha = 1$ and $\beta = 0.13$, by solving (6b) we obtain that $\{4.11\}$ is an optimal plain contract, under which the DM charges a price $q_\ell = 5.15$, yielding revenues $J^\dagger = J(\{4.11\}) = 3.26$. Under a contract with debt relief ($r = 1$ and $B > p^\star_T$), by solving (6b) we obtain $B = 4.2$, $q^R_\ell = 5.22$ and $J(\{4.2,1\}) = 3.28$ when the DM follows a single-period debt-covering strategy. Consequently, $J^\dagger < J(\{4.2,1\}) \le J^R$.
- For $d = 2$, $\alpha = 1$ and $\beta = 0.1$, by solving (6b) we obtain that $\{3.32\}$ is an optimal plain contract, under which the DM charges a price $q_\ell = 5.97$, yielding revenues $J^\dagger = J(\{3.32\}) = 4.74$. Under a contract with debt relief ($r = 1$ and $B > p^\star_T$), by solving (6b) we obtain $B = 10.2$, $q^R_\ell = 9.17$ and $J(\{10.2,1\}) = 2.7$ when the DM follows a single-period debt-covering strategy; $B = 13.3$, $q^R_\ell = 9.18$ and $J(\{13.3,1\}) = 2.64$ when the DM follows a two-period strategy. Thus, the optimal contract with debt relief has $r = 0$, in particular, $\{3.32,0\}$, and $J^\dagger = J^R$ for this instance.

To complete the proof of this claim, we now revisit the possibility of the DM charging a price that is lower than $B - p^\star_T$ at $T-1$, under an optimal contract with debt relief, $r = 0$ and $B > p^\star_T$. Under such circumstances, whether the DM makes a sale at $T-1$ or not, his outstanding debt at the beginning of T would remain higher than $p^\star_T$, and thus trimmed to that level. Because the DM becomes indifferent about the price he charges at $T-1$, we assume that he charges a revenue-maximizing price. We distinguish two cases:

– When $B - p_T^\star > p_T^\star$, the revenue-maximizing price that is lower than $B - p_T^\star$ is precisely $p_{T-1}^\star(2) = p_T^\star$. Thus, the debt value can be written in this case as

$$\tilde{D}(\{B,1\}) = \lambda(p_T^\star) \cdot \left[p_T^\star + \lambda\left(p_T^\dagger\left(p_T^\star,1\right)\right)p_T^\star\right] + (1 - \lambda(p_T^\star)) \cdot \lambda(p_T^\dagger(p_T^\star,1)) \cdot p_T^\star.$$

Similarly, expected revenues are

$$\tilde{J}(\{B,1\}) = V_T(0,1) + \lambda(p_T^\dagger(p_T^\star,1)) \cdot p_T^\dagger(p_T^\star,1).$$

Suppose now that such a contract is optimal. It will therefore yield higher revenues than a debt relief contract $\{B',1\}$ that induces a price $q_\ell^R(B') > p_T^\star$. In other words,

$$\Big\{(\beta,B,B',d)\ :\ \tilde{J}(\{B,1\}) > J(\{B',1\}),$$
$$\tilde{D}(\{B,1\}) = d,\ B > 2p_T^\star,\ D(\{B',1\}) = d,\ B' > p_T^\star,\ \beta > 0,\ d > 0\Big\}$$

is non-empty, where the expressions for J and D are as in the single-period debt-covering strategy case above. However, one can readily use sum-of-squares techniques to show that the above set is, in fact, empty.

