Optimal multi-horizon portfolios with forward-looking expectations and loss aversion: an application to sovereign wealth funds

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ABSTRACT

This paper presents a framework for portfolio optimization that makes three departures from the traditional mean-variance approach. First, we optimize the portfolio over multiple horizons, reflecting the belief that long-term investors care about intertemporal gains and losses, as well as cumulative performance, rather than simply long-run performance (expressed as a terminal value at the end of the optimization period). Second, rather than approximate through variance, which includes upside performance too, we account for loss aversion by simulating more severe shock events than those captured in historical samples, as well as through a specification of investor utility that sharply penalizes loses beyond a specified threshold. Finally, our framework allows investors to express forward-looking expectations (or make Bayesian adjustments) around how future performance may differ from those observed in the past. We demonstrate the value of the framework and how it could be implemented through a consideration of the problem faced by sovereign wealth funds with long-term investment horizons. While this implementation exercise is illustrative, we find that these adjustments – which more realistically capture the observed behaviour of sovereign wealth funds as long-term investors than the traditional mean-variance heuristic – result in meaningful shifts in optimal portfolio weights.

Introduction

Most sovereign wealth funds have a mandate to provide savings and income on an intergenerational basis. Consequently, they are rare examples of investors that can approach portfolio construction from a long-term perspective. Indeed, sovereign wealth funds frequently reference their long investment horizons as a defining attribute, often presenting it as a structural advantage over other investors. The New Zealand Superannuation Fund, one of the world's most sophisticated sovereign wealth funds, for example, states: "As a long-term investor we can exercise more control over the Fund's capital than investors with shorter investment horizons. We have the luxury of being able to pick our investment horizon and are less likely to need to sell assets in response to short-term falls in market value than they are. This is an important competitive advantage or endowment."[1]

The perceived advantages of such long investment horizon include sovereign wealth funds' ability to engage in contrarian or countercyclical investing, invest in illiquid assets that can be difficult to sell quickly and are expected to therefore deliver an illiquidity premium over time, engage in innovative long-term partnerships with like-minded investors, benefit from a wider selection of assets than those available to short-term investors, invest in multi-decade infrastructure projects, and tailor investment strategies to benefit from and protect the portfolio against slow-moving, long-term investment themes (Bolton, Samama and Stiglitz, 2012; and Orr, 2013).

Despite the benefits of a comparatively long investment horizon, long-term investors are not impervious to short- and medium-term losses. Since Samuelson (1979), it has been understood that investors with long horizons should be primarily concerned with cumulative wealth maximization, rather than the expected probability or frequency of gains and losses. Using a dynamic programming approach, Ang (2014) shows that "long-run investors are first and foremost short-run investors. They do everything that short-run investors do, and they can do more because they have the advantage of a long horizon. The effect of the long horizon enters through the indirect

[1] New Zealand Superannuation Fund. "How we invest: long-term investing", available at: www.nzsuperfund.nz/how-we-invest/endowments/long-term-investing

utility in each one-period optimization problem" (Ang, 2014: 121). Similarly, a former head of the New Zealand sovereign wealth fund argues that a long horizon "means much more than just buying long-dated assets. We think of it more as being able to repeat investment activity for as long as the institution exists. So, we can make both high-frequency investments and long-dated investments; we can choose the horizon on which we want to pursue particular investment activities."[2]

Capturing the advantages afforded by long investment horizons, therefore, is a much more complex undertaking than merely optimizing portfolios based on long-term average returns and correlations between major asset classes, with a focus on maximizing the terminal value at the of the optimization period. The traditional mean-variance optimization framework, for example, provides a useful heuristic for thinking about key elements of portfolio construction, but fails to account for several important dynamics that matter for long-term investors. We argue in this paper that a series of departures from the assumptions and inputs of mean-variance type portfolio optimization frameworks can more accurately capture the way sovereign wealth funds invest in practice and how they perceive and frame the risk of losses.

The first set of departures in our framework pertain to the need to consider the portfolio-optimization problem across multiple horizons, rather than just the long term. This is important due to the cumulative-wealth maximization point of Samuelson and Ang, discussed earlier; but also because of the empirical fact that relationships between leading asset classes are unstable over time, with correlations (and hence diversification benefits) varying depending on the market and/or macroeconomic regime (Barberis, 2000; Ang and Bekaert, 2004; Kritzman and Li, 2010; and Kinlaw, Kritzman, Page and Turkington, 2021). Even if there is weak predictability to these changes in returns and correlations between asset classes, long-term investors can still account for this information in their *ex ante* portfolio-construction process to achieve improvements in risk-return efficiency at the portfolio level; and if there is higher predictability, they can potentially outperform static strategic benchmarks by dynamically "tilting" or tactically allocating exposures. Our approach, as discussed in Sections 2 and 3 of the paper, involves constructing a mixed-

[2] McKinsey & Co. (2017). "Sovereign wealth funds and pensions: the future is collaborative," Interview with Adrian Orr, available at: www.mckinsey.com/industries/private-equity-and-principal-investors/our-insights/sovereign-wealth-funds-the-future-is-collaborative

frequency return sample that blends short-term returns with long-term returns, thus enabling us to capture the nuances and dynamic changes in correlations across the multiple return horizons that matter to long-term investors.

Our second innovation pertains to the conceptualization and specification of investor utility. In a cumulative long-term, what matters to investors is not only the average returns, correlations, and probability/frequency of losses, but also the timing, sequencing, and magnitude of total losses (and subsequent recoveries). A more realistic conception of investor utility, particularly for sovereign wealth funds, should consider long-term investor's deep aversion to shocks whose magnitude or timing could be regarded as an existential threat, because wealth levels may never fully recover. Below we discuss both rational and behaviorally determined reasons for loss aversion, which leads to us to introduce a kinked utility function that sharply penalizes loses that exceed a specified threshold.

The third and final departure in our framework draws on the fact that long-term investors are likely to have "views" or form expectations about future market dynamics and investment themes. Such forward-looking expectations typically vary from observed history, as the Australian Future Fund stated in a recent paper on what it termed the "New Investment Order." Listing no less than ten "paradigm shifts" affecting long-term investors, the Australian sovereign wealth fund notes "the investment thinking that has delivered strong returns over recent decades needs to be revisited." The fund concludes by calling for a "new approach that tests long-held assumptions and questions the conventional wisdom that has guided institutional investors over recent decades."[3]

In current industry terminology, the formulation of such medium- to long-term views is often presented as "thematic investing." In more technical language, it may be referred to as Bayesian adjustments or Bayesian updating. What we wish to account for in our portfolio optimization framework is the possibility that long-term allocators believe that the future will look different from the past. Investors may believe that structural changes to financial markets and the real economy will alter the correlation between leading asset classes, and that asset-class and sectoral

[3] Future Fund (2021). "A New Investment Order", Position Paper, September 2021, available at: www.futurefund.gov.au/news-room/position-paper---a-new-investment-order

performance will drift away from their observed historical trends. A popular current investment theme, for example, is built around the idea that the combination of repeated large-scale fiscal stimulus programs, zero interest rates, central bank asset purchases, and disruptions to global supply chains and trade will result in a period of sustained inflation. Other popular themes or drift scenarios under discussion by sovereign wealth funds include the impact of climate change and climate-change adaptation policies, the increasing popularity of ESG investments, technological innovation and disruption, and various long-term demographic shifts (often linked to the rise of emerging markets and emerging-market consumers, and the aging of Western societies).

The bar for including such themes into the portfolio construction process (or indeed as a factor in "bottom-up" security selection) is high: first, long-term investors need to correctly identify the underlying medium- to long-term theme; second, they need to anticipate its differential impact on various asset classes; and, finally, they need to convince themselves that capital markets have not (yet) fully priced in the anticipated impact, but will do so over time. Despite this high bar, it is exactly the long-term nature of their investment horizon that has led many sovereign wealth funds to conclude that a thematic overlay is essential to their portfolio construction and management process. Again, the New Zealand Superannuation Fund's articulation is an instructive summary of this mindset: "Thematic investing is about identifying and investing into return streams positively exposed to the themes and avoiding those negatively exposed. Our investment thesis is that these exposures will not usually be fully priced by markets given they are 'slow burn' in nature and subject to uncertainty. Thematic investing is therefore very much aligned with the Fund's long-term investment horizon."[4]

The remainder of the paper is structured as follows. First, we provide an overview of long-term sovereign wealth funds' most salient features and how these inform their portfolio construction problem. We argue that while sovereign wealth funds are indeed able to invest for the long term and absorb certain types of short-term losses, their long-term investment objectives must still be pursued with due consideration of intertemporal risks that long-term sovereign wealth funds are averse to. We also consider how leading sovereign wealth funds are incorporating views on

[4] New Zealand Superannuation Fund. "How we invest: themes", available at: www.nzsuperfund.nz/how-we-invest/themes

medium- to long-term investment themes around climate-change adaptation, inflation risks, and technological innovation into their portfolios. The second section of the paper outlines the logic, practical appeal, and methodology of our multi-period optimization framework, called Full-Scale Optimization. We start with a conceptual overview of the ways in which the approach improves on mean-variance portfolio optimization, before proceeding with a discussion of our methodology. The final section discusses the application of Full-Scale Optimization, possible applications, and extensions of this approach, and draws out the most salient policy implications for sovereign wealth funds with intergenerational investment mandates.