– When $B - p_T^\star \le p_T^\star$, the revenue-maximizing price that is lower than $B - p_T^\star$ is precisely $B - p_T^\star$. Thus, the debt value can be written in this case as

$$\tilde{D}(\{B,1\}) = \lambda(B - p_T^\star) \cdot \left[B - p_T^\star + \lambda\left(p_T^\dagger\left(p_T^\star,1\right)\right)p_T^\star\right] + (1 - \lambda(B - p_T^\star)) \cdot \lambda(p_T^\dagger(p_T^\star,1)) \cdot p_T^\star.$$

Similarly, expected revenues are

$$\tilde{J}(\{B,1\}) = \lambda(B - p_T^\star) \cdot (B - p_T^\star) + \lambda(p_T^\dagger(p_T^\star,1)) \cdot p_T^\dagger(p_T^\star,1).$$

Suppose now that such a contract is optimal. It will therefore yield higher revenues than a debt relief contract $\{B',1\}$ that induces a price $q_\ell^R(B') > p_T^\star$. In other words,

$$\Big\{(\beta,B,B',d)\ :\ \tilde{J}(\{B,1\}) > J(\{B',1\}),$$
$$\tilde{D}(\{B,1\}) = d,\ B \le 2p_T^\star,\ B > p_T^\star,\ D(\{B',1\}) = d,\ B' > p_T^\star,\ \beta \in (0,1],\ d > 0\Big\}$$

is non-empty, where the expressions for J and D are as in the single-period debt-covering strategy case above. However, one can readily use sum-of-squares techniques to show that the above set is, in fact, empty.

Claim 3: $J^R < J^A$. We now compare contracts with debt relief and contracts with debt amortization. We first claim that, in equilibrium, any debt amortization contract $\{B, \theta\}$ must satisfy $B \leq p_T^\star$. To that end, consider two contracts $\{B, \theta\}$ and $\{B', \theta'\}$ such that $B \leq p_T^\star < B'$. The DM's pricing policy for a given debt amortization contract is fully characterized in Proposition G.1. When $B \leq p_T^\star$, the DM charges a price $q_\ell^1(B) \geq B$ in period $T-1$, and the price $p_T^\star$ is always charged in period T. Therefore, the value of the debt and the expected revenues under the contract $\{B, \theta\}$ can be written as:

$$D\big(\{B,\theta\}\big) = \lambda\big(q_\ell^1(B)\big)B + \big[1 - \lambda\big(q_\ell^1(B)\big)\big]\lambda(p_T^\star)B \tag{G-32a}$$

$$J\big(\{B,\theta\}\big) = \lambda\big(q_\ell^1(B)\big)q_\ell^1(B) + \lambda(p_T^\star)p_T^\star. \tag{G-32b}$$

It is worth noting that the expressions above are independent of θ.

Under contract $\{B', \theta'\}$, two cases can arise, depending on whether $B' \leq \tilde{B}$ (see Proposition G.1):

- if $B' \leq \tilde{B}$, the DM charges a price $q_\ell^2(B') \geq B'$ at $T-1$, and $p_T^\star < B'$ is always charged at T. The value of the debt and the expected revenues are thus:

$$\begin{aligned} D'_\ell\big(\{B',\theta'\}\big) &:= \lambda\big(q_\ell^2(B')\big)B' + \big[1 - \lambda\big(q_\ell^2(B')\big)\big]\lambda(p_T^\star)p_T^\star \\ J'_\ell\big(\{B',\theta'\}\big) &:= \lambda\big(q_\ell^2(B')\big)q_\ell^2(B') + \lambda(p_T^\star)p_T^\star. \end{aligned}$$

  As with the contract $\{B, \theta\}$, these expressions do not depend on θ'. Using Proposition G.1 to express D, J, D'_ℓ and J'_ℓ as functions of $B, \theta, B', \theta', \alpha$ and β, we can then verify through sums-of-squares techniques that the set

$$\Big\{(\alpha, \beta, B, B', d) \,:\, J'_\ell(\{B',\theta'\}) > J(\{B,\theta\}),\; D'_\ell(\{B',\theta'\}) = d,\; D(\{B,\theta\}) = d, \\ B' \geq p_T^\star,\; B \leq p_T^\star,\; \alpha \in [0,1],\; \beta > 0, d > 0\Big\}$$

  is always empty, which proves that the contract $\{B, \theta\}$ dominates $\{B', \theta'\}$.