I. Sovereign wealth funds as long-term investors

In the broadest terms, sovereign wealth funds are defined as investment funds "owned or controlled by the government [to] hold, manage, or administer assets primarily for medium- to long-term macroeconomic and financial objectives" (International Monetary Fund, 2008). Within this broad conception, however, there is a significant degree of heterogeneity amongst global sovereign wealth funds along several dimensions. Sovereign wealth funds differ in terms of their purpose, which includes fiscal and external stabilization, long-term savings and inter-generational wealth distribution, income generation, and developmental investment. Accordingly, they differ in terms of their investment styles, which range from highly liquid short-term stabilization funds to long-term permanent- or endowment fund models, pursued through strikingly different long-term asset allocations and execution models.

Finally, there are considerable differences in how sovereign wealth funds are capitalized, with the common sources of funds including state proceeds from commodity production and exports, such royalties, taxes, profits, export earnings; general or commodity-based fiscal surpluses; foreign exchange reserves; and privatization proceeds. Funding procedures for sovereign wealth funds run the gamut from entirely *ad hoc* to strictly rule based. It follows that there cannot be a single optimal portfolio solution for all sovereign wealth funds, and what is required is a flexible framework that can be tailored to idiosyncratic (and multi-dimensional) risk tolerances, opportunities, and structural advantages.

The functions, features and constraints of long-term sovereign wealth funds

To focus the discussion, we ignore both stabilization funds, where the portfolio largely consists of short-term assets, and sovereign development funds, whose investment mandates are not strictly commercial. We focus, therefore, on long-term sovereign wealth funds, whose defining characteristics are framed by their performance of the following functions:

- First, they are vehicles for the *long-term investment* of public savings, thus preserving or growing a share of otherwise transitory windfalls for future generations. The prototypical case is a sovereign wealth fund managing a temporary revenue windfall from oil or another commodity. The fund is a vehicle for the continuous transformation of a share of oil wealth and revenue into financial wealth and income. This transformation occurs because policymakers wish to diversify the overall national endowment away from oil and into financial assets. Under an appropriate set of rules, this permanent endowment will outlast the period of commodity extraction, thus promoting intergenerational equity.

- Second, they contribute towards the *provision of income*, which can be permanent and eventually replace other sources of public revenue, under the appropriate set of fiscal rules and governance arrangements. In this sense, sovereign wealth funds contribute to the national budget in much the same way an endowment portfolio contributes to annual revenues of a university. The fund is a permanent financial endowment of the sovereign, providing a stabilizing stream of regular fiscal income. Using a steady spending-growth rule or something like a percentage-of-market-value spending policy, the provision of income can be permanent, sustainable, and uncorrelated with oil revenue. If the sovereign wealth fund portfolio is constantly growing through regular contributions from oil income during the period of oil extraction, permanent income from the fund complements – and eventually replaces – oil revenue.

- Finally, through a combination of saving and spending policies, this kind of sovereign wealth fund promotes *macroeconomic and fiscal stability*, providing counter-cyclical income or transfers that enhance resilience to internal and external shocks. The fund can also become a recipient of sustainable revenue booms that would otherwise be misallocated and destabilizing. For example, the fund may receive larger inflows of oil revenue in boom periods and make higher-than-average contributions to the budget in bust periods, when oil prices and revenues collapse.

The three major functions outlined above are not mutually exclusive, but they do need to be balanced through the calibration of spending and saving policies. Famous sovereign wealth funds, such as those in Norway, Alaska, Wyoming, and the Middle East, for example, perform several inter-related functions: they contribute to the stability of fiscal revenues, support inter-generational transfers of finite oil revenues and the sustainability of resource-financed public spending, and help counteract Dutch disease and commodity dependence. It is worth emphasizing that some of these functions are not exclusive to funds established through the transfer of resource revenues but can be generalized to other long-term sovereign wealth funds, capitalized through other sources of public surplus.[5] These sovereign wealth funds are also part of a process of transforming one form of wealth into a financial portfolio, to provide a combination of current or future income or serve as a buttress against shocks. A form of self-insurance against debt, banking, and balance-of-payments crises, for example, played an important role in foreign exchange reserve accumulation, which spawned several sovereign wealth funds, in many Asian countries in the aftermath of the Asian financial crisis of 1997-1998 (Aizenman and Marion, 2004). Further, sovereign wealth funds such as GIC and Temasek provide regular transfers to the general budget.

Several implications emerge for the portfolio construction process. First, this type of sovereign wealth fund is the prototypical manager of a long-term portfolio: the fund is intended to be an intergenerational, permanent endowment that lasts beyond the lifecycle of oil production. That said, the long-term objective is balanced by the fact that the fund has implied liabilities. The provision of fiscal income to the budget in the same way a university endowment or foundation has a spending rule or policy, makes the sovereign wealth fund at least somewhat averse to both short-term and sustained losses. Sovereign wealth funds that mimic the permanent-endowment type structure may be particularly averse to severe losses in their first few years of inception, both out of a concern for institutional credibility and because it can be shown arithmetically that losses and overspending at the early stages of an infinitely lived endowment will have a permanent effect on the capital growth of the endowment and hence permanently lower the sustainable income stream from the fund (Kaufman and Woglom, 2005).

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[5] Non-commodity sovereign wealth funds also invest public windfalls for the benefit of both current and future generations. The China Investment Corporation, Korea Investment Corporation, and the Government Investment Corporation (GIC) of Singapore are capitalized through a share of excess foreign exchange reserves arising from exchange-rate management and associated trade surpluses. Temasek Holdings in Singapore and Khazanah Nasional in Malaysia became strategic state holding companies with broad portfolios of state-owned companies, many of which have been sold, publicly listed and privatized generating proceeded for reinvestment; while the Australian Future Fund and the New Zealand Superannuation Fund a share of general fiscal surpluses to help meet anticipated future increases in public pensions and social security funding costs.

Moreover, there are constraints and preferences that lie outside this fiscal framework. The public servants and political appointees who manage and oversee sovereign wealth funds also have behaviorally determined bounds for the total magnitude of losses they are able to incur, no matter how compelling the expectation that such losses will be offset by subsequent gains. Sovereign wealth fund managers and trustees suffer from loss aversion, as per the behavioral insights from prospect theory, as per Kahnemann and Tversky (1979): losses are feared and disliked more than gains of the same magnitude. There are also rational explanations from such loss aversion: political-economy and institutional factors, such as "career-" or "reputation risk."

Sovereign wealth fund stakeholders – investment managers, trustees, political office holders and policymakers – will undoubtedly exhibit significant loss aversion. They are likely to be more sensitive than they "need" to be from a purely theoretical perspective about acute short-term losses. This need not be solely due to self-interested and myopic "career-risk management" but could also emanate from a valid concern over public and political support for the fund as a public institution (particularly around the fund's inception), a feature that is often called "legitimacy" in the sovereign wealth fund context (Clark, Dixon and Monk, 2013). Loss aversion defines an appetite for risk – and for specific types of intertemporal risks – that is not robustly captured in a mean-variance framework, with its quadratic specification of investor utility. As discussed below, we therefore introduce a kinked utility curve function to capture loss aversion more accurately.

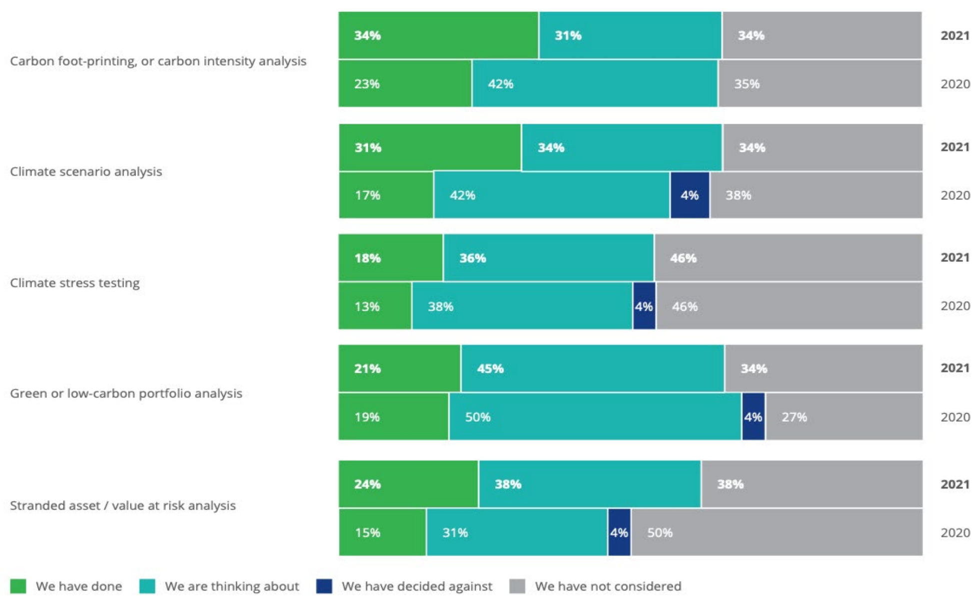
Thematic views, expectations, and Bayesian adjustment: the essence of long-term investing?

Most sovereign wealth funds do not exclusively rely on historical data and observed returns and correlations when constructing portfolios. Indeed, for many sovereign wealth funds, anticipating the ways in which the future may be different to the past – and identifying long-term risks and opportunities that are not (yet) fully priced in – is fundamental to long-term investing. At the very least, even in the likely event that future trends and their impact on markets cannot be predicted, sovereign wealth fund managers and Boards have a responsibility to assess how different scenarios may affect the fund's portfolio, risk profile, and performance. In the tradition of Bayesian inference, they may wish to assign different probabilities to various events, trends, and scenarios

– frequently updating them as more information becomes available – and then regularly conducting "stress tests" of their impact on the portfolio over the long term.

Which themes do sovereign wealth funds typically focus on? Today, almost all sovereign wealth funds focus on the risk of climate change and climate-change mitigation policies, and how best to hedge against risks and capture opportunities arising from the long-term global energy transition (see Figure 1). Indeed, they may have an even larger incentive to do so than other long-term investors, given that their underlying funding and the macroeconomic risks of their host government are so intrinsically tied to oil revenues (van den Bremer, van der Ploeg and Wills, 2016). References to accounting for the long-term impact of climate change, climate-change adaptation, and the energy transition are near-ubiquitous amongst sovereign wealth funds and has grown significantly in recent years.

Figure 1: Sovereign Wealth Fund Survey Responses on Climate-Change Policies



Source: IFSWF

According to a survey conducted by the International Forum for Sovereign Wealth Funds of its member institutions in 2021, for example, 71% of respondents noted that they had adopted an ESG investment approach and fewer than 10% said that they did not consider climate change in their investment approach.[6] Only one year earlier, the corresponding figures were that only 24% of respondents said they included ESG considerations in their investment process, while 48% said they considered climate change in their investment process. While only 9% of respondents reported that they were mandated to address climate change, 65% of sovereign wealth funds were proactively including climate-change considerations in their investment approach, with a majority of respondents stating that they were motivated in doing so by the belief that it would enhance returns and/or reduce risk to the portfolio. Around one-third of sovereign wealth funds stated that they were engaging in climate-change scenario analysis in the 2021 survey. For many sovereign wealth funds, this particular thematic focus is explicitly linked to their long-term investment horizon, as well as the perceived mispricing of risks and opportunities, as exemplified by the New Zealand Superannuation Fund's belief that "the market currently underprices carbon risks – a shorthand for the various risks posed by the impact of climate change" and that this mispricing will only be resolved "over the long time horizon that matters for the [fund's] investment purposes."[7]

A brief review of the investment policies, research output and stated thematic views of leading sovereign wealth funds reveals several areas of common focus, in addition to climate change and adaptation.[8] The most common areas of non-climate focused thematic overlap amongst sovereign wealth funds include:

i. *Inflation risk:* many sovereign wealth funds have expressed concerns over whether the experiences of roughly the past four decades, characterized by a near-inexorable decline in inflation, central bank policy rates and Treasury yields serve as a reliable guide to the future. The combination of large-scale fiscal stimulus packages and ultra-low policy rates and

[6] International Forum of Sovereign Wealth Funds. (2021). "In Full Flow: Sovereign wealth funds mainstream climate change", November 2021, available at: www.ifswf.org/sites/default/files/IFSWF_InFullFlow.pdf

[7] New Zealand Superannuation Fund. (2019). "Climate Change Investment Strategy", How We Invest White Paper, March 2019, available at: www.nzsuperfund.nz/assets/documents-sys/Guardians-of-NZ-Super-Climate-Change-White-Paper-March-2019.pdf

[8] Our analysis is focused on the New Zealand Superannuation Fund, the Australian Future Fund, the Alberta Heritage Savings Trust Fund, the Government Investment Corporation of Singapore, the Canadian Public Pension Investments and the Norwegian Government Pension Fund Global.

central bank asset purchases, coupled with structural factors such as ageing populations in the advanced economies and risks to global supply chains and trade, has led many sovereign wealth funds to question how sustained higher inflation rates and lower real yields on government bonds may affect their portfolios – and indeed whether the traditional role of government bonds as a diversifier of equity risk can be assumed to continue.[9]

ii. *Technological innovation and disruption:* it is very common for long-term investors to attempt to analyze and anticipate the impact of disruptive technology innovation, the digital economy, fintech, robotics, and other technological innovations on their portfolios and leading asset classes and sectors (or indeed to seek an outright exposure to the positive trends expected from such megatrends, often through their Venture Capital allocations).

iii. *The rise of emerging-market consumers, middle-class demographics, and urbanization:* several sovereign wealth funds favor an overweight allocation to emerging markets. This case rests not only on the expectation of higher per capita growth rates, but also on such factors as growing consumer discretionary spending patterns, relatively underdeveloped capital markets that results in potential financial inefficiencies that can enhance investor returns, and potential portfolio diversification benefits from less correlated emerging markets. The thematic case for emerging markets is typically further buttressed by the rapid urbanization. While these trends can be reasonably easily identified, they may not be fully priced into asset values in markets dominated by short-term investors, as these themes likely take years to play out. Sovereign wealth funds have argued that they can access this theme – often with targeted positions *within* an emerging-market allocation – by investing in sectors with positive urban-consumer correlations, such as health- and educational-service providers, high-quality protein producers, and urban and student housing.

Given the pervasive practice of anticipating future trends and deviations from observed historical patterns, it would be helpful to incorporate into the portfolio-construction framework the ability to express or assess the impact of certain thematic views other investor-specific beliefs about long-term drifts in asset prices, provided we can find a plausible proxy to capture the impact of the theme. The framework we employ in this paper is sufficiently flexible to allow investors to assess the impact on portfolio construction and asset allocation of different levels of conviction they may have about the significance of a particular theme, and hence the magnitude of the drift scenario. Further, as our approach allows for optimization across multiple horizons using a mixed-frequency

[9] The Australian Future Fund provides a useful summary of this view: "Government bonds have been the defensive anchor of investment portfolios for over 30 years with the traditional 60/40 equity bond portfolio relying on negative correlation between the two asset classes. Bonds provided the opportunity to add to returns while having downside protection to the equity risk in their portfolios. The world now looks different. Nominal bond yields are significantly lower so the scope for bonds to pay off is reduced. Investors have ended up paying to benefit from bond rallies rather than being paid. If inflation begins to rise the bond-equity correlation may prove much less beneficial going forward." See: www.futurefund.gov.au/news-room/position-paper---a-new-investment-order

return sample, the fact that shocks and long-term drifts or themes play out over different time periods – all of which matter to the long-term investor – can be accounted for. We now turn to a conceptual overview of our multi-horizon optimization approach.

II. Optimizing Over Multiple Horizons: A Conceptual Overview

Our methodology comprises three major steps, which we describe conceptually in this section and through an illustrative application in the following section. The first step is to construct a mixed-frequency return sample that properly blends short-term returns with long-term returns. Following Kritzman and Turkington (2021), we draw on inputs from three sources: observed historical data, simulated shock scenarios, and imposed medium- to long-term drift scenarios, based on forward-looking Bayesian adjustments of how certain investment themes may alter the observed historical performance of asset classes and equity-market sectors. The second major step is to specify investor utility, which in our framework involves a kinked utility function, such that loss aversion is accounted for through the penalization of losses. The final stage is to add up the utilities across short-term and long-term returns for as many portfolios as necessary to identify the portfolio with the highest expected utility, using a technique called Full-Scale Optimization.