- if $\tilde{B} < B' \leq \frac{2\alpha}{\beta}$, the DM charges a price $q_m^\theta(B') \leq B'$ at $T-1$. Provided that this results in a sale, the DM then charges a price $p_T^\dagger\big(B' - q_m^\theta(B')\big)$ at T; otherwise, the price $p_T^\star$ is charged. The debt value and the expected revenues thus become:

$$\begin{aligned} D'_m\big(\{B',\theta'\}\big) &:= \lambda\big(q_m^\theta(B',\theta')\big)\Big[q_m^\theta(B',\theta') + \lambda\Big(p_T^\dagger\big(B' - q_m^\theta(B',\theta')\big)\Big) \cdot \big(B' - q_m^\theta(B',\theta')\big)\Big] \\ &\quad + \big[1 - \lambda\big(q_m^\theta(B',\theta')\big)\big]\lambda(p_T^\star)p_T^\star \\ J'_m\big(\{B',\theta'\}\big) &:= \lambda\big(q_m^\theta(B',\theta')\big)\Big[q_m^\theta(B',\theta') + \lambda\Big(p_T^\dagger\big(B' - q_m^\theta(B',\theta')\big)\Big) \cdot p_T^\dagger\big(B' - q_m^\theta(B',\theta')\big)\Big] \\ &\quad + \big[1 - \lambda\big(q_m^\theta(B',\theta')\big)\big]\lambda(p_T^\star)p_T^\star. \end{aligned}$$

Note that two sub-cases can be further distinguished depending on θ', which determines $q_m^\theta(B',\theta')$ per Proposition G.1. For each case, using Proposition G.1 to express D, J, D'_m, J'_m as functions of $B,\theta,B',\theta',\alpha,\beta$ and θ', we can verify again through sums-of-squares techniques that the set

$$\Big\{(\alpha,\beta,B,B',\theta',d)\ :\ J'_m(\{B',\theta'\}) > J(\{B,\theta\}),\ D'_m(\{B',\theta'\}) = d,\ D(\{B,\theta\}) = d,\\ B' \geq p_T^\star,\ B \leq p_T^\star,\ \alpha\in[0,1],\ \beta>0, d>0\Big\}$$

is also empty, which confirms that the contract $\{B,\theta\}$ again dominates the contract $\{B',\theta'\}$.

We now proceed to complete the proof that $J^R < J^A$. Based on the argument above, we only need to consider contracts with debt amortization $\{B,\theta\}$ with $B \leq p_T^\star$. Consider a contract allowing debt relief $\{B',r\}$, with $r\in\{0,1\}$. Based on the argument in Claim 2, we distinguish two cases:

- For $r=1$, let $D'(\{B',1\})$ and $J'(\{B',1\})$ denote the corresponding debt value and expected revenues, respectively. The expressions for D' and J' are available from Case 2, and are respectively given by either (G-28)-(G-29) or (G-30)-(G-31), depending on whether the DM plans on covering the debt with one or two sales. In each case, we can use sums-of-squares techniques to verify that the set

$$\Big\{(\alpha,\beta,B,B',d)\ :\ J'(\{B',1\}) \geq J(\{B,\theta\}),\\ D'_m(\{B',1\}) = d,\ D(\{B,\theta\}) = d,\ B\leq p_T^\star,\ \alpha\in[0,1],\ \beta>0, d>0\Big\}$$

  is always empty, where $D(\{B,\theta\})$ and $J(\{B,\theta\}$ are given by (G-32a)-(G-32b). This proves that a contract with debt amortization always strictly dominates one with debt relief here.

- For $r=0$, the debt relief contract becomes a plain contract. Based on the analysis in the proof of Proposition 4.1, the debt value $D'(\{B',0\})$ for this contract is given by:

$$D'(\{B',0\}) = \begin{cases} \lambda\big(q_\ell(B')\big)B' + \big[1-\lambda\big(q_\ell(B')\big)\big]\lambda\big(p_T^\dagger(B',1)\big)B', & \text{if } B'\leq \widehat{b}\\ \lambda\big(q_m(B')\big)\big[q_m(B') + \lambda\big(p_T^\dagger(B'-q_m(B'),1)\big)\cdot(B'-q_m(B'))\big] & \\ \qquad +\big[1-\lambda\big(q_m(B')\big)\big]\lambda\big(p_T^\dagger(B',1)\big)B', & \text{if } B'>\widehat{b}, \end{cases}$$

  where $q_\ell(B')$ and $q_m(B')$ are given by (E-5) and (E-6), respectively. Similarly, the expected revenues $J'(\{B',0\})$ are:

$$J'(\{B',0\}) = \begin{cases} \lambda\big(q_\ell(B')\big)\big[q_\ell(B') + \lambda(p_T^\star)p_T^\star\big] \\ \qquad\qquad + \big[1-\lambda\big(q_\ell(B')\big)\big]\lambda\big(p_T^\dagger(B',1)\big)p_T^\dagger(B',1), & \text{if } B' \le \widehat{b} \\ \lambda\big(q_m(B')\big)\big[q_m(B') + \lambda\big(p_T^\dagger(B'-q_m(B'),1)\big)\cdot p_T^\dagger(B'-q_m(B'),1)\big] \\ \qquad\qquad + \big[1-\lambda\big(q_m(B')\big)\big]\lambda\big(p_T^\dagger(B',1)\big)\cdot p_T^\dagger(B',1), & \text{if } B' > \widehat{b}. \end{cases}$$

For each of the two cases above, we can use sums-of-squares techniques to verify that the set

$$\Big\{(\alpha,\beta,B,B',d) \,:\, J'(\{B',0\}) \ge J(\{B,\theta\}), \\ D'(\{B',0\}) = d,\, D(\{B,\theta\}) = d,\, B \le p_T^\star,\, \alpha\in[0,1],\, \beta>0, d>0\Big\}$$

is always empty, where $D(\{B,\theta\})$ and $J(\{B,\theta\}$ are given by (G-32a)-(G-32b). This proves that a contract with debt amortization always strictly dominates a plain contract, and completes the proof of **Claim 3**.

Claim 4: $J^A < J^\star$. Consider an arbitrary debt amortization contract $\{B,\theta\}$. If $B \le \frac{\alpha}{2\beta}$, by Proposition G.1, the price charged in period $T-1$ will be $\tilde{p}_{T-1}(B,2) > p_{T-1}^\star(2) = \frac{\alpha}{2\beta}$, and thus $J(\{B,\theta\}) < J^\star$. Conversely, if $B > \frac{\alpha}{2\beta}$, then even if the DM actually charged $\frac{\alpha}{2\beta}$ in period $T-1$, the price in period T upon making a sale would be $p_T^\dagger\big(B-\frac{\alpha}{2\beta}\big) > p_T^\star$, so that again $J(\{B,\theta\}) < J^\star$. Thus, $J^A < J^\star$. ☐

Proposition G.1. *Consider a contract with debt amortization $\kappa = \{B,\theta\}$, and let b_T denote the DM's outstanding debt in period T. The DM's prices are given by:*

$$\forall\, y,\ \tilde{p}_T(b_T,y) = \begin{cases} p_T^\dagger(b_T,y), & \textit{if } b_T \le (1-\theta)B \\ p_T^\star(1), & \textit{otherwise} \end{cases}$$