Step 1: Constructing a mixed-frequency sample

As with most portfolio optimization methods, our approach starts with historical data on observed asset-price movements and relationships. However, unlike most popular portfolio-construction models, we also wish to treat shock outcomes differently than ordinary returns, as well as gradually emerging thematic drifts in returns. These are expected to cause asset values to drift in one direction or another over time but are invisible from higher frequency observations and not yet fully priced into current asset values. The key innovation of this approach is the blending of return samples in a non-arbitrary way, allowing for the balancing of concerns around both short- and long-term outcomes. The challenge is to create a unified return sample that simultaneously captures three types of scenarios: history, shocks, and thematic drifts, keeping in mind that we observe these scenarios at different frequencies.

History: a starting point

Historical returns are simply replications of a large segment of history, which in most portfolio-construction approaches serve as the sole source of information to guide expectations of the future. Most popular portfolio construction models, notably mean-variance optimization, use as their input merely the statistical summaries of historical samples, rather than accounting for each return individually. Our approach is to include a large historical sample of individual returns of the past five decades, observed at monthly and five-year frequencies as a major component of our overall return sample. This is a robust starting point for the construction of our overall sample, as the historical period used includes multiple monetary and fiscal regimes; significant variations and cycles in interests rates, inflation rates and energy prices; various shock events, including housing and financial crises (not least the 1987 stock market crash, the dotcom bubble, the 2008 global financial crisis, and the COVID-19 crash), as well as significant geopolitical events, and structural changes associated with globalization and the rise of China and other emerging markets.

Shocks

Forward-looking investors can augment the historical record with shock scenarios that are of special concern to them. Sovereign wealth funds suffer from loss aversion, just as other investors do – despite their comparatively long-term investment horizons. Certain shocks cannot be tolerated, as they constitute an existential threat to the institution or its management. Moreover, as noted in the preceding section, many long-term sovereign wealth funds contribute regular investment income to their government owners, thereby creating an implied liability that exacerbates aversion to losses. There are various approaches one might consider to take shocks into account. Our approach is to account for severe short-term shocks by calibrating the utility function to penalize the shock selloffs more severely than losses that occur under more ordinary circumstances.

Drifts: expectations-driven thematic scenarios

Given sovereign wealth funds' efforts to identify themes that may cause returns to drift differently than the pattern of historical returns, it is a useful addition to a portfolio construction process to allow for scenario-based expectations. We consider three scenarios. First, any meaningful analysis of long-term investment themes should attempt to capture the effects of climate change and climate-change mitigation – as this a major concern for a majority of sovereign wealth funds. The second drift scenario we include in our application is that of a persistent rise in inflation, which has recently emerged as a major concern for all investors. Finally, we impose drifts on historical returns due to the assumed impact of technological innovation. We could examine the individual and combined effect of several additional long-term investment themes on expected drifts in returns (as well as iterating between different magnitudes and timing of impact) – as we would encourage practitioners to do. However, to keep the illustration relatively simple, we focus only on these three drift scenarios.

Step 2: Describing sovereign wealth fund utility

The mixed-frequency sample can be used in combination with any utility function. However, the versatility and practical relevance of the mixed-frequency sample is best illustrated in combination with a specification of utility that can capture higher moments of the return distribution than simply the mean and variance. Specifically, given the preceding discussion around sovereign wealth funds' loss aversion, we prefer to specify a kinked utility function, which captures the fact that investors have greater aversion to losses below a certain threshold than they do above that threshold (indicated by the location of the kink in the utility curve).

Step 3: Portfolio formation using Full-Scale Optimization

The final step is to identify optimal portfolios, given the multi-period sample and utility function. To do so, we use a technique called Full-Scale Optimization, first introduced by Adler and Kritzman (2007). This approach involves calculating a portfolio's utility for every period in the mixed-frequency sample, while considering as many asset mixes as necessary to identify the weights that yield the highest expected utility, given the specified utility function. The process starts with an asset mix that one would reasonably expect, based on experience or judgment, to yield an optimal or near-optimal solution. We then compute the sum of the utilities associated with the composite historical, shock, and drift scenarios and store this value. Next, we substitute another asset mix, compute its utility, and store that value. We proceed in this fashion until we have computed utility for enough asset mixes that we are confident one of them yields the maximum utility or a level of utility that is sufficiently close to the maximum. We then rank the utilities and identify the asset mix with the highest sum of utilities. This approach considers all features of the mixed-frequency return sample, including kurtosis, skewness, and any other peculiarities of the sample.

III. Optimizing a Multi-Horizon Portfolio: Implementation

In this section, we illustrate how to construct an optimal multi-horizon portfolio using our approach, according to the process described conceptually in Section 2 and summarized here in Table 1.

Table 1: Multi-Period Portfolio Construction: Summary of Methodology

Step One: **Constructing a Mixed-Frequency Sample**	Step Two: **Specifying Investor Utility**	Step Three: **Identifying Optimal Portfolio Weights**
Sub-step 1.1: Identify the asset classes and equity sectors to be considered for inclusion in the portfolio (Section 3.1.1.) Sub-step 1.2: Construct a sample of historical five-year returns and monthly returns for the chosen asset classes and economic sectors (Section 3.1.2.) Sub-step 1.3: Identify long-term drifts and estimate their effects on cumulative and monthly asset class and sector returns (Section 3.1.3.) Sub-step 1.4: Identify the historical shock experiences and segment the returns associated with the drawdown phases of these shocks (Section 3.1.4.)	Step 2: Specify the utility function to be applied to different segments of the return sample (Section 3.2.)	Step 3: Optimize across asset class returns to determine asset class weights and then again across the equity sector returns to determine weights within the equity component (Section 3.3.)

3.1. Constructing the mixed-frequency sample with drift effects

The first of the major steps is to construct a return sample. This process starts with the specification of the opportunity set through the selection of eligible assets and sectors. After this, we introduce mixed frequencies, while also imposing shock and drift scenarios on the data.

3.1.1. Selecting asset classes and economic sectors

All portfolio constriction processes start with the specification of eligible assets or the investment universe. We start with selection 10 asset classes and 10 economic sectors as our investible universe, which we then group into growth assets, defensive assets, and opportunistic assets, as shown in Table 2. The assignment of sectors and even asset classes to these three groupings may change depending on the drift effects that investors perceive to be to be underway at a particular point in time. This is compatible with the notion and practice of tactical asset allocation, thematic or strategic tilting, and dynamic risk management, as practiced by several sophisticated sovereign wealth funds.

Table 2: Asset Classes and Economic Sectors

Growth Assets	Defensive Assets	Opportunistic Assets
U.S. Stocks	Cash	Global ESG stocks
Non-U.S. Stocks	Treasury Bonds	Commodities
Emerging Markets Stocks	TIPS	Real Estate
	Corporate Bonds	
Growth Sectors (stocks)	Defensive Sectors (stocks)	Opportunistic Sectors (stocks)

Note that the selection of the asset classes and sectors shown in Table 2 is intended for illustrative purposes. The U.S. focus is merely one of convenience, driven by data availability. A sovereign wealth fund outside of America could simply use a different menu of asset classes and economic sectors, subject to the availability and quality of the required high-frequency and long-term data. For a description of how to incorporate illiquid assets into the portfolio construction process, see Kinlaw, Kritzman and Turkington (2013)

3.1.2. Mixed frequency return sample

Because we optimize in two stages, we construct two return samples: one for asset class returns and one for U.S. equity sector returns. Given the computational intensity of our optimization algorithm (see Section 3.3), it is more efficient to optimize this way. It is critical that the return sample has the proper balance of high-frequency and low-frequency returns, which is to say that the return sample must include all the high-frequency returns that make up all the low-frequency returns. If, for example, the sample omitted some of the high-frequency returns, the larger low-frequency returns would have an outsized effect on the solution. The greater number of high-frequency returns balances the larger size of the low-frequency returns. This balance must be preserved even if the sample is comprised of overlapping low-frequency returns. Table 3 shows a stylized rendering of the mixed frequency return sample for three hypothetical asset classes and 50 years of monthly and five-year returns, before it is adjusted to incorporate the drift effects.

Table 3: Stylized Mixed Frequency Asset Class Return Sample (Without Drifts)

Periods	Asset Class Returns		
	Asset Class 1	Asset Class 2	Asset Class 3
Month 1	0.83%	0.42%	0.21%
Month 2	1.25%	0.67%	0.17%
Month 3	-0.42%	0.17%	0.18%
↓			
Month 600	0.55%	0.42%	0.19%
1st 5 Years	64.53%	28.34%	13.30%
2nd Five Years	110.72%	48.98%	10.73%
3rd Five Years	-22.16%	10.51%	11.62%
↓			
10th Five Years	51.03%	29.28%	11.88%

3.1.3. Long-term drifts

The next step is to quantify the long-term drift effects that investors arrive at through forward-looking expectations and incorporate them into the mixed frequency return sample. In technical terms, this is referred to as Bayesian adjustment. Table 4 shows how the stylized mixed frequency return sample changes based on the expectation that Asset Class 1 will drift 10% cumulatively below the five-year returns that occurred historically, while Asset Class 2's five-year cumulative returns will drift 10% higher than historically. These five-year incremental drifts are converted to monthly effects and added to the relevant monthly returns in the stylized mixed frequency return sample, as shown in Table 4.

Table 4: Stylized Mixed Frequency Asset Class Return Sample with Drift Effects

Periods	Asset Class Returns		
	Asset Class 1	Asset Class 2	Asset Class 3
Month 1	0.65%	0.58%	0.21%
Month 2	1.07%	0.83%	0.17%
Month 3	-0.60%	0.33%	0.18%
↓			
Month 600	0.38%	0.58%	0.19%
1st 5 Years	54.53%	38.34%	13.30%
2nd Five Years	100.72%	58.98%	10.73%
3rd Five Years	-32.16%	20.51%	11.62%
↓			
10th Five Years	41.03%	39.28%	11.88%

In our empirical illustration, we assume three long-term drift effects, based on the themes of climate change, a persistent rise in inflation, and technological innovation. Table 5 shows our qualitative assessment of the effects of these three themes. Again, these qualitative assessments are for illustrative purposes. Sovereign wealth fund managers and Boards can impose their own views about how these themes would impact asset class and sector returns.

Table 5: Qualitative Assessment of Drift Effects

Climate Change Assumptions	Technological Innovation Assumptions	Inflation Assumptions
Investors will continue to channel funds to ESG stocks.	Stocks and ESG stocks will benefit from higher productivity and the application of new technologies to climate mitigation.	Cash will benefit from higher interest rates.
Investors will favor non-U.S. stocks to U.S. stocks given relative social values.	Emerging markets will benefit from the ability to import new technologies.	Treasury bonds and corporate bonds will suffer from higher interest rates.
Investors will favor U.S. Treasury bonds over corporate bonds to mitigate risk.	Commodities will benefit from adopting new technologies, which will boost their profitability and, in some cases, increase demand for natural resources.	TIPS will benefit from higher inflation.
Materials sector will benefit from continual disaster recovery and climate risk mitigation.	Communication services and Information Technology will directly benefit from the application of new technologies.	Consumer discretionary will suffer from higher interest rates as consumers retrench.
Energy sector will suffer from shift to clean alternatives and more regulation, while Financials will suffer from insurance losses.	Discretionary spending, Financials, Healthcare, and Industrials will benefit from productivity, efficiency and cost savings due to innovation.	Financials will benefit from an increase in banks' net interest income and earnings.
Emerging markets will suffer from heightened exposure to climate risk.		Utilities will suffer as their share prices drop to maintain competitive yields.

Next, we convert these qualitative assessments into quantitative effects. There are many ways to do this, but in our illustration, we first construct a scorecard in which we assign a +1 for a positive drift effect, a 0 for no effect, and a -1 for a negative drift effect to the U.S. equity sectors and asset classes, outside of U.S. stocks. For this asset class, we take the average of the U.S. equity sector effects to preserve internal consistency. We then sum these effects across the three thematic drifts of climate change, inflation, and technological innovation. We then apply a return multiplier of 10% to these scores to derive the five-year cumulative drift effect for each sector and asset class, which we add to each five-year return in our mixed frequency return sample. We add the corresponding monthly return to all the monthly returns in our mixed frequency sample. Table 6 shows this scorecard approach.

Table 6: Drift Effect Scorecard

	Drift Effect Scores					
	Climate	Technological Innovation	Inflation	Total	5-Year Drift	Monthly Drift
US Sectors						
Communication services	0.0	1.0	0.0	1.0	10.0%	0.16%
Consumer discretionary	-1.0	1.0	-1.0	-1.0	-10.0%	-0.18%
Consumer staples	-1.0	0.0	0.0	-1.0	-10.0%	-0.18%
Energy	-1.0	0.0	0.0	-1.0	-10.0%	-0.18%
Financials	-1.0	1.0	1.0	1.0	10.0%	0.16%
Healthcare	0.0	1.0	0.0	1.0	10.0%	0.16%
Industrials	-1.0	1.0	0.0	0.0	0.0%	0.00%
Information technology	0.0	1.0	0.0	1.0	10.0%	0.16%
Materials	1.0	0.0	0.0	1.0	10.0%	0.16%
Utilities	-1.0	0.0	-1.0	-2.0	-20.0%	-0.37%
Asset Classes						
US Stocks	-0.5	0.6	-0.1	0.0	0.0%	0.00%
Non-US stocks	0.0	0.0	-0.1	-0.1	-1.0%	-0.02%
Emerging stocks	-1.0	1.0	-0.1	-0.1	-1.0%	-0.02%
Global ESG stocks	0.7	1.3	-0.2	1.9	19.0%	0.29%
US Treasury bonds	1.0	0.0	-1.0	0.0	0.0%	0.00%
US TIPS	1.0	0.0	1.0	2.0	20.0%	0.30%
US corporate bonds	0.0	0.0	-1.0	-1.0	-10.0%	-0.18%
Real estate	0.0	0.0	1.0	1.0	10.0%	0.16%
Commodities	0.0	1.0	1.0	2.0	20.0%	0.30%
Cash	0.0	0.0	1.0	1.0	10.0%	0.16%

Sector return multiplier = 10%
Asset class return multiplier = 10%

Clearly, it is impossible to know precisely how these drift effects will impact returns. Our purpose here is to illustrate how a mechanism for tilting a portfolio in response to these long-term drift effects can be established and integrated into a multi-horizon portfolio optimization process. Our demonstration is illustrative and simplistic; however, the approach is compatible with investors' efforts to map identified investment themes into differentiated expected drifts in medium- to long-term asset-class and sector returns.

3.1.4. Shocks

The final sub-step in the construction of the mixed-frequency return sample is to identify the monthly returns that are associated with the drawdown phases of shock experiences. In our stylized return sample, we designate Month 3 as a shock return. We designate the returns associated with the drawdown phases of the Long-Term Capital Management Crisis, the collapse of the dotcom bubble, the Global Financial Crisis, and the COVID-19 shock as sharp drawdown events.

3.2. Specifying utility function for different segments of return sample

Having constructed a mixed-frequency return sample drawing on a combination of history, shocks, and imposed drifts on future returns, the second major step in our process is to specify a function that captures the utility of a prototypical sovereign wealth fund. We use a kinked utility function to convert a portfolio return into the utility of each monthly and five-year return, as shown in Equation 1.

$$U_{kinked}(R) = \begin{cases} \frac{(1+R)^{1-\gamma}-1}{1-\gamma}, for\ r \geq k \\ \frac{(1+R)^{1-\gamma}-1}{1-\gamma} - \omega(k-R), for\ r < k \end{cases} \tag{1}$$

The term $U_{\text{kinked}}(R)$ is expected utility, R is the return of the portfolio, k is the location of the kink, γ determines the curvature of the function above the kink, and ω is the slope of the function below the kink. With this utility function, investor satisfaction drops precipitously when returns fall below the level of the kink; while, when returns are above the kink, investor satisfaction conforms to a power utility function. This utility function is designed to express a stronger aversion to losses below the kink than above the kink.

This specific utility function is not required, but it affords the investor a great deal of flexibility to capture nuances in investor preferences, in line with the discussion in Section 1 of this paper regarding the type of loss aversion sovereign wealth funds likely have despite their comparatively long investment horizon. We calibrate this utility function as shown in Table 7 for the different segments of the return sample. Setting γ = to 0 is equivalent to assuming investors have log-wealth utility above the kink.