$$\tilde{p}_{T-1}(B,2) = \begin{cases} q_\ell^1(B) := \frac{1}{4}\Big[B(2-\alpha)+\frac{2\alpha}{\beta}\Big], & \textit{if } 0 < B \le \frac{\alpha}{2\beta} \\ q_\ell^2(B) := \frac{4\beta B+\alpha(4-\alpha)}{8\beta}, & \textit{if } \frac{\alpha}{2\beta} < B \le \tilde{B} \\ q_m^\theta(B,\theta), & \tilde{B} < B \le \frac{2\alpha}{\beta}, \end{cases}$$

$$\textit{where}\ \ q_m^\theta(B,\theta) := \begin{cases} \frac{\alpha+\beta B}{3\beta}, & \textit{if } \theta \le \frac{1}{3}+\frac{\alpha}{3\beta B} \\ \theta B, & \textit{otherwise}, \end{cases}$$

and $\tilde{B} \in \left[\frac{\alpha}{2\beta}, \frac{\alpha}{\beta}\right]$ *depends on* α, β *(and* θ*, if* $\theta > \frac{1}{3} + \frac{\alpha}{3\beta B}$*). Furthermore,*

$$\begin{aligned}
\tilde{p}_{T-1}(B,2) &> p^\star_{T-1}(2) = \frac{\alpha}{2\beta} \\
\tilde{p}_{T-1}(B,2) &\geq B, \quad \forall\, B \in (0, \tilde{B}) \\
\tilde{p}_{T-1}(B,2) &< B, \quad \forall\, B \in \big(\tilde{B}, \frac{2\alpha}{\beta}\big].
\end{aligned}$$

Proof of Proposition G.1. The price charged in period T is either $p^\star_T(1)$ (if $b_T > (1-\theta)B$, i.e., the DM loses control) or $p^\dagger_T(b_T)$ (if the DM retains control). In the latter case, the value function at time T would thus be given by the (proof of) Proposition 4.1. We separate the analysis for period $T-1$ into different cases, depending on the value of B.

Case 1. $B \in [0, \frac{\alpha}{2\beta}]$. When no sale occurs at $T-1$, the DM loses control but still achieves a positive expected payoff of $\lambda\big(p^\star_T(1)\big)\big(p^\star_T(1) - b\big) = \frac{\alpha(\alpha-2\beta B)}{4\beta}$. Therefore, $\tilde{p}_{T-1}(B,2) \in \arg\max_p f(p)$, where

$$f(p) := \begin{cases} f_\ell(p) := \lambda(p)\Big[p - B + \frac{\alpha}{2\beta}\lambda\big(\frac{\alpha}{2\beta}\big) - \frac{\alpha(\alpha-2\beta B)}{4\beta}\Big], & p \in [B, \frac{\alpha}{\beta}] \\ f_m(p) := \lambda(p)\Big[\frac{\lambda^2(B-p)}{4\beta} - \frac{\alpha(\alpha-2\beta B)}{4\beta}\Big], & p \in [\theta B, B) \\ f_h(p) := \lambda(p)\Big[\frac{\alpha[\alpha-2\beta(B-p)]}{4\beta} - \frac{\alpha(\alpha-2\beta B)}{4\beta}\Big], & p \in [0, \theta B). \end{cases}$$

We analyze each of the maximization problems separately, and compare their optimal values.

- For $\max_{p\in[B,\frac{\alpha}{\beta}]} f_\ell(p)$, note that f_ℓ is a concave, quadratic function, achieving its maximum at

  $$q^1_\ell(B) := \frac{1}{4}\Big[B(2-\alpha) + \frac{2\alpha}{\beta}\Big]. \tag{G-33}$$

  Furthermore, $B \leq q^1_\ell(B) \leq \frac{\alpha}{\beta}$ always holds, so that $q^1_\ell(B)$ is also the optimal constrained decision.

- For $\max_{p\in[\theta B,B)} f_m(p)$, note that f_m is cubic. By solving the equation $(f_m)'(p) = 0$, we obtain the critical points $\frac{-\alpha+2\beta B\pm\sqrt{7\alpha^2-10\alpha\beta B+\beta^2B^2}}{3\beta}$. It can be checked that for $B \in [0, \frac{\alpha}{2\beta}]$, the critical point given by a plus sign takes a value larger than B and corresponds to a local maximum, while the other critical point is negative, and corresponds to a local minimum. Therefore, the optimal decision is always $p \to B$, resulting in an optimal value $f_m(B)$.

- For $\max_{p\in[0,\theta B)} f_h(p)$, note that f_h is concave, quadratic, with a maximum achieved at $\frac{\alpha}{2\beta}$. Since $\frac{\alpha}{2\beta} \geq B \geq \theta B$, it is optimal to take a price $p \to \theta B$, resulting in a value of $f_h(\theta B)$.

We now compare the values above. It can be checked that $f_m(B) = f_\ell(B)$, and $f_m(\theta B) - f_h(\theta B) = \frac{1}{4}\beta B^2(1-\theta)^2(\alpha - \theta\beta B) \geq 0$ (when $B \leq \frac{\alpha}{2\beta}$). Therefore, the DM's price in **Case 1** is always $q^1_\ell(B)$. To complete this case, it can be readily checked that $q^1_\ell(B) \geq p^\star_{T-1} = \frac{\alpha}{2\beta}$, and that $q^1_\ell(B) \geq B$.

Case 2: $B \in (\frac{\alpha}{2\beta}, \frac{\alpha}{\beta}]$**.** When no sale occurs at $T-1$, the DM achieves a payoff of zero. Therefore, $\tilde{p}_{T-1}(B,2) \in \arg\max_p f(p)$, where

$$f(p) := \begin{cases} f_\ell(p) := \lambda(p)\big[p - B + \frac{\alpha}{2\beta}\lambda\big(\frac{\alpha}{2\beta}\big)\big], & p \in [B, \frac{\alpha}{\beta}] \\ f_m(p) := \frac{\lambda(p)\lambda^2(B-p)}{4\beta}, & p \in [\theta B, B) \\ f_h(p) := \frac{\lambda(p)\alpha[\alpha - 2\beta(B-p)]}{4\beta}, & p \in \big[B - \frac{\alpha}{2\beta}, \theta B\big) \\ 0, & p \in \big[0, B - \frac{\alpha}{2\beta}). \end{cases} \tag{G-34}$$

We again analyze the maximization problems separately, and compare their optimal values to determine the DM's pricing decision.

- The problem $\max_{p \in [B, \frac{\alpha}{\beta}]} f_\ell(p)$ parallels the one considered in **Case 1**. In particular, note that f_ℓ is a concave quadratic that achieves its maximum at the value $q_\ell^2(B) := \frac{4\beta B + \alpha(4-\alpha)}{8\beta}$. While $q_\ell^2(B) \le \frac{\alpha}{\beta}$ always holds, note that $q_\ell^2(B) \ge B$ if and only if $B \le \frac{\alpha(4-\alpha)}{4\beta}$. Thus, the optimal price and expected payoff in this case are respectively given by

$$p_\ell(B) = \begin{cases} q_\ell^2(B), & B \in \big(\frac{\alpha}{2\beta}, \frac{\alpha(4-\alpha)}{4\beta}\big] \\ B, & B \in \big(\frac{\alpha(4-\alpha)}{4\beta}, \frac{\alpha}{\beta}\big] \end{cases} \qquad F_\ell(B) = \begin{cases} \frac{(\alpha(4+\alpha) - 4B\beta)^2}{64\beta}, & B \in \big(\frac{\alpha}{2\beta}, \frac{\alpha(4-\alpha)}{4\beta}\big] \\ \frac{\alpha^2(\alpha - \beta B)}{4\beta}, & B \in \big(\frac{\alpha(4-\alpha)}{4\beta}, \frac{\alpha}{\beta}\big]. \end{cases} \tag{G-35}$$