Table 7: Utility Function Calibration

	K	γ	ω
Monthly returns	-0.01	0	1.5
Shock returns	-0.08	0	2.5
5-Year returns	0.15	0	2.0

It is useful to emphasize that the calibration of the utility function is one of the levers that investors can deploy to manage the allocation of the portfolio. The other levers are the assessment and calibration of the drift effects and the simulation of shock events. The calibration chosen for our empirical illustration assumes that investors are more averse to monthly losses associated with the drawdown phases of shocks than they are to monthly losses that occur during non-shock periods, and that investors are more averse to sustained losses than they are to short-term losses. Both these forms of loss aversion can be easily reconciled with the observed, real-world behavior and incentives of sovereign wealth funds. That said, our framework is not dependent on this specification of the utility or the specific calibration we use in this application.

3.3. Optimizing across asset classes and equity sectors

The final step in our process is to use an optimization procedure called Full-Scale Optimization to determine the asset class weights. This procedure, first introduced by Adler and Kritzman (2007), starts by calculating the portfolio return of a candidate asset class mix for each monthly and five-year period. It then converts these periodic portfolio returns into utility values, given the different specifications of the utility function that are applied to the different segments of the mixed frequency return sample. Next, it sums all the utilities for the monthly and five-year returns and stores this value. It then repeats this process for another asset mix and carries on in this fashion until enough portfolios have been evaluated to ensure that one of the portfolios offers the highest possible utility.

Table 8: Full-Scale Optimization

First Trial									
Periods	Asset Class Returns			Portfolio Weights			Portfolio Return	Utility Function	Utility
	Asset Class 1	Asset Class 2	Asset Class 3	Asset Class 1	Asset Class 2	Asset Class 3			
Month 1	0.65%	0.58%	0.21%	65%	35%	0%	0.63%	U = f(W)	0.0060
Month 2	1.07%	0.83%	0.17%	65%	35%	0%	0.99%	U = f(W)	0.0074
Month 3	**-0.60%**	**0.33%**	**0.18%**	**65%**	**35%**	**0%**	**-0.27%**	**U = f'(W)**	**-0.0034**
↓									
Month 600	0.38%	0.58%	0.19%	65%	35%	0%	0.45%	U = f(W)	0.0043
Sum of monthly utilities									2.1362
1st 5 Years	64.35%	28.50%	13.30%	65%	35%	0%	51.81%	U = f''(W)	0.4921
2nd Five Years	110.54%	49.14%	10.73%	65%	35%	0%	89.05%	U = f''(W)	0.6679
3rd Five Years	-22.34%	10.67%	11.62%	65%	35%	0%	-10.78%	U = f''(W)	-1.2132
↓									
10th Five Years	50.85%	29.44%	11.88%	65%	35%	0%	43.36%	U = f''(W)	0.4119
Sum of five-year utilities									0.8969
Sum of monthly and five-year utilities									3.0331
Second Trial									
Periods	Asset Class Returns			Portfolio Weights			Portfolio Return	Utility Function	Utility
	Asset Class 1	Asset Class 2	Asset Class 3	Asset Class 1	Asset Class 2	Asset Class 3			
Month 1	0.65%	0.58%	0.21%	60%	35%	5%	0.61%	U = f(W)	0.0058
Month 2	1.07%	0.83%	0.17%	60%	35%	5%	0.94%	U = f(W)	0.0071
Month 3	**-0.60%**	**0.33%**	**0.18%**	**60%**	**35%**	**5%**	**-0.23%**	**U = f'(W)**	**-0.0029**
↓									
Month 600	0.38%	0.58%	0.19%	0.65	0.35	0	0.45%	U = f(W)	0.0043
Sum of monthly utilities									2.1263
1st 5 Years	64.35%	28.50%	13.30%	60%	35%	5%	49.25%	U = f''(W)	0.4679
2nd Five Years	110.54%	49.14%	10.73%	60%	35%	5%	84.06%	U = f''(W)	0.6305
3rd Five Years	-22.34%	10.67%	11.62%	60%	35%	5%	-9.09%	U = f''(W)	-1.0222
↓									
10th Five Years	50.85%	29.44%	11.88%	65%	35%	0%	43.36%	U = f''(W)	0.4119
Sum of five-year utilities									1.2202
Sum of monthly and five-year utilities									3.3465

The computational intensity of this process is significant, but there are efficient search algorithms that converge to a solution fairly quickly. For our empirical illustration, we employed a genetic optimization algorithm followed by a pattern search to identify the full-scale optimal allocation. A genetic algorithm is a numerical method for solving constrained and unconstrained optimization problems based on natural selection. It is partly inspired by Charles Darwin's theory of evolution and therefore relies on operating concepts such as mutation, crossover, and selection.[10] Table 8 extends our stylized return sample to demonstrate Full-Scale Optimization, using this approach, showing two hypothetical trials out of the many thousands of trials that may be required to arrive at a solution.

Notice that the sum of monthly utilities for the first trial is slightly greater than the sum of monthly utilities for the second trial. However, the sum of five-year utilities is significantly greater for the second trial than the first trial, rendering the asset mix used in the second trial superior to the asset mix used in the first trial. The more conservative asset mix may have greater utility because perhaps the asset classes have less desirable diversification properties, based on cumulative five-year returns, than their monthly returns suggest. Intuitively, we can think here of corporate bonds, which are typically relatively closely correlated with government bonds over shorter horizons but more correlated with equities in the long run. Part of the appeal of Full-Scale Optimization is that it captures co-movement at different return frequencies.

For our empirical illustration, we apply Full-Scale Optimization to 10 asset classes to arrive at the asset class weights; and then again to 10 U.S. equity sectors to arrive at sector weights within equity markets. Based on the five-year and monthly drifts shown in Table 6 and the utility specifications presented in Table 7, the optimal mix of asset classes is given by the right-most

[10] The algorithm first creates a random initial population of asset class weights, then scores and scales the population. At each iteration, a selection rule, mutation rule, and crossover rule are applied. A selection rule randomly selects the individual variables, called parents, that contribute to the population of the next generation. The crossover rule combines two parents to form children for the next generation. The mutation rule applies random changes of the parent weights to form a child. We use the genetic algorithm to guide the search toward the general region in which the solution lies.

Once we have narrowed the search based on the genetic algorithm, we employ a pattern search that uses an adaptive mesh technique to refine the search for the optimal portfolio. Unlike classical optimization techniques, this technique is a direct search method that does not need information about gradients or higher derivatives to search for an optimal point. In a direct search algorithm, each iteration computes a sequence of portfolios (the mesh) surrounding a portfolio solution (current point) to search for values of the utility function that are higher than the current point. When a point in the mesh is found to have a higher utility than the current point, it becomes the new center point, and a new mesh is evaluated. This process is repeated for a specified number of iterations or until the optimal point is found.

column in Table 9. As a comparison, we also report the weights of three alternative portfolios, as follows:

i. **The Log-Wealth Portfolio**: derived from applying Full-Scale Optimization only to the monthly returns and with a single specification of the utility function equal to log-wealth utility. It therefore does not consider drift effects that occurred historically nor does it incorporate assumptions about how climate change and inflation may impact returns in a forward-looking manner. Nor does it consider nuances in the utility function such as greater aversion to shock losses or sustained losses.

ii. **The Multi-Horizon Log-Wealth Portfolio**: adds one consideration to the Log-Wealth portfolio, namely that it includes both monthly and five-year returns and therefore considers historical drift effects. However, it does not consider nuances in investor preferences nor the assumed effects of climate change and inflation going forward.

iii. **The Multi-Horizon Optimal Full-Scale Portfolio (without Drift Effects)**: considers all the features of the Multi-Horizon Optimal Full-Scale optimization, except for views about the future drift effects of climate change and inflation.

iv. **The Multi-Horizon Optimal Full-Scale Portfolio with Drift Effects**: accounts for all the richness of the multi-horizon optimization process, including the mixed frequency return sample, the nuanced description of investor preferences and utility, and the Bayesian views regarding the drift effects of climate change, technological innovation, and inflation. It is important to keep in mind that the inclusion of multiple horizons enhances the portfolio construction process beyond the mere fact that it captures differences in higher moments and co-movement between high and low frequency returns. It also enables investors to specify different risk preferences for different segments of the return sample.

The most conspicuous distinction of the Multi-Horizon Optimal Full-Scale Portfolio with Bayesian Views from the other portfolios is the shift from U.S. stocks to Global ESG stocks. This shift occurs because Global ESG stocks are a close substitute for U.S. stocks (they have a 57.49% exposure to U.S. stocks), yet they are assumed to have a favorable drift effect, owing to concerns about climate change and benefits from technological innovation. The fact that ESG stocks and U.S. stocks co-vary tightly at high frequencies, but ESG stocks are assumed to drift upwards over the long term, explains why they are preferred to U.S. stocks in a portfolio optimization framework that takes long-term drifts into account. As discussed below, our framework does, however, provide signals as to *within* equity sector over- and underweights.