- For $\max_{p \in [\theta B, B)} f_m(p)$, note that f_m is cubic. By solving the equation $(f_m)'(p) = 0$, we obtain the critical points $B - \frac{\alpha}{\beta}$ and $q_m^\theta(B) := \frac{\alpha + \beta B}{3\beta}$. When $B \in (\frac{\alpha}{2\beta}, \frac{\alpha}{\beta}]$, it can be checked that the former critical point is always negative and corresponds to a local minimum, while $q_m^\theta(B)$ corresponds to a local maximum, and always satisfies $q_m^\theta(B) < B$. Furthermore, $q_m^\theta(B) \ge \theta B$ if and only if $\theta \le \frac{1}{3} + \frac{\alpha}{3\beta B}$. Therefore, f_m is increasing on $[\theta B, q_m^\theta(B)]$ and decreasing for $p \ge q_m^\theta(B)$, so that the optimal price and payoff are respectively given by

$$p_m(B) = \begin{cases} q_m^\theta(B), & \theta \le \frac{1}{3} + \frac{\alpha}{3\beta B} \\ \theta B, & \text{otherwise} \end{cases} \qquad F_m(B) = \begin{cases} \frac{(2\alpha - \beta B)^3}{27\beta}, & \theta \le \frac{1}{3} + \frac{\alpha}{3\beta B} \\ \frac{(\alpha - \theta\beta B)(\alpha - (1-\theta)\beta B)^2}{4\beta}, & \text{otherwise.} \end{cases} \tag{G-36}$$

- For $\max_{p \in [B - \frac{\alpha}{2\beta}, \theta B)} f_h(p)$, note that f_h is concave, quadratic, with a maximum achieved at $q_h^\theta(B) := \frac{\alpha + 2\beta B}{4\beta}$. Under $B \in (\frac{\alpha}{2\beta}, \frac{\alpha}{\beta}]$, it can be checked that $q_h^\theta(B) \ge B - \frac{\alpha}{2\beta}$ always holds, and $q_h^\theta(B) < \theta B$ if and only if $\theta > \frac{1}{2} + \frac{\alpha}{4\beta B}$. Thus, the optimal price and payoff are given by

$$p_h(B) = \begin{cases} \theta B, & \theta \le \frac{1}{2} + \frac{\alpha}{4\beta B} \\ q_h^\theta(B), & \text{otherwise} \end{cases} \qquad F_h(B) = \begin{cases} \frac{\alpha(\alpha - 2(1-\theta)\beta B)(\alpha - \theta\beta B)}{4\beta}, & \theta \le \frac{1}{2} + \frac{\alpha}{4\beta B} \\ \frac{\alpha(3\alpha - 2\beta B)^2}{32\beta}, & \text{otherwise.} \end{cases} \tag{G-37}$$

We now compare the optimal values in the problems above. Letting $\mathcal{C}$ denote the set of constraints $\{\alpha \ge 0,\, \alpha \le 1,\, \beta \ge 0,\, B \ge \frac{\alpha}{2\beta},\, B \le \frac{\alpha}{\beta},\, \theta \ge 0,\, \theta \le 1\}$, it can be checked that:

$$\big\{(\alpha,\beta,\theta,B) \,:\, F_h(B) > F_m(B),\;\; \mathcal{C}\big\} = \emptyset.$$

This requires testing three different conditions, depending on θ, and the corresponding sets can always be shown to be empty using sums-of-squares techniques. Therefore, $F_m(B) \ge F_h(B)$ always holds. Similarly, it can also be tested using sums-of-squares techniques that:

$$\begin{aligned}
\emptyset &= \Big\{(\alpha,\beta,\theta,B) \,:\, F_\ell(B) \ge F_m(B),\;\; B \ge \frac{\alpha(4-\alpha)}{4\beta},\;\; \mathcal{C}\Big\} \\
\emptyset &= \Big\{(\alpha,\beta,\theta,B) \,:\, F'_\ell(B) > F'_m(B),\;\; B \le \frac{\alpha(4-\alpha)}{4\beta},\;\; \mathcal{C}\Big\} \\
\emptyset &= \Big\{(\alpha,\beta,\theta) \,:\, F_\ell\Big(\frac{\alpha}{2\beta}\Big) \le F_m\Big(\frac{\alpha}{2\beta}\Big),\;\; \alpha \ge 0,\, \alpha \le 1,\, \beta \ge 0,\, \theta \ge 0,\, \theta \le 1\Big\} \\
\emptyset &= \Big\{(\alpha,\beta,\theta) \,:\, F_\ell\Big(\frac{\alpha(4-\alpha)}{4\beta}\Big) > F_m\Big(\frac{\alpha(4-\alpha)}{4\beta}\Big),\;\; \alpha \ge 0,\, \alpha \le 1,\, \beta \ge 0,\, \theta \ge 0,\, \theta \le 1\Big\}.
\end{aligned}$$

These results imply that there exists a $\tilde{B}$ such that $F_\ell(B) > F_\ell(B)$ for $B \in \big[\frac{\alpha}{2\beta}, \tilde{B}\big)$ and $F_\ell(B) \le F_m(B)$, $\forall B \in [\tilde{B}, \frac{\alpha(4-\alpha)}{4\beta}]$. This exactly yields the pricing policy in the statement of the proposition. To complete the proof, note that $\tilde{p}_{T-1}(B,2) \ge \frac{\alpha}{2\beta}$ is immediate, and also $q_\ell^2(B) \ge B$, $q_m^\theta(B) \le B$.

Case 3: $B \in (\frac{\alpha}{\beta}, \frac{2\alpha}{\beta}]$**.** In this case, the DM must rely on two sales, and his payoff is always zero when failing to make the intermediate sale. Therefore, $\tilde{p}_{T-1}(B,2) \in \arg\max_p f(p)$, where

$$f(p) := \begin{cases} 0, & p > \frac{\alpha}{\beta} \\ f_m(p) := \frac{\lambda(p)\lambda^2(B-p)}{4\beta}, & p \in \big[\theta B,\, \frac{\alpha}{\beta}\big] \\ f_h(p) := \frac{\lambda(p)\alpha[\alpha - 2\beta(B-p)]}{4\beta}, & p \in \big[B - \frac{\alpha}{2\beta}, \theta B\big) \\ 0, & p < B - \frac{\alpha}{2\beta}. \end{cases}$$

The two optimization problems for f_m and f_h are identical to those considered in (G-34) for Case 2. Therefore, the price that maximizes f_m is $p_m(B)$ and the corresponding payoff is $F_m(B)$, as per (G-36). Since $p_m(B) \le \frac{\alpha}{\beta}$, this remains optimal in Case 3, as well. For the problem of maximizing f_h, a candidate price is $p_h(B)$ given by (G-37). However, in addition to the conditions in Case 2, it can be readily checked that $p_h(B) \geq B - \frac{\alpha}{2\beta}$ holds if and only if $B \leq \frac{3\alpha}{2\beta}$. Letting $F_h(B) = f_h\big(\max\{p_h(B), B - \frac{\alpha}{2\beta}\}\big)$ and using sums-of-squares techniques, it can be verified that:

$$\Big\{(\alpha, \beta, \theta, B) \,:\, F_h(B) > F_m(B), \quad \alpha \geq 0,\, \alpha \leq 1,\, \beta \geq 0,\, B \geq \frac{\alpha}{\beta},\, B \leq \frac{2\alpha}{\beta},\, \theta \geq 0,\, \theta \leq 1\Big\} = \emptyset.$$

This requires testing three cases, depending on θ and B; the feasible set in each case is empty. This proves that the DM's price is $q_m(B)$. To complete the proof, note that $\frac{\alpha}{2\beta} \leq p_m(B) \leq \frac{\alpha}{\beta} \leq B$. □