Table 9: Multi-Horizon Optimal Full-Scale and Log-wealth Portfolios[11]

	Mean-Variance (Monthly)	Mean-Variance (5 Year)	Multi-Horizon Optimal Full-Scale without Bayesian Views	Multi-Horizon Optimal Full-Scale with Bayesian Views
US stocks	53.24%	46.78%	54.68%	**0.00%**
Non-US stocks	0.00%	0.00%	0.00%	**0.00%**
EM stocks	0.00%	8.22%	6.25%	**10.00%**
Global ESG stocks[12]	0.00%	0.00%	0.00%	**51.77%**
US Treasury bonds	33.68%	6.45%	0.00%	**0.00%**
US TIPS bonds	0.00%	38.55%	25.83%	**32.08%**
US corporate bonds	9.88%	0.00%	7.08%	**0.00%**
Real estate	3.21%	0.00%	6.25%	**6.25%**
Commodities	0.00%	0.00%	0.00%	**0.00%**
Cash	0.00%	0.00%	0.00%	**0.00%**

Table 10 shows the historical performance of the optimal asset allocation for the Mean-Variance portfolios and the Multi-Horizon Full-Scale portfolios. The historical performance serves as a historical description of the portfolios and does not include any prospective views. The results underscore the value of a multi-horizon approach. It reveals that from a monthly perspective, the mean-variance portfolios appear superior to the full-scale portfolios. However, when we consider cumulative long-term performance, the full-scale portfolios dominate the mean-variance portfolios. Moreover, it is important to recognize that their higher volatility is upside volatility – as evidenced by their significantly positive skewness, which is a good thing. Mean-variance analysis does not distinguish between upside and downside volatility, whereas as full-scale optimization does (due to the kinked utility function). It is also important to recognize that the results reported in Table 10 are backward looking and are not influenced by how the drift effects may unfold. If the drift

[11] As constraints on the optimization process, we add typical investment guidelines for cash, emerging markets, and peripheral asset classes with a maximum allocation of 5%, 10%, and 10% respectively. The peripheral asset classes are real estate and commodities.

[12] We use MSCI ACWI ESG Universal index for Global ESG stocks which has a total of 57.49% US exposure as reported by MSCI, February 2022.

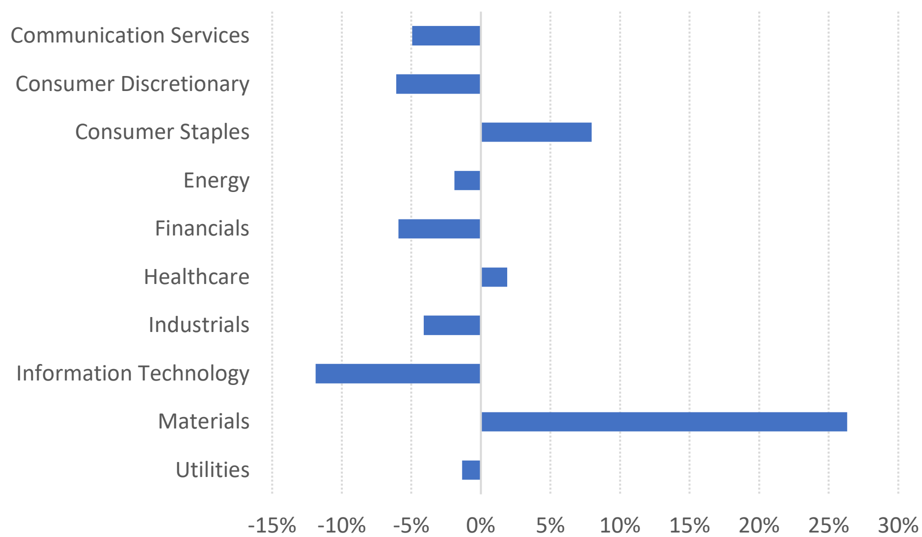
assumptions materialize as assumed, the full-scale portfolio that accounts these thematic effects will outperform mean variance by an even greater margin.

Table 10: Comparing historical performance of mean variance and multi-horizon portfolios

			Multi-Horizon Optimal Full-Scale	
	Mean-Variance (Monthly)	Mean-Variance (5 Year)	without Bayesian Views	with Bayesian Views
Monthly Performance				
Average	0.7%	0.7%	0.7%	0.7%
Volatility	2.4%	2.6%	3.1%	3.3%
Skewness	-0.59	-0.87	-0.86	-0.98
Frequency of Loss	33.8%	33.8%	32.8%	34.1%
Frequency > 10% Loss	0.00%	0.3%	0.7%	0.7%
Worst Month	-9.9%	-13.5%	-15.0%	-17.7%
10th Percentile	-2.3%	-2.6%	-3.1%	-3.4%
90th Percentile	3.4%	3.5%	4.2%	4.2%
Best Month	7.6%	7.9%	9.1%	9.2%
Downside Volatility	2.6%	2.9%	3.5%	3.8%
5 Year Performance				
Average	38.1%	39.7%	43.2%	41.5%
Volatility	20.8%	18.6%	24.9%	26.8%
Skewness	0.32	0.67	0.47	1.13
Frequency of Loss	0.8%	0.4%	1.3%	0.8%
Worst 5 Years	-7.6%	-4.5%	-12.1%	-3.7%
10th Percentile	14.2%	20.8%	16.0%	14.2%
90th Percentile	67.6%	69.3%	78.1%	82.1%
Best 5 Years	99.3%	100.4%	124.7%	127.3%
Downside Volatility	5.6%	4.5%	7.8%	2.6%

Once we solve for the asset class weights, we run a new Full-Scale Optimization to solve for U.S. sector weights. Figure 2 shows the over- and under-weights of U.S. sectors relative to their capitalization exposure within Global ESG stocks. Because we only have data for U.S. sector returns, these active exposures apply only to the U.S. component of Global ESG Stocks. For example, the optimal Multi- Horizon Portfolio has a 51.77 % allocation to Global ESG Stocks, while Global ESG Stocks include a 57.49% exposure to U.S. ESG stocks. Therefore, each adjustment to the sector allocations as a percentage of the total portfolio is scaled by 0.2976 (0.5177 x 0.5749). If we had access to sector returns for all ESG Stocks, we would instead scale the active sector exposures by the weighting in ESG stocks to arrive at the net absolute exposure to equity sectors.

Figure 2: Multi-Horizon Optimal US Sector Active Exposures



Again, we wish to reiterate that this empirical analysis is meant only to illustrate the multi-horizon portfolio construction process. We do not claim that our calibration of the utility function or the assessment of the drift effects represents all investors or even the average investor. Indeed, even our choice of asset classes and sectors is only for illustrative purposes. The comparative weights that we show in Table 9 and Figure 2, therefore, are merely meant to illustrate that the portfolio weights will differ in response to the various considerations that go into the multi-horizon portfolio construction process.

The multi-horizon process for constructing long-term portfolios can be an invaluable tool for clarifying and assessing the impact of various risk sensitivities, preferences, and expectations about future capital market and economic themes. The paper demonstrates the impact on optimal portfolio weights that arises from such conceptually simple steps as using a mixed frequency return sample that captures both short- and long-term correlations between major asset classes, varying description of investor preferences and utility, and the inclusion of simple, plausible Bayesian expectations regarding the drift effects of climate change and inflation. These are all factors that matter to sovereign wealth funds in practice, and the failure to account for them in the portfolio construction process can result is materially sub-optimal asset allocation.

IV. Potential applications and extensions

It is helpful to consider how sovereign wealth funds can practically approach the use of our framework in a real-world setting, including a number of possible extensions and institution-specific refinements. The framework could be refined and customized to become the main portfolio construction model for a sovereign wealth fund, or it can be used to help formalize and structure various critical decision-making processes amongst leading stakeholders.

Probabilistic scenario analysis

Our framework can serve as a tool for structured discussions and analysis of how beliefs and expectations about future trends would affect asset allocation and portfolio construction. As noted at the start of the paper, many sovereign wealth funds argue that anticipating the ways in which the future may be different to the past – and identifying long-term risks and opportunities that are not (yet) fully priced in by the capital markets – is fundamental to their role as long-term investors. If that is the case, it is clear that some sort of framework and systematic approach is required whenever sovereign wealth funds, and indeed other long-term institutional investors, depart from solely relying on past data and experience to construct portfolios.

Our approach allows for variations in scenarios, which is way to test the impact on portfolio construction of varying degrees of conviction around the magnitude of themes and their implications for markets. One could, for example, repeat the same scenarios with different drift terms, as long as the correct balance between short- and long-term returns is preserved, and then modify the size of the drift-return samples to reflect different views of their relative probabilities. Similarly, our fixed-frequency set-up is sufficiently flexible to allow the drift scenario to play out differently across different horizons. One could account for expectations of a "J-curve effect" stemming from an anticipated theme: where, for example, climate change might affect markets in one direction over the short- to medium term, before reversing its impact over longer horizons.

Even if future trends and their impact on markets cannot be predicted, sovereign wealth fund managers and Boards have a duty to assess how different scenarios may affect the fund's portfolio, risk profile and performance. In the tradition of Bayesian inference, they may wish to assign different probabilities to various events, trends, and scenarios, and then periodically update them as new information becomes available. As the probabilities of different long-term scenarios shift, the Board can use our framework to conduct periodic impact assessments on the portfolio. Our framework can be refined to fit institution-specific contexts and help structure discussions between major sovereign wealth fund stakeholders (government owners, Boards, and management) around the impact of various scenarios.

Stress testing

Investors often use Value-at-Risk (VaR) or similar models to measure exposure to loss by estimating the largest loss a portfolio might experience at different confidence levels. The typical approach for calculating VaR is to use the full-sample covariance matrix to compute the portfolio's standard deviation and analyze the probability distribution *at the end* of the investment horizon – similar to way inputs are used in mean-variance portfolio construction. However, this ignores interim dynamics both in the construction and risk management setting. The potential exposure to losses can be more reliably assessed by estimating covariances from sub-periods within the larger sample when markets are experiencing excessive volatility and long-term correlations become unstable, as per Ang and Bekaert's (2004) "regime changes" or Kritzman and Li's (2010) "turbulence" measures, which are helpful for simulating shock scenarios in our framework.

Of course, actual risks – and specifically potentially existential risks – do not manifest as some long-run average over the full long-term sample period. Rather, they concentrate during peak stress periods *within* long-term horizon. To be clear, the claim is not that including such simulated shock scenarios in a mixed-frequency optimization framework necessarily improves the sovereign wealth fund's ability to predict when shocks might occur. Rather, we argue, the mixing short- and long-term samples and utilities is a better captures the consequences of such events and their implications for optimal portfolio weights for a given specification of utility.

Customizing risk aversion

Another attractive feature of our framework is that it gives investors the ability to specify risk aversion and preferences in a highly customized manner. In this paper, we have used a kinked utility function that combines a log-wealth utility conception with more severe aversion to losses above a specified threshold (the location of the kink). However, our framework could be usefully expanded to account for highly idiosyncratic and investor-specific aversions to particular losses. For example, one may argue that an oil-based sovereign wealth fund has an intuitive hedging demand against risks and exposures correlated with oil shocks. If macroeconomic and fiscal risks of the country are closely tied to volatile oil prices, for example, it could be that the sovereign wealth fund would prefer a portfolio that performs relatively well during periods of oil-price collapses.[13] A preference for hedging oil-related risks can be incorporated in the portfolio-construction process as an aversion to acute losses (where one some simulate excessive losses correlated with negative oil-price shocks); and/or as drift scenario (a medium- to long-term thematic view) in which the portfolio is structured to hedge against the expected impact of a multi-decade energy transition away from fossil fuels.

Investor-specific hedging demands and aversions to specific losses can be generalized to other sovereign wealth funds. For example, one could assume that Asian sovereign wealth funds wish to hedge against major disruptions in global trade flows and export demand. Our framework could help identify possible hedging options and the cost (in terms of expected returns or loss of diversification) from employing them.

[13] In practice, this hedging demand is seldom explicitly stated in the investment mandates and policies of oil-based sovereign wealth funds. Indeed, one could argue that a sovereign wealth fund should simply aim to maximize returns relative to a general specification of loss tolerance, irrespective of its funding sources or the macro-fiscal risk of its sovereign owner. Our point here is merely to emphasize that both concepts can be accounted for in our portfolio construction framework.

Critical analysis of diversification benefits

Finally, our framework provides a “sense check” around the merits of including or overweighting different asset classes, sectors, and factors to preserve or enhance portfolio diversification. Currently, for example, as concerns mount over whether government bonds will continue to serve their traditional purposes of income generation and the diversification of equity risk in benchmark portfolios (due to low global policy rates, large central bank bond purchases, and rising inflation), many funds are investing in or considering corporate bonds, real estate, and alternative assets classes. Similarly, most sovereign wealth funds have significantly increased their exposure to real estate and private equity (and indeed other non-listed assets).

The analysis that informs the case for the inclusion or overweighting of asset classes, sectors and factors for diversification benefits often relies mainly on historical, long-term relationships. However, as we argue and demonstrate in this paper, it is import that even long-term investors consider multiple investment horizons. To underline this point, consider the case of corporate bonds, which are often mooted as a replacement for government bonds. While corporate bonds may offer a spread in returns over government bonds (due to higher credit risk and lower liquidity), their contribution to diversification is not straightforward. Corporate bonds are typically highly correlated with government bonds over the shorter horizons, but then more closely correlated with equities in the long run. For the long-term sovereign wealth fund with multi-horizon utility, it is the *combined* short-, medium- and long-term correlations to other assets in the portfolio that matters, as well as the stability of those relationship during periodic stress episodes.

Similarly, the case for investments in real estate needs to be carefully assessed, as its overall correlation and average drawdowns are likely to matter less to the long-term investor than its sensitivity and correlation to interest rate increases and banking crises – that is, how the asset class, sector or factor performs in times of general market stress. Private equity needs to be carefully assessed for the risk of merely mimicking a concentrated equity positions in small- to mid-cap listed stocks, combined with leverage, a combination of factors that exposes to private-equity investor to various overall and within-period risks that are not necessarily evident from the smoothed and infrequently reported results on the performance of private equity funds.

Our multi-horizon portfolio optimization framework, with its used of mixed frequency samples and sensitivity of shocks and thematic trends, is a useful way for investors to assess the true diversification benefits of different asset classes, sectors, or factors under consideration by the Board or investment committee of a sovereign wealth fund.

Conclusion

The primary point we demonstrate in this paper is that a number of conceptually simple adjustments that sovereign wealth funds should make to traditional portfolio-optimization frameworks result in meaningful changes to optimal portfolio weights – and improved long-run risk-return characteristics of optimized portfolios. We focus on three major departures from the mean-variance optimization approach: (i) using a mixed frequency return sample that captures both short- and long-term relationships between major asset classes and enables intertemporal utility maximization, (ii) adjusting the specification of utility to reflect different degrees of loss aversion for different segments of the return sample, and (iii) allowing future asset classes and sector returns to drift away from historical observations based on forward-looking, thematic expectations.

Our implementation of the proposed framework, in line with what we characterize as the typical long-term portfolio optimization problem of a sovereign wealth fund, is relatively basic, with the focus being on demonstrating proof of concept. The inclusion of multiple horizons enhances the portfolio construction process beyond the mere fact that it captures differences in higher moments and co-movement between high and low frequency returns. It also enables investors to specify different risk preferences for different segments of the return sample.

Our results underline the importance of thinking deeply about unique, institution-specific advantages, risk tolerances and preferences. This includes the fact that long-term investors, such as sovereign wealth funds, do not care only about long-term performance or the terminal value of wealth at the end of the optimization period, with no concern for portfolio dynamics that occur in the interim. Moreover, if sovereign wealth funds regard thematic investing as an important and even definitive element of long-term investing, significant resources need to be deployed on research and analysis around the identification of investment themes, the articulation of forward-looking expectations and how these are expected to result in drifts in future asset class and sector trends. We show that even modest drifts result in significant changes in optimal portfolio weights.

In practice, the portfolio that best suits a particular sovereign wealth fund will ultimately depend on the fund's owners and Board's choice of eligible assets and the calibration of the various inputs, including an investor-specific expression of utility and the parameterization of forward-looking expectations. These are not simple undertakings, but they are invaluable discussions to have – ideally as part of a periodically updated portfolio-optimization process – between key sovereign wealth fund stakeholders, such as the government, the Board, and the executive. Our framework is a tool for framing such discussions amongst stakeholders, which can help sovereign wealth funds enhance long-term risk-adjusted returns by constructing portfolios that more realistically and accurately capture their specific risks, preferences, and views about the future.

The authors express their gratitude to Cel Kulasekaran for his thoughtful comments and computational assistance.

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